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Exercise 1 of Theoretische Physik II: Elektrodynamik
Fourier analysis

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Exercise 1 (8 points): *Einstein summation convention*

The totally antisymmetric Levi-Civita symbol ϵ_{ijk} in three dimensions is defined by

$$\epsilon_{ijk} := \begin{cases} 1 & (i,j,k) \text{ even permutation of } (1,2,3) \\ -1 & (i,j,k) \text{ odd permutation of } (1,2,3) \\ 0 & \text{else} \end{cases}$$

1. Calculate the following important expressions using the Einstein summation convention for a twice continuously differentiable function $\phi(\mathbf{x})$ as well as a twice continuously differentiable vector field $\mathbf{A}(\mathbf{x})$: (i) $\nabla \times (\nabla \times \mathbf{A})$, (ii) $\nabla \cdot (\nabla \times \mathbf{A})$, (iii) $\nabla \times \nabla\phi$, (iv) $\nabla \cdot (\phi\mathbf{A})$. (2 points)

2. Prove the following important identities: (2 points)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$
$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

3. Calculate the expression ∇r as well as, assuming $r \neq 0$, $\nabla(1/r)$ and $\Delta(1/r)$, where $r = |\mathbf{x}|$. (2 points)

4. For which functions $f(r)$ is the vector field $\mathbf{A}(\mathbf{x}) = f(r)\mathbf{x}$ divergenceless on the region $\mathbf{R}^3\backslash\{0\}$? (2 points)

Exercise 2 (2 points): *Integral theorems*

Calculate expressions 1.1.(ii) und 1.1.(iii) using the theorems of Gauß and Stokes. (2 points)

Exercise 3 (8 points): *Fourier analysis*

1. Consider the function $f(x) = |x|, x \in [-\pi, \pi]$ and its periodic continuation on $\mathbf{R}$. Calculate the real Fourier series. Using this result, derive the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(3 points)

2. Derive the following identity in an analogous way, considering the function $f(x) = x^2, x \in [-\pi, \pi]$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(This is exactly $\zeta(2)$, where $\zeta(n) = \frac{1}{\Gamma(n)} \int_0^\infty du \frac{u^{n-1}}{e^u - 1} = \sum_{k=1}^{\infty} \frac{1}{k^n}, n \in \mathbf{N}$ is the Riemannian Zeta-function.) (3 points)

3. Calculate the Fourier transformation of a Gaussian packet of width a: $f(x) = \frac{1}{\sqrt{a}} e^{-\frac{x^2}{2a^2}}$. What is the width of the Fourier transformed function? (You may use the fact that $\int_{-\infty}^{\infty} dx\, exp[-\alpha(x + i\beta)^2] = \sqrt{\frac{\pi}{\alpha}}, \alpha \in \mathbf{R}$, a result which can be derived in complex analysis.) (2 points)