University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Yi-Fan Wang, Leonardo Chataignier and Tim Schmitz

www.thp.uni-koeln.de/gravitation/courses/rci2021.html
ver. 1.01

Third exercise sheet on Relativity and Cosmology I

Winter term 2020/21

Release: Mon, Nov. 23rd **Submit**: Mon, Nov. 30th on ILIAS **Discuss**: Thu, Dec. 3rd

Exercise 6 (10 points): *Covariant Maxwell equations*

Recall from classical electromagnetism the Maxwell equations (in Gaussian units with $c = 1$):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{J}\,; \qquad \vec{\nabla} \cdot \vec{B} = 0\,, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,. \tag{1}$$

In terms of the scalar (Φ) and vector ($\vec{A}$) potentials, the electric and magnetic fields are $\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. In components, eq. (1) reads

$$\sum_{i=1}^{3} \partial_i E^i \equiv \partial_i E^i = 4\pi\rho\,, \qquad \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon^{ijk}\partial_j B_k - \partial_t E^i \equiv \epsilon^{ijk}\partial_j B_k - \partial_t E^i = 4\pi J^i\,; \tag{2a}$$

$$\partial_i B^i = 0\,, \qquad \epsilon^{ijk}\partial_j E_k + \partial_t B^i = 0\,, \tag{2b}$$

where ϵ^{ijk} is the Levi-Civita pseudo-tensor, $i, j, \ldots = 1, 2, 3$, and Einstein notation is assumed, as has been explained in eq. (2a). Let $A^\mu := (\Phi, \vec{A})$ be the four-potential, $j^\mu := (\rho, \vec{J})$ the four-current, $\mu, \nu, \ldots = 0, 1, 2, 3$, and define the field-strength tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3}$$

6.1 Show that the expressions

$$\sum_{\nu=0}^{4} \partial_\nu F^{\mu\nu} \equiv \partial_\nu F^{\mu\nu} = 4\pi j^\mu\,, \tag{4a}$$

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{4b}$$

correspond to eqs. (2a) and (2b) respectively.
Hint: $\epsilon_{ijk}\epsilon^{ilm} = \delta_j{}^l \delta_k{}^m - \delta_j{}^m \delta_k{}^l$.

6.2 Show that eq. (4a) leads to the continuity equation $0 = \partial_t \rho + \vec{\nabla} \cdot \vec{J} \equiv \partial_\mu j^\mu$. How does the continuity equation look like in a Lorentz-boosted reference frame?

Exercise 7 (6 points): *Covariant Lorentz force*

Let $p^\mu := m u^\mu$ be the kinematic four-momentum, u^μ the four-velocity, τ the proper time; $\vec{P} := m\vec{v}$, $\vec{v} := \mathrm{d}\vec{x}/\mathrm{d}t$, and t the coordinate time.

7.1 Show that the spatial components of the covariant Lorentz four-force

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = f_\mu = q F_{\mu\nu} u^\nu \tag{5}$$

give in the non-relativistic limit the Lorentz force

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

7.2 What is the physical meaning of the time component f^0 of the covariant four-force in eq. (5)?

Exercise 8 (6 points): *Kottler–Møller coordinates*

Let g be a constant acceleration, (t, x, y, z) Cartesian coordinates in 4-dimensional Minkowski space. Consider the transformation ($c = 1$)

$$
\begin{aligned}
t &= \left(\frac{1}{g} + x'\right) \sinh g t' \,, \\
x &= \left(\frac{1}{g} + x'\right) \cosh g t' - \frac{1}{g} \,, \\
y &= y' \,, \qquad z = z' \,.
\end{aligned}
$$

9.1 Show that the transformation gives accelerated frames of reference with constant acceleration.

9.2 Calculate the components of the metric with respect to the coordinates (t', x', y', z').