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Third unofficial exercise sheet on Quantum Gravity

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Exercise 10: *Proca theory: constraints*

For field theories in the canonical formalism with conjugate "coordinates" and "momenta" (φ^q, π_p), the Poisson bracket is defined as

$$[f, g]_{\mathrm{P}} = \int \mathrm{d}^d x \left\{ \frac{\delta f}{\delta \varphi^q} \frac{\delta g}{\delta \pi_q} - \frac{\delta f}{\delta \pi_q} \frac{\delta g}{\delta \varphi^q} \right\}. \tag{1}$$

Consequently, the non-vanishing fundamental Poisson brackets are

$$\left[\varphi^q \left(x^k\right), \pi_p \left(y^k\right)\right]_{\mathrm{P}} =: \left[\varphi^q, \pi'_p\right]_{\mathrm{P}} = \delta^q{}_p \delta^d \left(x^k - y^k\right). \tag{2}$$

Consider the Proca [1] action in $(d+1)$ dimensions

$$S^{\mathrm{l}}[A_\mu] = \int \mathrm{d}t\, \mathrm{d}^d x\, \mathfrak{L} := \int \mathrm{d}^{d+1} x \left\{ -\frac{1}{4} \eta^{\mu\xi} \eta^{\nu\pi} F_{\mu\nu} F_{\xi\pi} - \frac{m^2}{2} \eta^{\mu\nu} A_\mu A_\nu \right\}, \tag{3}$$

where $m > 0$, $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. The fundamental Poisson brackets are

$$[A_\mu, \Pi^{\nu\prime}]_{\mathrm{P}} = \delta_\mu{}^\nu \delta^d \left(x^k - y^k\right), \qquad [A_\mu, A'_\nu]_{\mathrm{P}} = 0 = [\Pi_\mu, \Pi^{\nu\prime}]_{\mathrm{P}}. \tag{4}$$

1. Derive the canonical formalism with primary constraints

$$S^{\mathrm{P}}[A_\mu, \Pi^\nu, V_0] = \int \mathrm{d}t \left\{ \int \mathrm{d}^d x\, \Pi^\mu \dot{A}_\mu - H^{\mathrm{P}} \right\} = \int \mathrm{d}t\, \mathrm{d}^d x \left\{ \Pi^\mu \dot{A}_\mu - \mathfrak{H}^{\mathrm{P}} \right\}, \tag{5a}$$

$$\mathfrak{H}^{\mathrm{P}} = \mathfrak{H}^{\mathrm{s}} + V_0 \mathfrak{F}, \tag{5b}$$

$$\mathfrak{H}^{\mathrm{s}} = \frac{1}{2} \delta_{ij} \Pi^i \Pi^j + \frac{1}{4} \delta^{ik} \delta^{jl} F_{ij} F_{kl} + \frac{m^2}{2} \eta^{\mu\nu} A_\mu A_\nu + \Pi^i \partial_i A_0, \tag{5c}$$

$$\mathfrak{F} = \Pi^0. \tag{5d}$$

2. Imposing persistence of $\mathfrak{F}$ leads to the secondary constraint(s). Show that

$$[\mathfrak{F}, \mathfrak{H}^{\mathrm{P}\prime}]_{\mathrm{P}} = \left(m^2 A'_0 - \Pi^{i\prime} \partial'_i\right) \delta^d (x^\mu - y^\mu), \tag{6a}$$

$$[\mathfrak{F}, H^{\mathrm{P}}]_{\mathrm{P}} = m^2 A_0 + \partial_i \Pi^i =: \mathfrak{G}, \tag{6b}$$

where $\mathfrak{G}$ is a secondary constraint.

The Hamiltonian with primary constraint can now be written as

$$\mathfrak{H}^{\mathrm{P}} = \mathfrak{H}^{\mathrm{c}} - A_0 \mathfrak{G} + V_0 \mathfrak{F} + \partial_i \left(\Pi^i A_0\right), \tag{7a}$$

$$\mathfrak{H}^{\mathrm{c}} := \frac{1}{2} \delta_{ij} \Pi^i \Pi^j + \frac{1}{4} \delta^{ik} \delta^{jl} F_{ij} F_{kl} + \frac{m^2}{2} \left(\delta^{ij} A_i A_j + A_0^2\right). \tag{7b}$$

3. The rest of this exercise shows that $\mathfrak{F}$ and $\mathfrak{G}$ are all the constraints.

From Exercise 28 in Relativity I WS1819 one knows that $\frac{1}{4} \delta^{ik} \delta^{jl} F_{ij} F_{kl} = \frac{1}{2} \delta^{ik} \delta^{jl} \left(\partial_i A_j\right) F_{kl}$. Use this identity to show that

$$\left[\partial_i \Pi^i, \frac{1}{4} \delta^{ik} \delta^{jl} F'_{ij} F'_{kl}\right]_{\mathrm{P}} = 0. \tag{8}$$

Equipped with eq. (8), show that

$$[\mathfrak{G}, \mathfrak{H}^{\mathrm{c}\prime}]_{\mathrm{P}} = -m^2 \delta^{ij} A'_i \partial_j \delta^d \left(x^k - y^k\right); \qquad [\mathfrak{G}, H^{\mathrm{c}}]_{\mathrm{P}} = 0. \tag{9}$$

Exercise 11: *Proca theory: Dirac brackets*

Consider a constrained system, where all constraints are primary and second-class

$$S^{\mathrm{P}} = \int \mathrm{d}t \left\{ p_i \dot{q}^i - H^{\mathrm{P}} \right\} = \int \mathrm{d}t \left\{ H^{\mathrm{s}} + \lambda^a \Phi_a \right\} . \tag{10}$$

The evolution of a constraint reads

$$\dot{\Phi}_a = [\Phi_a, H^{\mathrm{P}}]_{\mathrm{P}} = [\Phi_a, H^{\mathrm{s}}]_{\mathrm{P}} + \lambda^b [\Phi_a, \Phi_b]_{\mathrm{P}} =: [\Phi_a, H^{\mathrm{s}}]_{\mathrm{P}} + \lambda^b S(\Phi)_{ab} . \tag{11}$$

Since the system is second-class,

$$(\det S(\Phi)_{ab})_{\Phi_a = 0} \neq 0 , \tag{12}$$

so that it has the matrix invert

$$S(\Phi)_{ac} S^{-1}(\Phi)^{cb} = S^{-1}(\Phi)^{bc} S(\Phi)_{ca} = \delta^b_a . \tag{13}$$

One may choose the Lagrange multipliers $\{\lambda^a\}$ as

$$\lambda^a = -S^{-1}(\Phi)^{ab} [\Phi_b, H^{\mathrm{s}}]_{\mathrm{P}} , \tag{14}$$

so that $\dot{\Phi}_a = 0$.
For the choice of $\{\lambda^a\}$ in eq. (14), the evolution of a dynamical variable reads

$$\dot{\omega} = [\omega, H^{\mathrm{s}}]_{\mathrm{P}} - [\omega, \Phi_a]_{\mathrm{P}} S^{-1}(\Phi)^{ab} [\Phi_b, H^{\mathrm{s}}]_{\mathrm{P}} =: [\omega, H^{\mathrm{s}}]_{D(\Phi)} , \tag{15}$$

where the *Dirac brackets* [2, 3] is defined as

$$[f, g]_{D(\Phi)} := [f, g]_{\mathrm{P}} - [f, \Phi_a]_{\mathrm{P}} S^{-1}(\Phi)^{ab} [\Phi_b, g]_{\mathrm{P}} . \tag{16}$$

It can be shown that this approach also works for second-class systems with secondary constraints. In the end, for generic second-class systems, one could work with Dirac brackets and H^{s}, so that the constraints are taken care of [4, sec. 2.3].

1. For Proca theory, one may write

   $$S(\Phi)_{ab} = T(\Phi)_{ab} \delta^d \left(x^k - y^k \right) , \tag{17}$$

   and the Dirac bracket also involves integrals over x and y.
   Show that

   $$T(\mathfrak{F}, \mathfrak{G}) = m^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{18}$$

   Argue that the system is second-class.

2. Show that the Dirac brackets in Proca theory reads

   $$[f, g]_{D(\mathfrak{F}, \mathfrak{G})} = [f, g]_{\mathrm{P}} - m^{-2} \int \mathrm{d}^d z \left\{ [f, \mathfrak{F}(z)]_{\mathrm{P}} [\mathfrak{G}(z), g]_{\mathrm{P}} - [f, \mathfrak{G}(z)]_{\mathrm{P}} [\mathfrak{F}(z), g]_{\mathrm{P}} \right\} . \tag{19}$$

   Argue and show that the only fundamental Dirac bracket which differs from the Poisson ones is [5]

   $$[A_0, A'_i]_{D(\mathfrak{F}, \mathfrak{G})} = m^{-2} \partial_i \delta^d \left(x^k - y^k \right) . \tag{20}$$

See overleaf.

Exercise 12: *Proca theory: physical variables*

The commutation relation of $\mathfrak{G}$ and $\mathfrak{F}$ suggests that one may use them as a conjugate pair of variables, say (α_0, p^0), namely

$$\alpha_0 = m^{-2}\mathfrak{G} = A_0 + m^{-2}\partial_i \Pi^i\,, \qquad p^0 = \mathfrak{F} = \Pi^0\,, \tag{21}$$

and the rest of the variables, say (α_i, p^i), might become regular.
This is indeed possible. Consider the following *type-3 generating functional for canonical transformations*

$$G_3 = G_3[\alpha_\mu, \Pi^\mu] = \int \mathrm{d}^d x \Big\{ -\Pi^0 \Big(\alpha_0 - m^{-2}\partial_i \Pi^i\Big) - \Pi^i \Big(\alpha_i - m^{-2}\partial_i \Pi^0\Big) \Big\}\,, \tag{22a}$$

so that the old coordinates and new momenta, expressed in terms of (α_μ, Π^ν), are

$$A_\mu = -\frac{\delta G_3}{\delta \Pi^\mu}\,, \qquad p^\mu = -\frac{\delta G_3}{\delta \alpha_\mu}\,. \tag{22b}$$

1. Show that the transformations for (α_0, p^0) are given by eq. (21), and those for (α_i, p^i) read

$$\alpha_i = A_i + m^{-2}\partial_i \Pi^0\,, \qquad p^i = \Pi^i\,; \tag{23}$$

2. Show that the Hamiltonian with primary constraints now reads

$$\mathfrak{H}^{\mathrm{p}} = \mathfrak{H}^{\mathrm{p}}\Big(\alpha_i, p^i; \alpha_0, p^0\Big) = \mathfrak{H}^{\mathrm{c}} + \mathfrak{H}^{\mathrm{u}} + p^0\Big(\delta^{ij}\partial_j \alpha_i + V_0\Big) + \partial_i\Big(-\delta^{ij}\alpha_j p^0 + \alpha_0 p^i - m^{-2} p^i \partial_j p^j\Big)\,, \tag{24}$$

$$\mathfrak{H}^{\mathrm{c}} = \mathfrak{H}^{\mathrm{c}}\Big(\alpha_i, p^i\Big) = \frac{1}{2}\delta_{ij} p^i p^j + \frac{1}{2m^2}\Big(\partial_i p^i\Big)^2 + \frac{1}{4}\delta^{ik}\delta^{jl}\phi_{ij}\phi_{kl} + \frac{m^2}{2}\delta^{ij}\alpha_i \alpha_j\,, \qquad \phi_{ij} := \partial_i \alpha_j - \partial_j \alpha_i\,, \tag{25}$$

$$\mathfrak{H}^{\mathrm{u}} = \mathfrak{H}^{\mathrm{u}}\Big(\alpha_0, p^0\Big) = \frac{1}{2m^2}\delta^{ij}\Big(\partial_i p^0\Big)\Big(\partial_j p^0\Big) - \frac{m^2}{2}\alpha_0^2\,. \tag{26}$$

3. Recognise that (α_0, p^0) is constrained to zero, and that the dynamics of (α_i, p^j) is determined solely by $\mathfrak{H}^{\mathrm{c}}$ in eq. (25).

   Remark. In other words, (α_i, p^j) have been separated as dynamical variables in a regular system and are therefore called *physical variable*.

4. Now go back to the old variables (A_μ, Π^ν). Solve the constraints $\mathfrak{F} = 0$, $\mathfrak{G} = 0$ for (A_0, Π^0) and insert the solution to $\mathfrak{H}^{\mathrm{p}}$ in eq. (7a). What do you find?

   Remark. Proca theory has its primary and secondary constrains in the *special form*, so that the approach is valid. For details, see [4, sec. 2.3].

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