NAKED SINGULARITIES

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Introduction

In preparation for my master thesis I give at the 20th of January a talk in the seminar 'Advanced Seminar on Relativity and Cosmology' about naked singularities. Here is a brief outline of the topics I want to talk about:

Outline

- Gravitational collapse and naked singularities
- Cosmic censorship
- Charging a black hole
- Summary and the Third law of Black-Hole-Dynamics

Gravitational collapse and naked singularities

The topic of naked singularities arises from the question: How does a Black Hole form? This leads us to the topic of so-called gravitational collapse. There exist a plenty of models to describe matter, that collapse into itself to form a black hole. One of these many models is the so-called Laimaître-Tolmann-Bondi model (1933) with the line element:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{(\partial_r R)^2}{1 + 2 \cdot E(r)}\mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2. \qquad (1)$$

This is a solution to Einstein's field equations for a spherical shell of dust under the influence of gravity, that is expanding or collapsing. Under certain initial conditions and assumptions one recovers the standard Schwarzschild solution or the Friedmann-Laimaître equations. But also so-called naked singularities can occur (these were found the first time numerically by Eardley et. al. in 1979). Now I want to specify what a naked singularity is:

Definition A naked singularity is a gravitational singularity, i.e. a point in spacetime which is infinitely large curvature, which is not hidden behind an event horizon (black hole horizon).

Now one could ask: Where is the problem with these 'objects'? The thing is, that the occurrence of such a non-hidden singularity would break down the predictability of general relativity itself. We could not say anything about any trajectory of particles that move within this spacetime. Since this inconsistency is really annoying and seems to be contradictionary to our real would, Penrose developed in 1969 the idea of the so-called **cosmic censorship**:

Cosmic censorship

There are indeed two different statements of the cosmic censorship:

Weak case There can be no singularity visible from future null infinity.

Strong case General relativity is a deterministic theory, i.e. the classical fate of all observers should be predictable from initial date.

Apart from the more or less general agreement in this conjecture, there are, up to now, no proves for this two conjectures.

Charging a black hole

In order to 'check' if the above statements are correct we want to consider the case of a non-rotating charged black hole, so-called Reissner-Nordström-black hole. It is given via the following line element

$$\mathrm{d}s^2 = -\left(\frac{\Delta}{r^2}\right)\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \qquad (2)$$

where $\Delta = r^2 - 2Mr + Q^2$, M is the mass of the black hole and Q denotes its charge. From this we can calculate, that the so-called event horizon is given by

$$r_\mathrm{H} = M + \sqrt{M^2 - Q^2} := M\left(1 + \sqrt{1 - \lambda^2}\right), \qquad (3)$$

where we introduced the so-called charge-to-mass parameter $\lambda = \frac{Q}{M}$. This question I want to consider now is: Can we enlarge Q (resp. λ) such that $\lambda > 1$ and therefore $r_\mathrm{H} \in \mathbb{C}$? This would correspond to a vanishing event horizon and a naked singularity.

The simplest idea is to trough into the black hole charges particles, i.e. we search for suitable values of the charge q, the mass m and the energy E of a particle such that:

The particle falls into the black hole from infinity and is not reflected The resulting charge-to-mass parameter λ is greater than before

In order to calculate this we will use the principle of stationary action and the Lorentz-force on a charged particle. Then the Lagrangian is given by

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}} + q\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A_\mu(x(s)), \qquad (4)$$

where $x(s) = \{x^\mu(s)\}$ is the trajectory of the particle, parametrized by an affine parameter s. Also we require that the trajectory of the particle is timelike, i.e. the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ has to be full filled. We will now calculate all equations of motion to get conditions for the parameters of the particle. Due to radial symmetry of the problem we will use spherical coordinates (t, r, ϕ, θ).

Since $\mathcal{L}$ is cyclic in t, we know that the energy is conserved:

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = -\frac{m}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\frac{\Delta}{r^2}\dot{t} - \frac{q\,Q}{r} = -\frac{m\,\Delta}{r^2}\dot{t} - \frac{q\,Q}{r} =: -E, \tag{5}$$

with $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}s}$. Since $\frac{\Delta}{r^2} \geq 0$ and $\dot{t} \geq 0$ for all $r \geq r_{\mathrm{G}}$ we get

$$E - \frac{q\,Q}{r^2} = m\frac{\Delta}{r^2}\dot{t} \geq 0 \quad \forall r \geq r_{\mathrm{H}}. \tag{6}$$

Also $\mathcal{L}$ is cyclic in ϕ and therefore we have conservation of angular momentum. We can set $\dot{\phi} = L = 0$. Now we calculate the Euler-Lagrange-equations for θ:

$$\frac{\mathrm{d}}{\mathrm{d}s}r^2\dot{\theta} = r^2\sin\theta\cos\theta\dot{\phi}^2 = 0. \tag{7}$$

Set $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. Applying this to the timelike condition we get

$$-\frac{\Delta}{r^2}\dot{t}^2 + \frac{r^2}{\Delta}\dot{r}^2 = -1 \tag{8}$$

and therefore

$$\dot{r}^2 = \frac{\left(E - \frac{q\,Q}{r}\right)^2}{m^2} - \frac{\Delta}{r^2}. \tag{9}$$

This is indeed the same as the Euler-Lagrange-equation for r.

Expanding this equation gives us

$$\frac{E}{2m} - \frac{m}{2} = \frac{1}{2}m\dot{r}^2 - \frac{m\,M}{r} + \frac{mQ^2}{2r^2} - \frac{q^2Q^2}{2mr^2} + \frac{E\,q\,Q}{mr} \tag{10}$$

$$:= \frac{1}{2}m\dot{r}^2 + V(r), \tag{11}$$

where we defined the effective potential $V(r) = -\frac{m\,M}{r} + \frac{mQ^2}{2r^2} - \frac{q^2Q^2}{2mr^2} + \frac{E\,q\,Q}{mr}$. Then we can deduce the conditions we need:

We need $\dot{r}^2 > 0$, so that the particle will always fall into the blackhole and will never stop. Together with the energy condition this tells us that

$$E - \frac{q\,Q}{r} > m\sqrt{\frac{\Delta}{r^2}} \geq 0 \quad \forall r \geq r_{\mathrm{H}}. \tag{12}$$

We also need that the energy of the particle is sufficient so overcome the potential barrier of $V(r)$, i.e. $E > \frac{q\,Q}{r}$ and therefore

$$E > \frac{q\,Q}{r_{\mathrm{H}}} = \frac{q\,Q}{M + \sqrt{M^2 - Q^2}}. \tag{13}$$

As a last condition we need that $E > m$ for $r \to \infty$

In order to satisfy condition 2, i.e. to get an enlargement of λ we also need $\frac{Q}{M} < \frac{Q+q}{M+E}$, which implies that $E < \frac{qM}{Q}$. By using the inequality from $\dot{r}^2 > 0$, this is also equivalent to the requirment

$$m < \sqrt{\frac{r^2}{\Delta}}\left(E - \frac{qQ}{r}\right) \quad \forall r \geq r_{\mathrm{H}}. \tag{14}$$

An analytical calculation shows, that the RHS of the above equation has its minimum at

$$r_{\mathrm{m}} = Q\left(\frac{Mq - QE}{Qq - ME}\right). \tag{15}$$

Finally, if one has $M > Q$ the following two conditions can occur:

$$\frac{qQ}{r_{\mathrm{H}}} < E < \frac{qQ}{M} < \frac{qM}{Q} \quad \text{or} \quad \frac{qQ}{r_{\mathrm{H}}} < \frac{qQ}{M} \leq E < \frac{qM}{Q}. \tag{16}$$

Summary and the Third law of Black-Hole-Dynamics

As we have shown in the above calculations: For $M > Q$ one can always find values E, m and q for a particle such it will enter through the event horizon and will increase its charge-to-mass parameter λ. But for $M = Q$ no particle can enter and therefore we can not reach $\lambda > 1$ by this method. That means, we proved in the case of a non-rotating black hole, that the cosmic censorship holds and no naked singularity can occur by 'overcharging' a black hole.

As a last side remark I want to mention that this is in perfect agreement with the Third law of Black-Hole-Dynamics. If κ denotes the surface gravity at the horizon, the law states that $\kappa = 0$ can not be reached within a finite steps. Therefore we also can not 'jump' over or reach $\kappa = 0$. This is here analogue to $\lambda = 1$.

References

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