University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Yi-Fan Wang and Leonardo Chataignier

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Exercise 4: *Relativistic charged particle as a regular system*

Consider a charged massive point particle in special relativity of $(d+1)$-dimensions, described by [1, sec. 16]

$$S\left[x^i\right] = \int_{t_A}^{t_B} \mathrm{d}t\, L\left(x^i, \dot{x}^j\right) := \int_{t_A}^{t_B} \mathrm{d}t \left\{-m\sqrt{1-\delta_{ij}\dot{x}^i\dot{x}^j} - q\Phi\left(x^j\right) + q\dot{x}^j A_i\left(x^k\right)\right\}, \qquad i,j,k = 1,\ldots,d, \tag{1}$$

where $\dot{x}^i := \mathrm{d}x^i/\mathrm{d}t$, m and q are the mass and electric charge, Φ and A_i the electric and vector potentials.

1. Show that the *canonical* d-momentum reads

$$P_i = P_i\left(x^j, \dot{x}^k\right) := \frac{\partial L}{\partial \dot{x}^i} = \delta_{ij}\frac{m\dot{x}^j}{\sqrt{1-\delta_{kl}\dot{x}^k\dot{x}^l}} + qA_i\,. \tag{2}$$

2. Show that the following partial inverse reads

$$\dot{x}^i = \overline{v}^i\left(x^j, P_k\right) = \delta^{ij}\frac{P_j - qA_j}{\sqrt{m^2 + \delta^{kl}(P_k - qA_k)(P_l - qA_l)}}\,. \tag{3}$$

   Remark. Such a partial inverse exists because the Hessian $M_{ij} := \frac{\partial P_i}{\partial \dot{x}^j} \equiv \frac{\partial^2 L}{\partial \dot{x}^j\,\partial \dot{x}^i}$ is a *regular* matrix, which is a condition for the *implicit function theorem* [2]; the system is *regular* because M_{ij} is regular.

3. Show that the canonical Hamiltonian of the particle reads

$$H^{\mathrm{c}} = H^{\mathrm{c}}\left(x^i, P_j\right) = \left\{\dot{x}^i P_i - L\left(x^i, \dot{x}^j\right)\right\}_{\dot{x}^i = \overline{v}^i\left(x^j, P_k\right)} = \sqrt{m^2 + \delta^{kl}(P_k - qA_k)(P_l - qA_l)} + q\Phi\,. \tag{4}$$

4. Derive the Hamilton's equations of motion in terms of Poisson brackets [3, sec. 42]

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = [\omega, H]_{\mathrm{P}}\,, \qquad \text{where } \omega = x^i, P_j, \quad [f,g]_{\mathrm{P}} := \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial P_i}\,. \tag{5}$$

   Why are the equations for $\dot{x}^i$ less interesting than those for $\dot{P}_i$?

Exercise 5: *Relativistic particle as a singular system: Lagrangian formalism*

It is an experimental fact that a physical system obeys the *Newton's principle of determinacy* [4, sec. 1.1]:

> The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.

In other words, for a mechanical system with s generalised coordinates $\left\{q^i\right\}$, an initial data $\left(q_0^i; \dot{q}_0^i; t_0\right)$ of $(2s+1)$ quantities uniquely determines the evolution of the system. We will show that the action for a point particle in eq. (6), as a singular system, *violates* this principle.
Consider a massive point particle in special relativity in $(1+1)$-dimensions, described by the action

$$S[t,x] = \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\nu) := \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{-m\sqrt{\dot{t}^2 - \dot{x}^2}\right\}, \tag{6}$$

where $\dot{x}^\mu := \mathrm{d}x^\mu/\mathrm{d}\lambda$. The system is *singular* because $M_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{x}^\nu \dot{x}^\mu}$ is singular.

1. Since $\{x^\mu\}$ are cyclic [3, sec. 14], the corresponding generalised momenta are integrals of motion and can serve as the velocity initial data. Show that these constants of motion can be chosen to be

$$E := \frac{m\dot{t}}{\sqrt{\dot{t}^2 - \dot{x}^2}}, \qquad p := \frac{m\dot{x}}{\sqrt{\dot{t}^2 - \dot{x}^2}}. \tag{7}$$

For simplicity, we assume in the following $E > 0$.

2. It is easy to check that (E, p) satisfies the relation

$$E = \sqrt{p^2 + m^2} > 0. \tag{8}$$

We could use β and $\sigma(\lambda)$ to recognise this, which are defined as

$$\sinh\beta := \frac{p}{m}, \qquad \sigma(\lambda) := \sqrt{\dot{t}^2 - \dot{x}^2} > 0. \tag{9}$$

Show that eq. (7) can be integrated as

$$t(\lambda) - t_0 = \cosh(\beta)\int_{\lambda_a}^{\lambda} \mathrm{d}\kappa\, \sigma(\kappa), \qquad x(\lambda) - x_0 = \sinh(\beta)\int_{\lambda_a}^{\lambda} \mathrm{d}\kappa\, \sigma(\kappa). \tag{10}$$

Remark 1. The initial data $(t_0, x_0; \beta; \lambda_0)$ of 4 quantities, which is *less than the expected number* of 5, is sufficient for the evolution of the system, and the initial data (E, p) *cannot be arbitrarily chosen*. This is a basic feature of *all constrained systems* in the Lagrangian formalism.

Remark 2. The evolution of the system is not uniquely determined: eq. (10) contains a *functional indeterminacy* $\sigma(\lambda)$. This is another feature for *gauge systems*.

Remark 3. The word *gauge* here is in a more general sense than it is in Yang(楊)–Mills theories!

3. In physics, one chooses $\sigma(\lambda) = \sigma \equiv$ const. for definiteness; in particular, $\sigma = 1$ makes λ the proper time. This is an example of 'gauge fixing', which *can already be imposed in eq.* (7).

What happens if one fixes the gauge earlier, i.e. in eq. (6)?

Exercise 6: *Relativistic charged particle as a singular system: vanishing Hamiltonian*

Consider a charged massive point particle in special relativity of $(d+1)$-dimensions, described by the action

$$S[x^\mu] = \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\nu) := \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{ -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu \right\}, \tag{11}$$

where $\dot{x}^\mu := \mathrm{d}x^\mu/\mathrm{d}\lambda$, A_μ is the four-potential.

1. Show that the action is invariant under a non-degenerate reparametrisation of the integral variable

$$\lambda \mapsto \lambda_f = f(\lambda), \qquad f'(\lambda) > 0. \tag{12}$$

As a weaker consequence, $L(x^\mu, \dot{x}^\nu)$ is a homogeneous function of degree 1 with respect to $\dot{x}^\nu$ [5],

$$L(x^\mu, k\dot{x}^\nu) = kL(x^\mu, \dot{x}^\nu), \qquad k > 0. \tag{13}$$

2. Show that the *canonical* $(d+1)$-momentum reads

$$P_\mu = P_\mu(x^\nu, \dot{x}^\rho) := \frac{\partial L}{\partial \dot{x}^\mu} = \eta_{\mu\nu}\frac{m\dot{x}^\nu}{\sqrt{-\eta_{\xi\pi}\dot{x}^\xi\dot{x}^\pi}} + qA_\mu. \tag{14}$$

3. Show that

$$M_{\mu\nu} := \frac{\partial P_\mu}{\partial \dot{x}^\nu} \equiv \frac{\partial^2 L}{\partial \dot{x}^\nu \dot{x}^\mu} = \frac{-\eta_{\mu\nu}\eta_{\xi\pi} + \eta_{\mu\xi}\eta_{\nu\pi}}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}}\dot{x}^\xi\dot{x}^\pi; \tag{15a}$$

$$M_{\mu\nu}\dot{x}^\nu = 0. \tag{15b}$$

Furthermore, use the implicit function theorem to argue that the partial inverse $\dot{x}^\mu = \overline{v}^\mu(x^\nu, P_\xi)$ does not exist.

Remark. The system is *singular* because the Hessian $M_{\mu\nu}$ is singular, which has just been proven.

4. Show that

$$\dot{x}^\mu P_\mu(x^\nu, \dot{x}^\rho) - L(x^\mu, \dot{x}^\nu) = 0 \,. \tag{16}$$

This holds for all systems, that has a Lagrangian $L(q^i, \dot{q}^j)$ homogeneous of degree 1 with respect to the generalised velicities $\dot{q}^i$ [6, sec. 3.1.1], which is `left as an exercise`.

Since the Hamiltonian-to-be vanishes, it seems that there is no way to understood the dynamics of the system in the Hamiltonian approach. In the following we will follow [7, ch. 2] and find a way around. More popular references include [8].

Exercise 7: *Extended analytical mechanics with velocity: kinematics*

Consider a time-independent mechanical system described by the *Lagrangian action*

$$S^{\mathrm{l}}\left[q^i\right] = \int_{t_A}^{t_B} \mathrm{d}t\, L\left(q^i, \dot{q}^j\right), \qquad i = 1, \ldots, s > 1 \,. \tag{17}$$

The system is called *regular* (*singular*) if $M_{ij} := \frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i}$ is regular (singular).
Define the following *Lagrangian*, *Hamiltonian* and *action with velocity*

$$\begin{aligned} S^{\mathrm{v}}\left[q^i, p_j, v^k\right] &= \int_{t_A}^{t_B} \mathrm{d}t \left\{L^{\mathrm{v}} + p_i\left(\dot{q}^i - v^i\right)\right\}, & L^{\mathrm{v}} &:= L\left(q^i, v^j\right); & \text{(18a)} \\ &= \int_{t_A}^{t_B} \mathrm{d}t \left\{p_i \dot{q}^i - H^{\mathrm{v}}\right\}, & H^{\mathrm{v}} = H^{\mathrm{v}}\left(q^j, p_k, v^l\right) &:= p_i v^i - L^{\mathrm{v}} \,. & \text{(18b)} \end{aligned}$$

1. Apply the variational principle for $\{p_i\}$ to the integral in eq. (18a) and show that it leads to eq. (17).
2. Apply the variational principle for $\left\{q^i, p_j, v^k\right\}$ to the integral in eq. (18b) and show that

$$\begin{aligned} \dot{\omega} &= [\omega, H^{\mathrm{v}}]_{\mathrm{P}}\,, \qquad \omega = q^i, p_j\,, \qquad [g, h]_{\mathrm{P}} := \frac{\partial g}{\partial q^i}\frac{\partial h}{\partial p_i} - \frac{\partial h}{\partial q^i}\frac{\partial g}{\partial p_i}\,; & \text{(19a)} \\ 0 &= \frac{\partial H^{\mathrm{v}}}{\partial v^i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v^i}\,. & \text{(19b)} \end{aligned}$$

3. For a regular system, eq. (19b) can be partially inverted to $v^i = \overline{v}^i(q^j, p_k)$. Show that the integral in eq. (18b) leads to the *Hamiltonian action*

$$S^{\mathrm{h}}\left[q^i, p_j\right] = \int_{t_A}^{t_B} \mathrm{d}t \left\{p_i \dot{q}^i - H^{\mathrm{c}}\right\}, \qquad H^{\mathrm{c}} = H^{\mathrm{c}}\left(q^i, p_j\right) := H^{\mathrm{v}}\big|_{v^i = \overline{v}^i(q^j, p_k)} \equiv \left(p_i \dot{q}^i - L\right)\Big|_{\dot{q}^i = \overline{v}^i(q^j, p_k)} \,. \tag{20}$$

   Do the Hamilton's equations follow from this action?
4. For a singular system, rank $M_{ij} = r < s$. One can expect that there are $(s - r)$ velocities, $\{v^u\}$, that *cannot* be solved from eq. (19b) and are *inexpressible* in terms of $(q^i, p_i; v^i)$, whereas r velocities, $\{v^e\}$, *can* be solved and are *expressible* in terms of $(q^i, p_i; v^i)$.

   (Optional) Argue that the assert above can be more accurate such that

$$v^e = \overline{v}^e\left(q^i, p_e; v^u\right), \tag{21}$$

   i.e. the expressible velocities are *independent* of $\{p_u; v^e\}$, where $\{p_u\}$ are the conjugate momenta of the generalised coordinates corresponding to the inexpressible velocities, and $\{v^e\}$ the expressible velocities.
5. Inserting eq. (21) into H^{v} yields the *Hamiltonian with primary constraint*

$$H^{\mathrm{p}} = H^{\mathrm{s}}\left(q^i, p_e\right) + v^u F_u\,, \qquad F_u = F_u\left(q^i, p_i\right) = p_u - f_u\left(q^i, p_e\right). \tag{22}$$

   $\{F_u\}$ are called *primary constraints* [9, 10], which are *linear* in the canonical momenta p_u in our formalism.

   (Optional) Derive eq. (22).

   Remark 1. The primary constraints originate from the definition of momenta and contain therefore no knowledge concerning the *dynamics* of the system.

   Remark 2. Because of the constraints (in general, $\{\Phi_a\}$), motions are confined in the submanifold $\Phi_a = 0$ in the phase space. This is a character for *all constrained systems* in the Hamilton's approach.

Exercise 8: *Linear action for a massive relativistic particle: primary constraints*

Consider a charged massive point particle in special relativity, described by the Lagrangian action

$$S^{\mathrm{l}}[x^\mu] = \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\nu) := \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{-m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu\right\}, \qquad m > 0\,. \tag{23}$$

1. Show that the Hamiltonian with velocity reads

$$H^{\mathrm{v}} = m\sqrt{-\eta_{\mu\nu}v^\mu v^\nu} + v^\nu(P_\mu - qA_\mu)\,. \tag{24}$$

2. Take v^0 as the inexpressible velocity. Show that the rest of the velocities, $\{v^i\}$, are expressible such that

$$v^i = \overline{v}^i\left(x^\mu, P_i; v^0\right) = \delta^{ij}\frac{P_j - qA_j}{\sqrt{m^2 + \delta^{kl}(P_k - qA_k)(P_l - qA_l)}}v^0\,. \tag{25}$$

3. Insert eq. (25) into S^{v} and show that

$$S^{\mathrm{P}}\left[x^\mu, P_\nu; v^0\right] := S^{\mathrm{v}}\left[x^\mu, P_\nu; v^0, v^i = \overline{v}^i\left(x^\mu, P_i; v^0\right)\right] = \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, \{P_\mu\dot{x}^\mu - H^{\mathrm{P}}\}, \tag{26a}$$

$$H^{\mathrm{P}} = H^{\mathrm{P}}\left(x^\mu, P_\mu; v^0\right) = v^0 F_0\,, \tag{26b}$$

$$F_0 = P_0 - f_0(x^\mu, P_i), \qquad f_0 = qA_0 - \sqrt{m^2 + \delta^{kl}(P_k - qA_k)(P_l - qA_l)}\,. \tag{26c}$$

Exercise 9: *Quadratic action for a relativistic particle: secondary constraints*

The evolution of a phase-space function $g(q^i, p_j)$ is determined by $\dot{g} = [g, H^{\mathrm{P}}]_{\mathrm{P}}$. Consistency requires that a constrain Φ persists in time, so that

$$0 \equiv \dot{\Phi} = [\Phi, H^{\mathrm{P}}]_{\mathrm{P}} = [\Phi, H^{\mathrm{s}}]_{\mathrm{P}} + v^u[\Phi, F_u]_{\mathrm{P}}\,, \tag{27}$$

New generations of constraints can be produced in this way, which are collectively called *secondary constraints*. Moreover, a Lie algebra of constraints $[\Phi_a, \Phi_b]_{\mathrm{p}}$ emerges. If the algebra closes,

$$[\Phi_a, \Phi_b]_{\mathrm{p}} = C^c_{ab}\Phi_c\,, \tag{28}$$

the constraints and the system are called *first-class* [9].
Consider a charged point particle in special relativity, described by the Lagrangian action [11, sec. 2.1]

$$S^{\mathrm{l}}[x^\mu, N] = \int \mathrm{d}\lambda\, \frac{N}{2}\left\{\eta_{\mu\nu}\frac{\dot{x}^\mu}{N}\frac{\dot{x}^\nu}{N} - m^2 + q\frac{\dot{x}^\mu}{N}A_\mu\right\}, \qquad N = N(t) > 0\,, \quad m \geqslant 0\,. \tag{29}$$

1. Derive the Euler–Lagrange equation for N. Furthermore, solve this equation for N and show that eq. (29) gives eq. (23) for $m > 0$.
2. Show that the Hamiltonian with velocity reads

$$H^{\mathrm{v}} = \frac{N}{2}\left(-\eta_{\mu\nu}\frac{v^\mu}{N}\frac{v^\nu}{N} + m^2 - q\frac{v^\mu}{N}A_\mu\right) + v^\nu P_\nu + v^N P_N\,. \tag{30}$$

3. Show that the Hamiltonian with primary constraints reads

$$H^{\mathrm{P}} = H^{\mathrm{s}} + v^N F_N =: -NH_\perp + v^N F_N\,, \tag{31a}$$

$$H_\perp = -\frac{1}{2}\left(\eta^{\mu\nu}(P_\mu - qA_\mu)(P_\nu - qA_\nu) + m^2\right), \quad F_N = P_N\,, \tag{31b}$$

4. Show that $H_\perp$ is a secondary constraint. Furthermore, show that there are no further constraints, and the system is first-class.

   Remark 1. The secondary constraint is already contained in H^{s} in this example.

   Remark 2. Confusingly, $H_\perp$ is often called the *Hamiltonian constraint* in the literature.

5. In this case, the Dirac quantisation rules for first-class systems [6, sec. 3.1.2] read

$$\widehat{F}_N \Psi = 0\,, \qquad \widehat{H}_\perp \Psi = 0\,. \tag{32}$$

Because H^{p} consists of constraints only and the canonical Hamiltonian H^{c} vanishes, there is no Schrödinger-type of equation. Consequently, *there is no time* in the equations.

Show that eq. (32) leads to the Klein–Gordon equation

$$\eta^{\mu\nu}(\hbar\partial_\mu - \mathrm{i}qA_\mu)(\hbar\partial_\nu - \mathrm{i}qA_\nu)\psi - m^2\psi = 0\,, \qquad \psi = \psi(x^\mu)\,. \tag{33}$$

In particular, the dependence of N drops out.

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