University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rci2223.html
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Ninth exercise sheet on Relativity and Cosmology I

Winter term 2022/23

Release: Thu, Dec. 15th **Submit**: Thu, Jan. 12th **Discuss**: Thu, Jan. 19th

Exercise 27 (6 points): *Klein–Gordon theory of scalar field*

Consider the action of a neutral Klein–Gordon field $\phi = \phi(x)$ with mass parameter m and potential $V(\phi)$,

$$S_{\mathrm{KG}}[\phi] := \int \mathrm{d}^4x \sqrt{|g|} \left\{ -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{m^2}{2}\phi^2 - V(\phi) \right\}.$$

27.1 Derive the Klein–Gordon field equation by the action principle.

27.2 Derive the symmetric energy-momentum tensor defined by

$$T_{\mu\nu} := -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}},$$

and calculate its trace.

In the following two exercises, consider *linearised general relativity* in a flat background with the ansatz

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\,\psi_{\mu\nu}(x),$$

where $\psi_{\mu\nu}$, $\psi_{\mu\nu,\rho}$ and $\psi_{\mu\nu,\rho,\sigma}$ are small perturbations of the same order.

Exercise 28 (9 points): *Linearised general relativity I: redundancy transformation*

Consider the infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} - 2 f^{\mu}(x),$$

where f^{μ} and $f^{\mu}{}_{,\nu}$ are of the same order as ψ.

28.1 Show that $\psi_{\mu\nu}$ transform as $\psi'_{\mu\nu}(x') = \psi_{\mu\nu}(x) + f_{\mu,\nu}(x) + f_{\nu,\mu}(x)$.

28.2 The *de Donder* or *harmonic condition* reads

$$\psi_{\mu\nu,}{}^{\nu} = \frac{1}{2}\psi^{\nu}{}_{\nu,\mu}.$$

Show that it can be realised by applying such a transformation.

28.3 Show that the linearised Riemann tensor is invariant under this transformation.

Exercise 29 (5 points): *Linearised general relativity II: Fierz–Pauli action*

Linearised general relativity can also be derived from an action, which has been given by *Fierz and Pauli*,

$$\begin{aligned} S_{\mathrm{FP}}[\psi_{\mu\nu}] &:= \int \mathrm{d}^4x \left\{ \frac{1}{2\varkappa}\left(-\psi^{\mu\nu,\sigma}\,\psi_{\mu\nu,\sigma} + 2\,\psi^{\mu\nu,\sigma}\,\psi_{\sigma\nu,\mu} + \psi^{\mu}{}_{\mu,\nu}\,\psi^{\rho}{}_{\rho,}{}^{\nu} - 2\,\psi^{\rho\nu}{}_{,\nu}\,\psi^{\sigma}{}_{\sigma,\rho} \right) + T_{\mu\nu}\,\psi^{\mu\nu} \right\} \\ &=: \int \mathrm{d}^4x \left\{ \mathcal{L}_{\mathrm{FP}} + T_{\mu\nu}\,\psi^{\mu\nu} \right\}, \end{aligned}$$

where $\varkappa := 8\pi G$; $T_{\mu\nu}$ is the *symmetric* energy-momentum tensor of matter, here playing the role of source, and is of the same order as $\psi_{\mu\nu}$.

29.1 Derive the equations of motion for $\psi_{\mu\nu}$ from S_{FP}, and show that they are equivalent to the linearised Einstein equations given in the lecture.

Hint. It might be quicker to apply the variational method directly, instead of using the Euler–Lagrange equations.

29.2 (bonus) Discard the source term. Calculate the *canonical* energy-momentum tensor of $\psi_{\mu\nu}$, defined by

$$t_{\mu\nu} := \frac{\delta S_{\text{FP}}}{\delta \psi_{\rho\sigma,}{}^{\nu}} \, \psi_{\rho\sigma,\,\mu} - \eta_{\mu\nu} \, \mathcal{L}_{\text{FP}} \, .$$

Remark. S_{FP} can be derived by expanding the Einstein–Hilbert action to the quadratic order, but the calculation is tedious by hand.