University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Yi-Fan Wang and Leonardo Chataignier

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# Sixth unofficial exercise sheet on Quantum Gravity

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## Exercise 18: *Arnowitt–Deser–Misner formalism: gauge transformation*

So far we have seen the gauge generators for electromagnetic field (Ex. 14) and the quadratic description of a charged relativistic particle (Ex. 17), which were based on somewhat *ad hoc* derivations. A *systematic* algorithm for finding generic gauge generators of first-class systems had been established in [1] by Castellani.

The Arnowitt–Deser–Misner formulation in the absence of matter is governed by the Hamiltonian action with primary constrants

$$S^{\mathrm{P}} = \int \mathrm{d}t\,\mathrm{d}^d x \left\{ p^{ij}\dot{h}_{ij} + \Pi\dot{N} + \Pi_i\dot{N}^i - N\mathfrak{H}_\perp - N^i\mathfrak{H}_i - V\Pi - V^i\Pi_i \right\} + \text{surface terms}\,, \tag{1a}$$

where $\{N, N^i, h_{ij}\}$ are the canonical positions; $\{\Pi, \Pi_i, p^{ij}\}$ are the corresponding conjugate momenta, which are all densities of weight 1; $\mathfrak{H}_\perp$ and $\mathfrak{H}_i$ are called the Hamiltonian and momentum constraints, given by (e.g. [2, sec. 4.2.2, 3, sec. E.2])

$$\mathfrak{H}_\perp := 2\varkappa \underset{\sim}{G}_{ijkl}p^{ij}p^{kl} - \frac{\overset{\approx}{h}{}^{1/2}}{2\varkappa}R[h] \equiv 2\varkappa\tilde{F}^{ijkl}h_{ij}h_{kl} - \frac{\overset{\approx}{h}{}^{1/2}}{2\varkappa}R[h]\,, \qquad \varkappa := 8\pi G\,, \tag{1b}$$

$$\mathfrak{H}_i := -2p_i{}^j{}_{|j} := -2h_{il}\left(p^{lj}{}_{,j} + \Gamma^l{}_{jk}p^{jk}\right), \tag{1c}$$

where $R[h]$ is the induced Ricci scalar on hypersurfaces, and

$$\underset{\sim}{G}_{ijkl} := \frac{1}{2\overset{\approx}{h}{}^{1/2}}\left(h_{ik}h_{lj} + h_{il}h_{kj} - h_{ij}h_{kl}\right) \equiv -\frac{\delta}{\delta h^{kl}}\left(\overset{\approx}{h}{}^{-1/2}h_{ij}\right), \tag{1d}$$

$$\tilde{F}^{ijkl} := \frac{1}{2\overset{\approx}{h}{}^{1/2}}\left(p^{ik}p^{lj} + p^{il}p^{kj} - p^{ij}p^{kl}\right). \tag{1e}$$

Applying the Castellani algorithm to the Arnowitt–Deser–Misner formulation gives the gauge generator [4]

$$G = -\int \mathrm{d}^d x \left\{ \left[ \xi^\perp\left(\mathfrak{H}_\perp + N_{,i}\Pi^i + \left(N\Pi^i\right)_{|j} + \left(N^i\Pi\right)_{,i}\right) + \dot{\xi}^\perp\Pi \right] + \left[ \xi^i\left(\mathfrak{H}_i + N^j{}_{,i}\Pi_j + \left(N^j\Pi_i\right)_{,j} + N_{,i}\Pi\right) + \dot{\xi}^i\Pi_i \right] \right\}. \tag{2a}$$

1. Show that the gauge transformations for the lapse and shift functions, as well as their conjugate momenta, are

$$\delta N = \xi^\perp{}_{,i}N^i - \dot{\xi}^\perp - \xi^i N_{,i}\,, \qquad \delta N^i = -\xi^\perp N_{,j}h^{ij} + \xi^\perp{}_{,j}Nh^{ij} - \xi^j N^i{}_{,j} + \xi^i{}_{,j}N^j - \dot{\xi}^i\,; \tag{3a}$$

$$\delta\Pi = -\left(\xi^\perp\Pi^i\right)_{,i} - \xi^\perp{}_{,i}\Pi^i - \left(\xi^i\Pi\right)_{,i}\,, \qquad \delta\Pi_i = -\xi^\perp{}_{,i}\Pi - \left(\xi^j\Pi_i\right)_{,j} - \xi^j{}_{,i}\Pi_j\,. \tag{3b}$$

   *Remark.* Although the lapse and shift functions are not dynamical so that their time evolution cannot be *solved* and have to be *imposed*, they do have a definite rule under gauge transformations.

2. Show that the gauge transformations for the metric on hypersurfaces are

$$\delta h_{ij} = -\mathcal{H}_{ij}\left(\xi^\perp, \xi^i\right), \tag{4a}$$

$$\mathcal{H}_{ij}(N, N_i) := 2\left(2\varkappa N\underset{\sim}{G}_{ijkl}p^{kl} + N_{(i|j)}\right). \tag{4b}$$

   *Remark.* One may refer to [2, sec. 4.2.2, 3, sec. E.2, 5, sec. 4.2.7] for the results.

3. *Argue* that the gauge transformations for the momenta conjugate to $h_{ij}$ are

$$\delta p^{ij} = -\mathcal{P}^{ij}\left(\xi^{\perp}, \xi^{i}\right), \tag{5a}$$

$$\begin{aligned}\mathcal{P}^{ij}\left(N, N^{i}\right) := 2\varkappa N\left(\frac{1}{2}h^{ij}\widetilde{F}^{klmn}h_{mn} - 2\widetilde{F}^{ijkl}\right)h_{kl} \\ + \frac{1}{2\varkappa}\left(-N\overset{\approx}{h}^{1/2}G^{ij}[h] + \widetilde{G}^{ijkl}N_{|k|l}\right) \\ - \left\{\left(h^{ki}p^{jl} + h^{kj}p^{il} - h^{kl}p^{ij}\right)N_{k}\right\}_{|l},\end{aligned} \tag{5b}$$

where $G^{ij}[h] := R^{ij}[h] - \frac{1}{2}R[h]h^{ij}$ is the induced $(2,0)$ Einstein tensor on the hypersurfaces,

$$\widetilde{G}^{ijkl} := \frac{1}{2}\overset{\approx}{h}^{1/2}\left(h^{ik}h^{lj} + h^{il}h^{kj} - 2h^{ij}h^{kl}\right) \equiv -\overset{\approx}{h}^{-1/2}\frac{\delta}{\delta h_{kl}}\left(\overset{\approx}{h}h^{ij}\right), \tag{5c}$$

*Remark.* One may refer to [3, sec. E.2, 5, sec. 4.2.7] for the argument.

4. One may wonder how the 'gauge' transformations in eqs. (3a) to (5c) in the Arnowitt–Deser–Misner formulation, generated by the first-class constraints, are related to those generated by diffeomorphism. Here is how one begins with.

   Remember that

$$\delta_{\text{diffeo}} g^{\mu\nu} = g^{\mu\nu}{}_{,\rho}\epsilon^{\rho} - g^{\rho\nu}\epsilon^{\mu}{}_{\rho} - g^{\mu\rho}\epsilon^{\nu}{}_{\lambda}, \tag{6}$$

   where $\epsilon^{\mu}$ is the 'gauge' parameter of diffeomorphism. Furthermore, $g^{00} = -N^{-2}$.

   a) Show that

$$\begin{aligned}\delta_{\text{diffeo}} N &= \dot{N}\epsilon^{0} + N_{,i}\epsilon^{i} + N\dot{\epsilon}^{0} - NN^{i}\epsilon^{0}{}_{,i} \\ &\equiv \partial_{t}\left(N\epsilon^{0}\right) + \left(\epsilon^{i} + N^{i}\epsilon^{0}\right)N_{,i} - N^{i}\left(N\epsilon^{0}\right)_{,i}.\end{aligned} \tag{7a}$$

   b) Compare eq. (7a) with eq. (3a) by setting $\delta N = \delta_{\text{diffeo}} N$ and show that

$$\xi^{\perp} = -N\epsilon^{0}, \qquad \xi^{i} = -\epsilon^{i} - N^{i}\epsilon^{0}. \tag{7b}$$

## References:


[1] L. Castellani, 'Symmetries in constrained Hamiltonian systems', Annals of Physics **143**, 357–371 (1982) 10.1016/0003-4916(82)90031-8.

[2] C. Kiefer, *Quantum gravity*, 3rd ed., Vol. 136, International Series of Monographs on Physics (Oxford University, 2012), ISBN: 9780199212521, 10.1093/acprof:oso/9780199212521.001.0001.

[3] R. M. Wald, *General relativity* (University of Chicago Press, 1984), 491 pp., ISBN: 0226870324.

[4] N. Kiriushcheva and S. Kuzmin, 'The Hamiltonian formulation of general relativity: myths and reality', Open Physics **9** (2008) 10.2478/s11534-010-0072-2, arXiv:0809.0097 [gr-qc].

[5] E. Poisson, *A relativist's toolkit, The mathematics of black-hole mechanics* (Cambridge University Press, 2004), ISBN: 9780521830911, 10.1017/cbo9780511606601.