

## Seventh exercise sheet on Relativity and Cosmology I

Winter term 2022/23

**Release:** Thu, Dec. 1<sup>st</sup>

**Submit:** Thu, Dec. 8<sup>th</sup>

**Discuss:** Thu, Dec. 15<sup>th</sup>

### Exercise 20 (6 points): *Riemannian normal coordinates*

Consider Riemannian normal coordinates  $y^\mu$  at a point  $P$ , which is set as the coordinate origin  $y^\mu = 0$ .

**22.1** Let  $R_{\mu\kappa\lambda\nu}$  be the Riemann tensor. Show that

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\kappa\lambda\nu}(0) y^\kappa y^\lambda + O(|y|^3).$$

Show to this end first that

$$\Gamma^\mu_{\rho\sigma,\kappa}(0) + \Gamma^\mu_{\sigma\kappa,\rho}(0) + \Gamma^\mu_{\kappa\rho,\sigma}(0) = 0.$$

**22.2** Give a physical interpretation of the equation.

### Exercise 21 (4 points): *Algebraic identities of the Riemann tensor*

**23.1** Let  $R_{ijkl}$  be the Riemann tensor. Show that the following algebraic identities hold

$$R_{ij(kl)} = 0, \quad R_{(ij)kl} = 0; \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

The last one is also called the *first* or *algebraic Bianchi identity*.

**23.2** (bonus) Give interpretations of the first two identities in terms of parallel transport.

### Exercise 22 (5 points): *Geometry and topology*

In Euclidean space  $\mathbb{R}^3$ , consider the cylindrical coordinates  $(\rho, \varphi, z)$ . Define a surface of revolution  $\mathcal{A}$  via

$$z^2 = [f(\rho)]^2, \quad \rho^2 = x^2 + y^2,$$

where  $f$  is a strictly positive smooth function on  $[0, a]$  with  $f(a) = 0$ ,  $f'(a^-) \rightarrow -\infty$  and  $f'(0^+) = 0$ .

**24.1** Determine the induced metric  $ds^2 = h_{ij} dy^i dy^j$  on  $\mathcal{A}$ .

**24.2** Let  $R$  be the Ricci scalar of  $\mathcal{A}$ . Evaluate the following integral explicitly

$$\int_{\mathcal{A}} d^2y \sqrt{h} R.$$

### Exercise 23 (5 points): *Geodesic deviation*

Consider two neighbouring geodesics with worldlines  $x^\mu(\tau)$  and  $x^\mu(\tau) + \zeta^\mu(\tau)$ , where  $\zeta^\mu(\tau)$  is considered to be small, so that quadratic and higher-order terms can be neglected. Let  $u^\mu = dx^\mu/d\tau$  be the velocity. Show that the relative acceleration satisfies

$$\frac{D^2 \zeta^\mu}{D\tau^2} = R^\mu{}_{\nu\kappa\lambda} u^\nu u^\kappa \zeta^\lambda,$$

where  $\frac{D}{D\tau} := u^\nu \nabla_\nu$ , which is known as the *geodesic deviation equation*.