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Fifth exercise sheet on Relativity and Cosmology II

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Exercise 45 (20 credit points): *Derivation of the Schwarzschild solution in Cartan calculus*

The aim of this exercise is to derive the Schwarzschild solution using the Cartan formalism.

The general spherically symmetric ansatz is given by:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\,e^{2a(r,t)}\,\mathrm{d}t^2 + e^{2b(r,t)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{1}$$

In terms of the pseudo-orthogonal coframe basis $\{\vartheta^i\}$, $i = 0, \ldots, 3$, the metric takes the form

$$\mathrm{d}s^2 = \eta_{ij}\,\vartheta^i \otimes \vartheta^j = -\,\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3\,. \tag{2}$$

In this exercise, numerals $(0, 1, 2, 3)$ and Latin indices $(i, j, k, l, \ldots)$ indicate an anholonomic basis, whereas coordinate (t, r, θ, ϕ) and Greek indices $(\mu, \nu, \lambda, \sigma, \ldots)$ refer to a holonomic basis. Holonomic and anholonomic indices can be converted into each other by using the *tetrad* defined via

$$\vartheta^i = e_\mu{}^i\,\mathrm{d}x^\mu\,. \tag{3}$$

45.1 Show that

$$\vartheta^0 = e^{a(r,t)}\mathrm{d}t\,, \quad \vartheta^1 = e^{b(r,t)}\,\mathrm{d}r\,, \quad \vartheta^2 = r\,\mathrm{d}\theta\,, \quad \vartheta^3 = r\sin\theta\,\mathrm{d}\phi \tag{4}$$

defines a suitable orthonormal coframe basis to describe the spherically symmetric ansatz.

45.2 Now, calculate the exterior derivatives $\mathrm{d}\vartheta^i$. Insert these into the first Cartan structure equation

$$\Theta^i := \mathrm{d}\vartheta^i + \omega^i{}_j \wedge \vartheta^j = 0 \tag{5}$$

and show that the connection $\omega^i{}_j$ in components reads

$$\omega^1{}_0 = a'\,e^{-b(t,r)}\,\vartheta^0 + \dot{b}\,e^{-a(t,r)}\,\vartheta^1\,, \qquad \omega^3{}_1 = \frac{e^{-b(t,r)}}{r}\,\vartheta^3\,, \qquad \omega^3{}_2 = \frac{\cot\theta}{r}\,\vartheta^3\,,$$

$$\omega^2{}_1 = \frac{e^{-b(t,r)}}{r}\,\vartheta^2\,, \qquad \omega^3{}_0 = 0\,, \qquad \omega^2{}_0 = 0\,,$$

where a prime and superscript dot denote derivatives with respect to r and t, respectively.

Hint: For any metric-compatible connection (like the Levi-Civita connection studied here) we have $\nabla g_{ij} = 0$ and therefore $\omega_{ij} = -\,\omega_{ji}$.

45.3 Calculate the curvature 2-forms $\Omega^i{}_j := \mathrm{d}\omega^i{}_j + \omega^i{}_a \wedge \omega^a{}_j$. Use the second Cartan structure equation

$$\Omega^i{}_j = \frac{1}{2}\, R^i{}_{jkl}\, \vartheta^k \wedge \vartheta^l \tag{6}$$

to show that the non-vanishing anholonomic components $R^i{}_{jkl}$ of the Riemann curvature tensor are

$$\begin{aligned}
-R^0{}_{101} &= e^{-2b}(a'^2 - a'b' + a'') + e^{-2a}(\dot{a}\dot{b} - \dot{b}^2 - \ddot{b}) = R^1{}_{010}\,, \\
R^0{}_{202} &= R^0{}_{303} = -\frac{a'\, e^{-2b}}{r}\,, \qquad R^1{}_{212} = R^1{}_{313} = \frac{b'\, e^{-2b}}{r}\,, \\
R^0{}_{212} &= R^0{}_{313} = -\frac{\dot{b}\, e^{-a-b}}{r} = -\,R^1{}_{202} = -\,R^1{}_{303}\,, \qquad R^3{}_{232} = \frac{1 - e^{-2b}}{r^2}\,.
\end{aligned}$$

45.4 Determine the anholonomic components of the Ricci tensor $R_{ij} = R^k{}_{ikj}$ as well as the Ricci scalar $R = \eta^{ij} R_{ij}$. Note that for the contraction of anholonomic indices the Minkowski metric has to be used.

45.5 Calculate the *holonomic* components of the Einstein tensor $G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\,\delta^\mu_\nu\, R$. In general, one can find the holonomic components of a tensor via the tetrad, $A_{\mu\nu} \overset{*}{=} e_\mu{}^i\, e_\nu{}^j\, A_{ij}$. However, for a diagonal metric, the holonomic diagonal components of a $(1,1)$-tensor coincide with the anholonomic diagonal components, so one can directly identify $A^i{}_i \overset{*}{=} A^\mu{}_\mu$, not summing over the respective indices.

45.6 Back to physics. Outside of the mass distribution we have a vacuum and therefore $T^\mu{}_\nu = 0$, i.e. $G^\mu{}_\nu = 0$. Use this to show that b depends only on r and that $a(r) = -b(r)$.

45.7 Finally, integrate the differential equation arising from $G^t{}_t = 0$ to find a relation between b and r and use the Newtonian limit to fix the integration constant. Show that

$$e^{-2b(r)} = 1 - \frac{2GM}{r}\,, \tag{7}$$

which concludes our derivation of the Schwarzschild solution.