

Sixth exercise sheet on Relativity and Cosmology I

Winter term 2022/23

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Exercise 17 (7 points): *Covariant derivative I: Metricity and torsion*

An affine connection $\tilde{\nabla}$, also known as a covariant derivative $\tilde{\nabla}_i$, can be defined by its connection coefficients $\tilde{\Gamma}^i_{kj}$, which transform in the same way as the Christoffel symbols. The covariant derivative of a tensor A^i_j is then

$$\tilde{\nabla}_i A^j_k := \partial_i A^j_k + \tilde{\Gamma}^j_{il} A^l_k - \tilde{\Gamma}^l_{ik} A^j_l.$$

$\tilde{\nabla}$ can be characterised by its non-metricity and torsion, defined as

$$Q_{ijk}(\tilde{\nabla}) := -\tilde{\nabla}_i g_{jk}, \quad T^i_{jk}(\tilde{\nabla}) := 2\tilde{\Gamma}^i_{[jk]} \equiv \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj}.$$

17.1 Let T^{\dots} be a (q, p) -tensor. Argue that $\tilde{\nabla}_i T^{\dots}$ is a $(q, p+1)$ -tensor. Are Q_{ijk} and T^i_{jk} tensors?

17.2 Let ∇ be the Levi-Civita connection in (pseudo-)Riemannian space, whose coefficients are given by the Christoffel symbols of the second kind Γ^i_{kj} . Show that the metric is covariantly constant, i.e. $\nabla_k g_{ij} = -Q_{kij}(\nabla) \equiv 0$, $\nabla_k g^{ij} \equiv 0$; furthermore, show that $T^i_{jk}(\nabla) \equiv 0$.

17.3 Let the non-metricity and torsion of $\tilde{\nabla}$ be zero. Show that $\tilde{\nabla}$ is necessarily Levi-Civita.

Exercise 18 (7 points): *Derivative of a determinant*

Let $g = \det g_{ij}$ be the metric determinant, d the dimension of the manifold; $\tilde{\epsilon}^{ij\dots m}$ and $\tilde{\epsilon}_{ij\dots m}$ the Levi-Civita symbols, taking values $+1$ (or -1) for even (or odd) permutations of the indices, denoted in the lecture as $\epsilon(ij\dots m)$.

18.1 From the definition of g , argue that

$$g = \frac{1}{d!} \tilde{\epsilon}^{ik\dots m} \tilde{\epsilon}^{jl\dots n} g_{ij} g_{kl} \dots g_{mn}, \quad \frac{1}{g} = \frac{1}{d!} \tilde{\epsilon}_{ik\dots m} \tilde{\epsilon}_{jl\dots n} g^{ij} g^{kl} \dots g^{mn},$$

which implies $\tilde{\epsilon}^{ij\dots k}$ ($\tilde{\epsilon}_{ij\dots k}$) is a tensor density of weight $+1$ (-1).

18.2 Express g_{ij} (and g^{ij}) in terms of d , g , $\tilde{\epsilon}_{ij\dots k}$ (or $\tilde{\epsilon}^{ij\dots k}$) and g^{ij} (or g_{ij}). Show that the determinant variation can be given by

$$\delta g = +g g^{ij} \delta g_{ij} = -g g_{ij} \delta g^{ij}.$$

18.3 Express $\partial g / \partial x^i$ in terms of g and the Christoffel symbols.

Exercise 19 (6 points): *Covariant derivative II: Vector densities*

19.1 Let V^i be a vector field and $\tilde{V}^i := \sqrt{|g|} V^i$ be the corresponding vector density of weight 1. Show that

$$\nabla_i V^i = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} V^i \right), \quad \text{and} \quad \nabla_i \tilde{V}^i = \partial_i \tilde{V}^i.$$

19.2 The Laplace–Beltrami operator for a scalar field ϕ is given by $\square \phi := \nabla^i \nabla_i \phi$.

- Rewrite $\square \phi$ such that the result only contains partial derivatives, instead of covariant derivatives.
- As an example, calculate the operator in spherical coordinates of a 3-dimensional Euclidean space.