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Fifth unofficial exercise sheet on Quantum Gravity

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Exercise 16: *Lagrange multipliers*

In this exercise we review the method of Lagrange multipliers and explain how it fits into the variational principle of constrained systems.
Consider an optimisation problem with an equality constraint:

$$\begin{aligned} &\text{maximise } f(x,y) = x^2 y\,, \\ &\text{subject to } g(x,y) = x^2 + 2y^2 - 6 = 0\,. \end{aligned} \tag{1}$$

The idea is that the *local extrema* of $f(x,y)$ subject to the constraint $g(x,y) = 0$ coincide with the *critical points* of another function (usually called the *Lagrangian*):

$$L(x,y,\lambda) := f(x,y) + \lambda g(x,y) = x^2 y + \lambda(x^2 + 2y^2 - 6)\,, \tag{2}$$

where λ is called the *Lagrange multiplier*.

1. Can you give a motivation for this definition based on the contour plots of $f(x,y)$ and $g(x,y)$?
2. Find the critical points of $L(x,y,\lambda)$ and the constrained maximum of $f(x,y)$.

Remark. In the case of constrained mechanical systems, the idea behind the method of Lagrange multipliers is similar, only now the optimisation problem is Hamilton's principle.

Exercise 17: *Quadratic action for a relativistic particle: gauge transformations*

Consider a charged point particle in special relativity, described by the actions [1, sec. 2.1]

$$S^{\mathrm{l}}[x^\mu, N] = \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, \frac{N}{2}\left\{ \frac{\eta_{\mu\nu}}{N^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - m^2 + \frac{q}{N}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}A_\mu \right\}, \qquad N = N(\lambda) \neq 0\,, \quad m \geqslant 0\,; \tag{3a}$$

$$\begin{aligned} S^{\mathrm{P}}\left[x^\mu, N; P_\mu, \varPi; v^N\right] &= \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{ P_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} + \varPi\frac{\mathrm{d}N}{\mathrm{d}\lambda} - NH_\perp - v^N\varPi \right\}, \\ H_\perp &:= \frac{1}{2}\left(\eta^{\mu\nu}(P_\mu - qA_\mu)(P_\nu - qA_\nu) + m^2 \right). \end{aligned} \tag{3b}$$

1. Consider the non-degenerate reparametrisation

$$\lambda \to \lambda_f = f(\lambda)\,, \qquad \frac{\mathrm{d}f}{\mathrm{d}\lambda} > 0\,, \qquad f(\lambda_{A,B}) = \lambda_{A,B}\,. \tag{4a}$$

Show that the transformation laws

$$x^\mu(\lambda) \to x^\mu_f\left(\lambda_f\right) = x^\mu(\lambda)\,, \qquad \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \to \frac{\mathrm{d}x^\mu_f}{\mathrm{d}\lambda_f} = \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda_f}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,, \qquad N(\lambda) \to N_f(\lambda_f) = \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda_f}N(\lambda)\,. \tag{4b}$$

imply that S^{l} given in in eq. (3a) is invariant under the reparametrisation given in eqs. (4a) to (4b).

2. Using eq. (4b), show that under an *infinitesimal* reparametrisation

$$\lambda \to \lambda_f = \lambda - \epsilon(\lambda)\,,\ \epsilon(\lambda_{A,B}) = 0\,, \tag{5a}$$

one obtains the following variations

$$\delta_{\epsilon(\lambda)}x^\mu(\lambda) := x^\mu_f(\lambda) - x^\mu(\lambda) = \epsilon(\lambda)\dot{x}^\mu + O\left(\epsilon^2\right), \tag{5b}$$

$$\delta_{\epsilon(\lambda)}N(\lambda) := N_f(\lambda) - N(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\epsilon(\lambda)N(\lambda)\right) + O\left(\epsilon^2\right), \tag{5c}$$

which imply that $x^\mu(\lambda)$ is a worldline scalar and $N(\lambda)$ is a worldline scalar density.

3. From S^{P}, one can show that any phase space quantity A obeys the evolution law

$$\frac{\mathrm{d}A}{\mathrm{d}\lambda} = \left[A, NH_\perp + v^N \Pi\right]_{\mathrm{P}} . \tag{6a}$$

   Show that

$$\dot{x}^\mu \approx N(\lambda)\eta^{\mu\nu}p_\nu , \tag{6b}$$

   where $p_\mu := P_\mu - qA_\mu$. Argue that this implies that p_μ is also a worldline scalar. Moreover, show that

$$\dot{p}_\mu \approx N(\lambda)\eta^{\rho\sigma}qF_{\mu\rho}p_\sigma , \tag{6c}$$

   where $F_{\mu\rho} = A_{\rho,\mu} - A_{\mu,\rho}$.

4. Using eq. (6a), show that

$$\dot{N}(\lambda) \approx +v^N , \tag{7a}$$

$$\dot{\Pi}(\lambda) = H_\perp \approx 0 . \tag{7b}$$

5. Let us now see how reparametrisations (“gauge transformations” for the point particle) can be generated in phase space. Using eqs. (5b) to (5c) and eqs. (6b) to (7b), show that

$$\delta_{\epsilon(\lambda)}x^\mu(\lambda) \approx [x^\mu, \xi(\lambda)H_\perp]_{\mathrm{P}} , \tag{8a}$$

$$\delta_{\epsilon(\lambda)}p_\mu(\lambda) \approx [p_\mu, \xi(\lambda)H_\perp]_{\mathrm{P}} , \tag{8b}$$

$$\delta_{\epsilon(\lambda)}N(\lambda) = \frac{\mathrm{d}\xi}{\mathrm{d}\lambda} \approx \left[N, \frac{\mathrm{d}\xi}{\mathrm{d}\lambda}\Pi\right]_{\mathrm{P}} , \tag{8c}$$

$$\delta_{\epsilon(\lambda)}\Pi(\lambda) \approx 0 , \tag{8d}$$

   where $\xi(\lambda) := \epsilon(\lambda)N(\lambda)$. Argue that the above equations imply that the reparametrisation-induced gauge transformation of any phase space function A can be written as

$$\delta_{\epsilon(\lambda)}A \approx [A, G]_{\mathrm{P}} , \tag{8e}$$

   where the quantity

$$G := -\xi(\lambda)H_\perp - \frac{\mathrm{d}\xi}{\mathrm{d}\lambda}\Pi \tag{8f}$$

   is called the *gauge generator*.

References:

[1] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*, Theoretical and Mathematical Physics (Springer, 2013), ISBN: 9783642294976, 10.1007/978-3-642-29497-6.