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First unofficial voluntary sheet on Quantum Gravity

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Exercise 1: *Actions for general relativity*

In a D-dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, consider the Γ^2- and Einstein–Hilbert actions [1]

$$S_{\Gamma^2}[g_{\mu\nu}] = \frac{1}{2\varkappa}\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-\tilde{\tilde{g}}}\, g^{\mu\nu}\left(\Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\rho\mu} - \Gamma^{\rho}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\nu\mu}\right), \tag{1}$$

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{2\varkappa}\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-\tilde{\tilde{g}}}\, R, \tag{2}$$

where $\varkappa := 8\pi G$, $\tilde{\tilde{g}} := \det g_{\mu\nu}$, $\Gamma^{\mu}{}_{\nu\rho}$ is the Christoffel symbol, and R the Ricci scalar.

1. Find the difference between S_{Γ^2} and S_{EH}.
2. Argue that applying the Hamilton's principle to S_{Γ^2} leads to the Einstein field equations.
3. S_{Γ^2} is not general invariant. Does it affect the classical dynamics?

Remark. S_{Γ^2} was proposed by Einstein [2]. For a historical discussion of S_{EH}, see [3, 4].

Exercise 2: *Boundary integral in Einstein–Hilbert action*

The Einstein–Hilbert action contains second derivatives, which could break the Hamilton's principle, that only works for Lagrangians containing at most first derivatives [5, sec. 1.1]. Adding a boundary integral fixes this problem [6, sec. 1.1.1], which we study here following [7].

Consider variation of S_{EH} in eq. (2) in a region $\mathcal{V} \subset \mathcal{M}$, where the boundary $\partial\mathcal{V}$ is smooth; for simplicity, it is also *space-like*, namely a tangential vector of $\partial\mathcal{V}$ is always space-like.

1. We know that $2\varkappa\,\delta S_{\mathrm{EH}} = \int_{\mathcal{V}} \mathrm{d}^D x \sqrt{-\tilde{\tilde{g}}}\, G_{\mu\nu}\,\delta g^{\mu\nu} + 2\varkappa\, I$, where $G_{\mu\nu}$ is the Einstein tensor. Use the generalised Stokes' theorem to argue that

   $$2\varkappa\, I[g_{\mu\nu}] = \int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x \sqrt{\tilde{\tilde{h}}}\, n_\mu \left(g^{\rho\sigma}\,\delta\Gamma^{\mu}{}_{\rho\sigma} - g^{\mu\nu}\,\delta\Gamma^{\rho}{}_{\rho\nu}\right), \tag{3}$$

   where n^μ is a normal vector field, $n^\mu n_\mu = -1$; $\tilde{\tilde{h}} = \det h_{ij}$, h_{ij} is the induced metric on $\partial\mathcal{V}$ in the *internal* holonomic basis, which we do not need here.

   Be aware that in the *external* holonomic basis, the induced metric reads $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, where n^μ is the tangential vector of $\partial\mathcal{V}$, $n^\mu n_\mu = -1$.

2. The final goal in this exercise is to separate $\nabla\delta g$ and δg in the integrand in eq. (3). Here is how Padmanabhan proceeded.

   Show that eq. (3) can be transformed to

   $$2\varkappa\, I[g_{\mu\nu}] = \int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x \sqrt{\tilde{\tilde{h}}}\left\{(\delta n^\mu + g^{\mu\nu}\,\delta n_\nu)_{;\mu} - \delta(2n^\mu{}_{;\mu}) + n_{\nu;\mu}\,\delta g^{\mu\nu}\right\}. \tag{4}$$

   Note that $\delta n^\mu = \delta(g^{\mu\nu}n_\nu) = \delta g^{\mu\nu}\, n_\nu + g^{\mu\nu}\,\delta n_\nu \neq g^{\mu\nu}\,\delta n_\nu$!

3. $\partial\mathcal{V}$ is a hypersurface, which will be studied in a later exercise. The result will show that

$$(\delta n^\mu + g^{\mu\nu}\,\delta n_\nu)_{;\mu} = (\delta n^\mu + g^{\mu\nu}\,\delta n_\nu)_{|\mu} + n_\mu n^\rho n_{\nu;\rho}\,\delta g^{\mu\nu}\,, \tag{5}$$

where $_|$ is the induced covariant derivative on $\partial\mathcal{V}$. Use eq. (5) and show that

$$2\varkappa\, I[g_{\mu\nu}] = \int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x\sqrt{\tilde{\tilde{h}}}\left\{(\delta n^\mu + g^{\mu\nu}\,\delta n_\nu)_{|\mu} - \delta\left(2n^\mu{}_{;\mu}\right) + \left(n_{\nu;\mu} + n_\mu n^\rho n_{\nu;\rho}\right)\delta g^{\mu\nu}\right\}. \tag{6}$$

4. Define (à la [6, eq. (4.45)])

$$K_{\mu\nu} := n_{\nu;\mu} + n_\mu n^\rho n_{\nu;\rho}\,, \qquad K := g^{\mu\nu}K_{\mu\nu}\,. \tag{7}$$

Be aware of the following properties

$$K_{\mu\nu} = K_{\nu\mu}\,, \quad n^\mu K_{\mu\nu} = 0\,, \quad K = n^\mu{}_{;\mu}\,; \qquad \delta\sqrt{\tilde{\tilde{h}}} = -\frac{1}{2}\sqrt{\tilde{\tilde{h}}}\,h_{\mu\nu}\,\delta h^{\mu\nu}. \tag{8}$$

Use eq. (8) and show that

$$2\varkappa\, I[g_{\mu\nu}] = \int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x\sqrt{\tilde{\tilde{h}}}\,(\delta n^\mu + g^{\mu\nu}\,\delta n_\nu)_{|\mu} - 2\varkappa\,\delta S_{\text{GHY}} - \int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x\sqrt{\tilde{\tilde{h}}}\,(Kh_{\mu\nu} - K_{\mu\nu})\,\delta h^{\mu\nu}\,, \tag{9}$$

$$S_{\text{GHY}} := \frac{1}{\varkappa}\int_{\partial\mathcal{V}} \mathrm{d}^{D-1}x\sqrt{\tilde{\tilde{h}}}\,K\,. \tag{10}$$

There might be some sign problems here. Please help me to correct them!

Remark 1. The variation of $\tilde{\tilde{g}}$ was `left as Exercise 18 in Relativity I WS1819`.
Remark 2. In eq. (9), the third integral vanishes if $\delta g^{\mu\nu}|_{\partial\mathcal{V}} = 0$ (more precisely, $\delta h^{\mu\nu} = 0$ is sufficient; δn^μ can be arbitrary); the first integral can be pushed to the boundary of $\partial\mathcal{V}$, i.e. $\partial^2\mathcal{V}$, which deserves further study (e.g. [8] and the references therein) but can be ignored here. The second term is what we use to cancel the second derivatives in S_{EH} and is usually called the *Gibbons–Hawking–York term*.

Exercise 3: *Fierz–Pauli action in vacuum*

The Fierz–Pauli action [9] (Might be wrong in sign!)

$$S_{\text{FP}}[f_{\mu\nu}] = \frac{1}{8\varkappa}\int_{\mathcal{M}} \mathrm{d}^D x\left\{\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\lambda\kappa}\left[f_{\rho\sigma,\lambda}\left(2f_{\kappa\nu,\mu} - f_{\nu\mu,\kappa}\right) - f_{\sigma\nu,\lambda}\left(2f_{\kappa\rho,\mu} - f_{\rho\mu,\kappa}\right)\right]\right\} \tag{11}$$

can be derived by expanding the metric around the flat one

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \qquad \delta g_{\mu\nu} \equiv f_{\mu\nu} \tag{12}$$

and expanding an action for the Einstein field equations to the second order.

1. For S_{Γ^2} in eq. (1), argue that the zeroth and first order terms in the expansion vanishes, and

$$S_{\Gamma^2}[\eta_{\mu\nu} + f_{\mu\nu}] = \frac{1}{2\varkappa}\int_{\mathcal{M}} \mathrm{d}^D x\left\{\eta^{\mu\nu}\left(\delta\Gamma^\rho{}_{\rho\sigma}\,\delta\Gamma^\sigma{}_{\nu\mu} - \delta\Gamma^\rho{}_{\nu\sigma}\,\delta\Gamma^\sigma{}_{\rho\mu}\right) + O\Big(\left(f_{\mu\nu}\right)^3\Big)\right\}. \tag{13}$$

2. Argue that expanding S_{EH} gives the same result as in eq. (13), up to boundary terms.

3. Use Riemannian normal coordinates to argue that

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}\eta^{\mu\lambda}\left(f_{\lambda\nu,\rho} - f_{\nu\rho,\lambda} + f_{\rho\lambda,\nu}\right) + O\Big(\left(f_{\mu\nu}\right)^2\Big) \qquad \text{for} \quad g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}\,. \tag{14}$$

4. Insert eq. (14) into eq. (13) and show that

$$S_{\Gamma^2}[\eta_{\mu\nu} + f_{\mu\nu}] = S_{\text{FP}}[f_{\mu\nu}] + \int_{\mathcal{M}} \mathrm{d}^D x\, O\Big(\left(f_{\mu\nu}\right)^3\Big). \tag{15}$$

5. Does eq. (11) reproduces [6, eq. (2.20)]?

Remark. By applying the Hamilton's principle, S_{FP} leads to the linearised Einstein equations, which was left as Exercise 33 in Relativity I WS1819. However, I do not find an easy way to write down an action given those equations.

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