

Fifth exercise sheet on Relativity and Cosmology I

Winter term 2022/23

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Submit: Thu, Nov. 24th

Discuss: Thu, Dec. 1st

Exercise 14 (6 points): *Curvature I*

For a nearly spherical body, the ratio of its Schwarzschild radius to its physical radius is a heuristic measure for the deviation of the geometry in the neighbourhood of the body from the flat Minkowski space-time.

- 14.1 Compare this ratio for a globular cluster of stars ($M \approx 10^6 M_\odot$, $R \approx 20$ pc), the Sun, the Earth, a neutron star ($M \approx M_\odot$, $R \approx 10$ km), a White Dwarf ($M \approx M_\odot$, $R \approx 10^4$ km) as well as for a proton and an electron. For the latter two, use their Compton wavelengths \hbar/mc as the (effective) radius.
- 14.2 Which mass would an elementary particle need to have, such that its Compton wavelength would be as large as its Schwarzschild radius? What size would its Schwarzschild radius then be?
- 14.3 The quantities appearing in these considerations are often expressed in terms of the so-called Planck units, which result from a unique combination of the natural constants G , c and \hbar . Calculate the Planck mass, the Planck length, the Planck time and the Planck energy in SI units.

Exercise 15 (9 points): *Curvature II*

In cylindrical coordinates (ρ, φ, z) of 3-dimensional Euclidean space, consider a surface of revolution with generatrix $z = \exp(-a^2 \rho^2)$.

- 15.1 Determine the induced metric on the surface.
- 15.2 Calculate the curvature scalar at the apex using three different methods:
 - a) Compare the circumferences and the areas (Bertrand–Diguët–Puisseux theorem).
 - b) Find the radius of the spherical shell, that best approximates the given surface around the apex, and use the known curvature of a sphere with radius R .

Exercise 16 (6 points): *Transformations of the Christoffel symbols*

Consider the Christoffel symbols of the first and second kinds

$$\Gamma_{ikj} := \frac{1}{2} (g_{ik,j} - g_{kj,i} + g_{ji,k}), \quad \Gamma^i{}_{kj} := g^{il} \Gamma_{lkj}.$$

and a general coordinate transformation $x^i \rightarrow x^{i'}(x^j)$

- 16.1 Derive the transformation of the symbols $\Gamma_{ikj} = \Gamma_{ikj}(\Gamma^{i'}{}_{k'j'})$ under the coordinate transformation. Do they constitute the components of a tensor?
- 16.2 Derive the corresponding transformation for $\Gamma^i{}_{kj} = \Gamma^i{}_{kj}(\Gamma^{i'}{}_{k'j'})$.