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Eleventh exercise sheet on Relativity and Cosmology I

Winter term 2022/23

Release: Thu, Jan. 19th **Submit**: Thu, Jan. 26th **Discuss**: Thu, Feb. 2nd

Exercise 34 (10 points): *An exact plane-wave solution I*

Consider the following *exact* metric given by *Bondi, Pirani and Robinson* (1959):

$$\mathrm{d}s^2 = L^2\left(\mathrm{e}^{+2\beta}\,\mathrm{d}x^2 + \mathrm{e}^{-2\beta}\,\mathrm{d}y^2\right) + \mathrm{d}z^2 - \mathrm{d}t^2 = L^2\left(\mathrm{e}^{+2\beta}\,\mathrm{d}x^2 + \mathrm{e}^{-2\beta}\,\mathrm{d}y^2\right) - \mathrm{d}u\,\mathrm{d}v\,,$$

where $L = L(u)$, $\beta = \beta(u)$ are the *background* and *wave factors*, and $u = t - z$, $v = t + z$ the *retarded* and *advanced light-cone* or *null coordinates*, respectively.

34.1 Show that

$$\Gamma^{v}{}_{xx} = \frac{\mathrm{d}}{\mathrm{d}u}\left(L^2\mathrm{e}^{+2\beta}\right) \qquad \Gamma^{v}{}_{yy} = \frac{\mathrm{d}}{\mathrm{d}u}\left(L^2\mathrm{e}^{-2\beta}\right);$$
$$\Gamma^{x}{}_{ux} = \Gamma^{x}{}_{xu} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(L^2\mathrm{e}^{+2\beta}\right) \qquad \Gamma^{y}{}_{uy} = \Gamma^{y}{}_{yu} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(L^2\mathrm{e}^{-2\beta}\right),$$

whereas all the other Christoffel symbols of the second kind vanish.

Hint. It might be quicker to apply the variational method directly, instead of using the definition.

34.2 Show that a *co-moving* particle, defined by $\dot{x} = \dot{y} = \dot{z} = 0$ and $\dot{t} = 1$, moves on a geodesic.

34.3 Derive the Ricci tensor.

Exercise 35 (6 points): *An exact plane-wave solution II*

The vacuum Einstein equation of the Bondi–Pirani–Robinson metric reduces to

$$L'' + (\beta')^2 L = 0\,, \qquad f' := \frac{\mathrm{d}f}{\mathrm{d}u}\,.$$

35.1 Solve the vacuum Einstein equation for $\beta' \equiv 0$, $L(0) = 1$ and $L'(0) = 0$.

35.2 For simplicity, let $\beta \equiv 0$, $L(0) = 1$ and $L'(0) = -1$. Solve the vacuum Einstein equation for these conditions. Employ the transformation to the new coordinates

$$\bar{t} = t - \frac{1}{2}(1 - t + z)\left(x^2 + y^2\right),$$
$$\bar{z} = z - \frac{1}{2}(1 - t + z)\left(x^2 + y^2\right),$$
$$\bar{x} = (1 - t + z)x\,, \qquad \bar{y} = (1 - t + z)y$$

to show that this solution is in fact just the Minkowski space, although containing a singularity.

35.3 Consider the linearised theory. For that purpose let β and β' be small and of the same order, and $L = 1 + \gamma$ where $\gamma(u)$ is of the same order as β^2. Show that the Bondi–Pirani–Robinson metric then takes a form that corresponds to a $+$-polarised wave in the linearised theory.

35.4 (bonus) Now go back to the full theory. Let $L \equiv 1$ for $u \leqslant -a < 0$, and $\beta' \equiv 0$ for $u < -a$ and $u \geqslant 0$. In other words, $\beta' \neq 0$ can only happen for $-a \leqslant u < 0$. Furthermore, assume that β' is such that the *singularity* appears at some $u > 0$.

Provide a *qualitative* discussion and a sketch of the solution for $L(u)$.

Hint: It is sufficient to consider the convexity of L on $[-a, 0)$.

35.5 (bonus) Recall the configuration in the lecture, where a beam of plane gravitational wave meets a ring of test particles, initially at rest in the $z = 0$ plane.

What happens to the ring after the wave has passed by, in the exact (**35.4**) and the linearised theories?

Exercise 36 (4 points): *Polarisation*

In a flat background spacetime, consider two Cartesian coordinate systems (t, x, y, z) and (t, x', y', z) that can be transformed into each other by a rotation with the angle θ around the z-axis.

36.1 Consider an electromagnetic wave that propagates in the z-direction. Let $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_{x'}$, and $\hat{\mathbf{e}}_{y'}$ be the unit polarisation vectors in the coordinate systems. Show that

$$\hat{\mathbf{e}}_{x'} = \hat{\mathbf{e}}_x \cos(\theta) + \hat{\mathbf{e}}_y \sin(\theta)\,, \qquad \hat{\mathbf{e}}_{y'} = -\,\hat{\mathbf{e}}_x \sin(\theta) + \hat{\mathbf{e}}_y \cos(\theta)\,.$$

36.2 Analogously, consider a linearised gravitational wave propagating in the z-direction. Let $\mathbf{e}_+$, $\mathbf{e}_\times$, $\mathbf{e}_{+'}$, $\mathbf{e}_{\times'}$ be the polarisation tensors in the coordinate systems. Show that

$$\mathbf{e}_{+'} = \mathbf{e}_+ \cos(2\theta) + \mathbf{e}_\times \sin(2\theta)\,, \qquad \mathbf{e}_{\times'} = -\,\mathbf{e}_+ \sin(2\theta) + \mathbf{e}_\times \cos(2\theta)\,.$$

36.3 (bonus) Let $|{\rightarrow}\rangle$ and $|{\leftarrow}\rangle$ be the quantum states of a spin-$\frac{1}{2}$ particle, whose spin is aligned or anti-aligned with respect to the x-direction, respectively, and analogously $|{\rightarrow}'\rangle$ and $|{\leftarrow}\rangle$ with respect to the x'-direction. Show that

$$|{\rightarrow}'\rangle = |{\rightarrow}\rangle \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\,|{\leftarrow}\rangle \sin\left(\frac{\theta}{2}\right), \qquad |{\leftarrow}'\rangle = \mathrm{i}\,|{\rightarrow}\rangle \sin\left(\frac{\theta}{2}\right) + |{\leftarrow}\rangle \cos\left(\frac{\theta}{2}\right).$$

37.4 (bonus) Write down the generalisation for the basis states of linear polarisation for a radiation field of arbitrary spin s.