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Sixth exercise sheet on Relativity and Cosmology I

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Exercise 17 (7 points): *Covariant derivative I: Metricity and torsion*

An *affine connection* $\tilde{\nabla}$, also known as a *covariant derivative* $\tilde{\nabla}_i$, can be defined by its *connection coefficients* $\tilde{\Gamma}^i{}_{kj}$, which transform in the same way as the Christoffel symbols. The covariant derivative of a tensor $A^i{}_j$ is then

$$\tilde{\nabla}_i\, A^j{}_k := \partial_i\, A^j{}_k + \tilde{\Gamma}^j{}_{il}\, A^l{}_k - \tilde{\Gamma}^l{}_{ik}\, A^j{}_l.$$

$\tilde{\nabla}$ can be characterised by its *non-metricity* and *torsion*, defined as

$$Q_{ijk}\left(\tilde{\nabla}\right) := -\tilde{\nabla}_i\, g_{jk}\,, \qquad T^i{}_{jk}\left(\tilde{\nabla}\right) := 2\,\tilde{\Gamma}^i{}_{[jk]} \equiv \tilde{\Gamma}^i{}_{jk} - \tilde{\Gamma}^i{}_{kj}\,.$$

17.1 Let $T^{\cdots}{}_{\cdots}$ be a (q,p)–tensor. Argue that $\tilde{\nabla}_i T^{\cdots}{}_{\cdots}$ is a $(q,p+1)$–tensor. Are Q_{ijk} and $T^i{}_{jk}$ tensors?

17.2 Let ∇ be the *Levi-Civita connection* in (pesudo-)Riemannian space, whose coefficients are given by the *Christoffel symbols of the second kind* $\Gamma^i{}_{kj}$. Show that the metric is *covariantly constant*, i.e. $\nabla_k g_{ij} = -Q_{kij}(\nabla) \equiv 0$, $\nabla_k g^{ij} \equiv 0$; furthermore, show that $T^i{}_{jk}(\nabla) \equiv 0$.

17.3 Let the non-metricity and torsion of $\tilde{\nabla}$ be zero. Show that $\tilde{\nabla}$ is necessarily Levi-Civita.

Exercise 18 (7 points): *Derivative of a determinant*

Let $g = \det g_{ij}$ be the metric determinant, d the dimension of the manifold; $\tilde{\epsilon}^{ij\ldots m}$ and $\underset{\sim}{\epsilon}_{ij\ldots m}$ the *Levi-Civita symbols*, taking values $+1$ (or -1) for even (or odd) permutations of the indices, denoted in the lecture as $\varepsilon(ij\ldots m)$.

18.1 From the definition of g, argue that

$$g = \frac{1}{d!}\tilde{\epsilon}^{ik\ldots m}\tilde{\epsilon}^{jl\ldots n} g_{ij}g_{kl}\cdots g_{mn}, \qquad \frac{1}{g} = \frac{1}{d!}\underset{\sim}{\epsilon}_{ik\ldots m}\underset{\sim}{\epsilon}_{jl\ldots n} g^{ij}g^{kl}\cdots g^{mn},$$

which implies $\tilde{\epsilon}^{ij\ldots k}$ ($\underset{\sim}{\epsilon}_{ij\ldots k}$) is a tensor density of weight $+1$ (-1).

18.2 Express g_{ij} (and g^{ij}) in terms of d, g, $\underset{\sim}{\epsilon}_{ij\ldots k}$ (or $\tilde{\epsilon}^{ij\ldots k}$) and g^{ij} (or g_{ij}). Show that the determinant variation can be given by

$$\delta g = +g g^{ij}\,\delta g_{ij} = -g g_{ij}\,\delta g^{ij}.$$

18.3 Express $\partial g/\partial x^i$ in terms of g and the Christoffel symbols.

Exercise 19 (6 points): *Covariant derivative II: Vector densities*

19.1 Let V^i be a vector field and $\tilde{V}^i := \sqrt{|g|}\, V^i$ be the corresponding vector density of weight 1. Show that

$$\nabla_i\, V^i = \frac{1}{\sqrt{|g|}}\,\partial_i\!\left(\sqrt{|g|}\, V^i\right), \qquad \text{and} \qquad \nabla_i\,\tilde{V}^i = \partial_i\,\tilde{V}^i\,.$$

19.2 The Laplace–Beltrami operator for a scalar field ϕ is given by $\Box\phi := \nabla^i\,\nabla_i\,\phi$.

a) Rewrite $\Box\phi$ such that the result only contains *partial* derivatives, instead of covariant derivatives.

b) As an example, calculate the operator in spherical coordinates of a 3-dimensional Euclidean space.