University of Cologne www.thp.uni-koeln.de/gravitation/courses/rcii19.html
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3rd exercise sheet on Relativity and Cosmology II

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Please note that you have two weeks for this exercise sheet. It gives twice the points as usual.

Exercise 44 (15 points): *Differential forms*

44.1 Consider an n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal co-basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n$.

Show that, in an arbitrary coordinate system $\{x^k\}$, the following holds:

$$\omega = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n\,, \tag{1}$$

where g denotes the determinant of the metric, whose components g_{ij} are given in these coordinates.

44.2 The *contraction* of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} := \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$.

Show that, with a given vector field v, the following holds:

$$\mathrm{d}[\omega(v)] = v^i{}_{,i}\,\omega\,. \tag{2}$$

44.3 Define $(\mathrm{div}_\omega v)\,\omega := \mathrm{d}[\omega(v)]$.

Show that, by using coordinates in which $\omega = f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$, the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f}\left(f v^i\right)_{,i}\,. \tag{3}$$

44.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Show that, in spherical coordinates, this volume form is given by $\omega = r^2 \sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$.

Use the result of **44.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi} \tag{4}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v^\theta\right) + \frac{\partial v^\phi}{\partial \phi}\,. \tag{5}$$

Exercise 45 (25 points): *Electrodynamics in flat space-time*

Differential forms are a convenient tool for field theories, as we will show in this exercise on the example of Maxwell electrodynamics in Minkowski space-time. We know that the electromagnetic field strength is given by the Faraday antisymmetric tensor $F_{\mu\nu}$, and the current is given by j^μ. In the language of exterior calculus, the $F_{\mu\nu}$ are components of the *Faraday* 2*-form* $\mathbf{F}$ describing an arbitrary electromagnetic field, given by

$$\mathbf{F} := \frac{1}{2} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -E_x\,\mathrm{d}t \wedge \mathrm{d}x - E_y\,\mathrm{d}t \wedge \mathrm{d}y - E_z\,\mathrm{d}t \wedge \mathrm{d}z + B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y\,, \tag{6}$$

while the j^μ are the components of the current 1-form $\mathbf{j}$, which is given by

$$\mathbf{j} := j_\mu\,\mathrm{d}x^\mu = -\rho\,\mathrm{d}t + j_x\,\mathrm{d}x + j_y\,\mathrm{d}y + j_z\,\mathrm{d}z\,. \tag{7}$$

See reverse.

45.1 The Hodge star operator $\star$ maps p-forms to $(4-p)$-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by $\mathbf{G} := \star\mathbf{F}$.

The Hodge star operator acts on the 2-form basis as follows

$$\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \frac{1}{2}\,\eta^{\mu\alpha}\eta^{\nu\beta}\underset{\sim}{\varepsilon}_{\alpha\beta\gamma\delta}\,\mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta\,, \tag{8}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the inverse Minkowski metric and $\underset{\sim}{\varepsilon}_{\alpha\beta\gamma\delta}$ the totally anti-symmetric Levi-Civita tensor density ($\underset{\sim}{\varepsilon}_{0123} = +1$). For example,

$$\star(\mathrm{d}t \wedge \mathrm{d}x) = -\mathrm{d}y \wedge \mathrm{d}z\,, \quad \text{etc.} \tag{9}$$

Show that the Maxwell 2-form is given by

$$\mathbf{G} = B_x\,\mathrm{d}t \wedge \mathrm{d}x + B_y\,\mathrm{d}t \wedge \mathrm{d}y + B_z\,\mathrm{d}t \wedge \mathrm{d}z + E_x\,\mathrm{d}y \wedge \mathrm{d}z + E_y\,\mathrm{d}z \wedge \mathrm{d}x + E_z\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{10}$$

Which formal relation between the components of $\mathbf{F}$ and $\mathbf{G}$ holds?

45.2 With these definitions, the Maxwell equations (in Gaussian units) can be written in a compact form: $\mathrm{d}\mathbf{F} = 0$ and $\mathrm{d}\mathbf{G} = 4\pi \star \mathbf{j}$.

a) Show that the equation $\mathrm{d}\mathbf{F} = 0$ corresponds to the two homogeneous Maxwell equations

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0\,. \tag{11}$$

b) Calculate $\star\mathbf{j}$ (a 3-form), which is the dual of the current 1-form $\mathbf{j}$. For that purpose use the following relations

$$\star\mathrm{d}x^\mu = \frac{1}{3!}\eta^{\mu\alpha}\underset{\sim}{\varepsilon}_{\alpha\beta\gamma\delta}\,\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta\,, \tag{12}$$

for example

$$\star\mathrm{d}t = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \quad \text{etc.} \tag{13}$$

c) Show that the equation $\mathrm{d}\mathbf{G} = 4\pi \star \mathbf{j}$ corresponds to the two inhomogeneous Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \quad \text{and} \quad \vec{\nabla}\times\vec{B} - \partial_t\vec{E} = 4\pi\vec{j}\,. \tag{14}$$

d) Which formal manipulations can you perform to turn eq. (11) into eq. (14) in the vacuum case?

45.3 The exterior derivative is nilpotent, i.e. the relation $\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = 0$ holds for any p-form $\boldsymbol{\omega}$.

Choosing $\boldsymbol{\omega} = \star\mathbf{j}$, show that this yields the continuity equation $\partial_t\rho - \vec{\nabla}\cdot\vec{j} = 0$.

45.5 Derive the inhomogeneous Maxwell equations from a variational principle. The action is given by

$$S[\mathbf{A}] = \int \left(\frac{1}{8\pi} \star \mathbf{F} \wedge \mathbf{F} + \mathbf{A} \wedge \star\mathbf{j}\right)\,, \tag{15}$$

where $\mathbf{F} = \mathrm{d}\mathbf{A}$ and $\mathbf{A} = A_\mu\,\mathrm{d}x^\mu$ is the electromagnetic 1-form potential.

Show in addition that the continuity equation implies that the action is invariant under the gauge transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathrm{d}f$, where f is a 0-form.

Hint: You can make use of the fact that $\star\alpha \wedge \beta = \alpha \wedge \star\beta$, where α and β are both k-forms.

45.6 ***Bonus exercise.*** Let $\boldsymbol{\omega}$ be a k-form. We define the *codifferential* δ via

$$\delta\boldsymbol{\omega} := (-1)^k \star \mathrm{d} \star \boldsymbol{\omega}\,. \tag{16}$$

This allows us to define the d'Alembert operator

$$\square := \mathrm{d}\delta + \delta\mathrm{d}\,. \tag{17}$$

Show the following statements:

- $\delta A = 0$ is the well-known Lorenz gauge condition $\partial_\mu A^\mu = 0$
- The Lorenz gauge can always be realized by a suitable gauge transformation.
- In the Lorenz gauge the inhomogeneous Maxwell equation can be written as $\square A = 4\pi\mathbf{j}$.

45.7 ***Bonus exercise.*** Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space-time?