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Exercise 13: *Scalar electrodynamics: first-class constraints*

Consider the Lagrangian action for the electromagnetic field coupled to a charged scalar field

$$S^{\mathrm{l}}_{\mathrm{ScED}}[\phi, \phi^*, A_\mu] = \int \mathrm{d}^{d+1}x \left\{ -\eta^{\mu\nu}\phi^*_{;\mu}\phi_{;\nu} - V(\phi^*\phi) - \frac{1}{4}\eta^{\xi\rho}\eta^{\pi\sigma}F_{\xi\pi}F_{\rho\sigma} \right\}, \tag{1a}$$

$$\phi_{;\mu} := \phi_{,\mu} - \mathrm{i}eA_\mu\phi\,, \qquad \phi^*_{;\mu} := \phi^*_{,\mu} + \mathrm{i}eA_\mu\phi^*\,. \tag{1b}$$

1. Show that

$$\Pi^i := \frac{\partial \mathfrak{L}^{\mathrm{v}}}{\partial V_i} = \delta^{ij}(V_j - A_{0,j}) = \delta^{ij}\, F_{0j}\big|_{\dot{A}_k = V_k} = \delta^{ij}\, E_j\big|_{\dot{A}_k = V_k}\,, \tag{2}$$

   where $\mathfrak{L}^v$ is the Lagrangian density with velocities, $E_i = -F_{0i} = F_{i0}$ is the electric field in $(3+1)$-decomposition.

2. Show that the action with primary constraints reads

$$\begin{aligned} S^{\mathrm{p}}_{\mathrm{ScED}}[\phi, \phi^*, A_\mu; \pi, \pi^*, \Pi^\nu; V_0] &= \int \mathrm{d}t \left\{ \int \mathrm{d}^d x \left(\Pi^\mu \dot{A}_\mu + \pi\dot{\phi} + \dot{\phi}^*\pi^* \right) - H^{\mathrm{p}} \right\} \\ &= \int \mathrm{d}t\, \mathrm{d}^d x \left\{ \Pi^\mu \dot{A}_\mu + \pi\dot{\phi} + \dot{\phi}^*\pi^* - \mathfrak{H}^{\mathrm{p}} \right\}, \end{aligned} \tag{3a}$$

$$\mathfrak{H}^{\mathrm{p}} = \mathfrak{H}^{\mathrm{c}} - A_0\mathfrak{G} + V_0\mathfrak{F} + \left(\Pi^i A_0\right)_{,i}, \tag{3b}$$

$$\mathfrak{F} = \Pi^0\,, \qquad \mathfrak{G} = \mathrm{i}q(\phi^*\pi^* - \pi\phi) + \Pi^i{}_{,i}\,, \tag{3c}$$

$$\mathfrak{H}^{\mathrm{c}} = \pi^*\pi + \frac{1}{2}\delta_{ij}\Pi^i\Pi^j + \delta^{ij}\phi^*_{;i}\phi_{;j} + V(\phi^*\phi) + \frac{1}{4}\delta^{ik}\delta^{jl}F_{ij}F_{kl}\,, \tag{3d}$$

   where (A_μ, Π^μ), (ϕ, π), (ϕ^*, π^*) are conjugate pairs of canonical variables, $\mathfrak{F}$ is the primary constraint.

3. In the literature, $\mathfrak{G}$ is sometimes called the *Gauss constraint*. Show that it is a secondary constraint, and there is no further constraint. Moreover, the constraint algebra is abelian,

$$[\mathfrak{F}, \mathfrak{G}]_{\mathrm{P}} = 0\,, \tag{4}$$

   so that the constraints are first-class.

4. Use the Dirac quantisation rules to write down the equations for the quantum wave functional.

Remark. For a story of *the Maxwellians*, see e.g. [1]

See overleaf.

Exercise 14: *Scalar electrodynamics: gauge transformation in phase space*

In the Hamiltonian formalism, infinitesimal gauge transformations are generated by the Poisson bracket with a 'gauge generator' (see e.g. [2, ch. 5]), which is widely believed to be the fist-class constraints. The statement is shown not to hold by the counterexample of electromagnetism [3].
In this exercise we recover the calculation and have a glimpse on *gauge transformations in phase space*. Consider the Lagrangian action in eq. (1a) for the electromagnetic field coupled to a charged scalar field.

1. We first use the Lagrangian approach. Show that the action in eq. (1a) is invariant under the *gauge transformation in configuration space*

$$\phi \to \mathrm{e}^{-\mathrm{i}e\Lambda}\phi\,, \qquad \phi^* \to \mathrm{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \qquad A_\mu \mapsto A_\mu - \Lambda_{,\mu}\,, \tag{5}$$

   where the 'gauge' is also in the Yang(楊)–Mills sense.

2. The condition $\eta^{\mu\nu}A_{\mu,\nu} = 0$ is called the *Lorenz gauge*. Is there any remaining functional indeterminacy? Does the Lorenz gauge render the initial value problem well-posed?

3. Now we go to the Hamiltonian formalism. Consider the generic gauge generator

$$\widetilde{G}(t) = \int \mathrm{d}^d x \left\{ \mathfrak{F}\left(x^k\right)\xi\left(t,x^k\right) + \mathfrak{G}\left(x^i\right)\epsilon\left(t,x^k\right)\right\}, \tag{6}$$

   containing two *independent* gauge parameters ξ, ϵ.

   Show that $\widetilde{G}(t)$ gives the following infinitesimal gauge transformations

$$\delta\phi\left(x^k\right) = \left[\phi\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = -\mathrm{i}e\phi\epsilon\,, \qquad \delta\phi^*\left(x^k\right) = \left[\phi^*\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = +\mathrm{i}e\phi\epsilon\,; \tag{7a}$$

$$\delta A_0\left(x^k\right) = \left[A_0\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = \xi\,, \qquad \delta A_i\left(x^k\right) = \left[A_i\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = -\epsilon_{,i}\,, \tag{7b}$$

$$\delta\pi\left(x^k\right) = \left[\pi\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = +\mathrm{i}e\phi\epsilon\,, \qquad \delta\pi^*\left(x^k\right) = \left[\pi^*\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = -\mathrm{i}e\phi\epsilon\,. \tag{7c}$$

$$\delta\Pi^0\left(x^k\right) = \left[\Pi^0\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = 0\,, \qquad \delta\Pi^i\left(x^k\right) = \left[\Pi^i\left(x^k\right), \widetilde{G}(t)\right]_{\mathrm{P}} = 0\,, \tag{7d}$$

4. How to recover the third expression in eq. (5) from eq. (7b)?

Remark 1. For a historical discussion of the gauge named after Ludvig Lorenz, see e.g. [4–6].
Remark 2. The variable A_0, usually considered as non-dynamical, also changes under a gauge transformation.

Exercise 15: *Scalar electrodynamics: the Kugo–Ojima terms for scalar electrodynamics*

For quantised Yang(楊)–Mills gauge theories, one can use the Faddeev–Popov trick (see e.g. [7]) to fix a gauge in the functional formalism. The trick can also be accommodated at the Lagrangian level by the Kugo(九後)–Ojima(小嶋) terms [8], so that the gauge fixing can be studied at the classical level, and in the Hamiltonian formalism as well.
For scalar electrodynamics, the Kugo(九後)–Ojima(小嶋) terms read

$$S^{\mathrm{l}}_\alpha\left[\phi,\phi^*,A_\mu,B\right] = S^{\mathrm{l}}_{\mathrm{ScED}} + S^{\mathrm{l}}_{\mathrm{KO},\alpha}\,, \qquad S^{\mathrm{l}}_{\mathrm{KO},\alpha} := \int \mathrm{d}^{d+1}x \left\{\frac{\alpha}{2}B^2 + B\eta^{\mu\nu}A_{\mu,\nu}\right\}. \tag{8}$$

1. We first use the Lagrangian approach. Show that the variation of S^{l}_α gives

$$\eta_{\mu\nu}\frac{\delta S^{\mathrm{l}}_\alpha}{\delta A_\nu} = \mathrm{i}e\left(\phi^*\phi_{;\mu} - \phi^*_{;\mu}\phi\right) + \eta^{\xi\pi}\left(A_{\mu,\pi,\xi} - A_{\pi,\mu,\xi}\right) - B_{,\mu}\,, \tag{9a}$$

$$\frac{\delta S^{\mathrm{l}}_\alpha}{\delta\phi^*} = -\left.\frac{\mathrm{d}V}{\mathrm{d}\Phi}\right|_{\Phi=\phi^*\phi}\phi + \phi_{;\mu;\nu}\,, \qquad \frac{\delta S^{\mathrm{l}}_\alpha}{\delta\phi} = -\left.\frac{\mathrm{d}V}{\mathrm{d}\Phi}\right|_{\Phi=\phi^*\phi}\phi^* + \phi^*_{;\mu;\nu}\,, \tag{9b}$$

$$\frac{\delta S^{\mathrm{l}}_\alpha}{\delta B} = \alpha B + \eta^{\mu\nu}A_{\mu,\nu}\,. \tag{9c}$$

2. In the literature, $\alpha \to 0^+$ leads to the Landau gauge, which is said to be classically equivalent to the Lorenz gauge. What happens if one inserts $\alpha \to 0^+$ in the equations of motion?

3. Solve $0 = \delta S^{\mathrm{l}}_{\alpha}/\delta B$ for $\alpha = 1$ for B, which is called the Feynman–'t Hooft gauge. Insert the solution to eq. (9a) in order to eliminate B.

4. Now we go to the Hamiltonian formalism. Show that the Hamiltonian density with primary constraints reads

$$\mathfrak{H}^{\mathrm{P}} = \mathfrak{H}^{\mathrm{s}} + V_0\mathfrak{F} + V_B\mathfrak{C}\,, \tag{10a}$$

$$\begin{aligned}\mathfrak{H}^{\mathrm{s}} &= \pi^*\pi + \frac{1}{2}\delta_{ij}\Pi^i\Pi^j - \mathrm{i}qA_0(\pi^* - \pi) + \Pi^i A_{0,i} \\ &\quad + \delta^{ij}\phi^*_{;i}\phi_{;j} + V(\phi^*\phi) + \frac{1}{4}\delta^{ik}\delta^{jl}F_{ij}F_{kl} - \frac{\alpha}{2}B^2 - B\delta^{ij}A_{i,j}\,,\end{aligned} \tag{10b}$$

$$\mathfrak{F} = \Pi^0 + B\,, \qquad \mathfrak{C} = \pi^B\,, \tag{10c}$$

where $\{\mathfrak{F}, \mathfrak{C}\}$ are primary constraints, satisfying

$$[\mathfrak{F}, \mathfrak{C}']_{\mathrm{P}} = -\delta^d\left(x^k - y^k\right). \tag{11}$$

5. Show that

$$[\mathfrak{F}, H^{\mathrm{P}}]_{\mathrm{P}} = \mathfrak{G} + V_B\,, \qquad \mathfrak{G} := \mathrm{i}q(\pi^* - \pi) + \Pi^i{}_{,i}\,; \tag{12a}$$

$$[\mathfrak{C}, H^{\mathrm{P}}]_{\mathrm{P}} = \mathfrak{R} - V_0, \qquad \mathfrak{R} := \alpha B + \delta^{ij}A_{i,j}\,, \tag{12b}$$

and the Hamiltonian density with primary constraints can be written as

$$\mathfrak{H}^{\mathrm{P}} = \mathfrak{H}^{\mathrm{c}} - A_0\mathfrak{G} - B\mathfrak{R} + V_0\mathfrak{F} + V_B\mathfrak{C} + \left(\Pi^i A_0\right)_{,i}, \tag{13a}$$

$$\mathfrak{H}^{\mathrm{c}} = \pi^*\pi + \frac{1}{2}\delta_{ij}\Pi^i\Pi^j + \delta^{ij}\phi^*_{;i}\phi_{;j} + V(\phi^*\phi) + \frac{1}{4}\delta^{ik}\delta^{jl}F_{ij}F_{kl} + \frac{\alpha}{2}B^2\,. \tag{13b}$$

6. Persistence of the primary constraints requires $\dot{\Phi} = \left[\Phi, H^{\mathrm{P}}\right]_{\mathrm{P}} \approx 0$, $\Phi = \mathfrak{F}, \mathfrak{C}$, which holds if one chooses $V_B = -\mathfrak{G}$, $V_0 = \mathfrak{R}$. According to different sources [2, sec. 3.4, 9, sec. 4.2.2], the algorithm to find constraints *might* terminate already at the primary stage.

   How would you proceed to arrive at a canonical description of the system that is ready to be quantised?

References:

[1] D. L. Sengupta and T. K. Sarkar, 'Maxwell, Hertz, the Maxwellians, and the early history of electromagnetic waves', IEEE Antennas and Propagation Magazine **45**, 13–19 (2003) 10.1109/map.2003.1203114.

[2] H. J. Rothe and K. D. Rothe, *Classical and quantum dynamics of constrained Hamiltonian systems* (World Scientific, Apr. 2010), ISBN: 9789814299640, 10.1142/7689.

[3] J. B. Pitts, 'A first class constraint generates not a gauge transformation, but a bad physical change: the case of electromagnetism', Annals of Physics **351**, 382–406 (2013) 10.1016/j.aop.2014.08.014, arXiv:1310.2756 [gr-qc].

[4] J. van Bladel, 'Lorenz or Lorentz?', IEEE Antennas and Propagation Magazine **33**, 69–69 (1991) 10.1109/map.1991.5672647.

[5] J. van Bladel, 'Lorenz or Lorentz? [addendum]', IEEE Antennas and Propagation Magazine **33**, 56–56 (1991) 10.1109/map.1991.5672657.

[6] R. Nevels and C.-S. Shin, 'Lorenz, Lorentz, and the gauge', IEEE Antennas and Propagation Magazine **43**, 70–71 (2001) 10.1109/74.934904.

[7] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory* (Addison-Wesley, 1995), ISBN: 9780201503975, http://physics.weber.edu/schroeder/qftbook.html.

[8] T. Kugo and I. Ojima, 'Manifestly covariant canonical formulation of the Yang–Mills field theories. i, General formalism', Progress of Theoretical Physics **60**, 1869–1889 (1978) 10.1143/ptp.60.1869.

[9] D. M. Gitman and I. V. Tyutin, *Quantization of fields with constraints* (Springer, 1990), ISBN: 9783642839405, 10.1007/978-3-642-83938-2.