

Ninth exercise sheet on Relativity and Cosmology II

Summer term 2021

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Submit: Mon, Jun. 28th on ILIAS

Discuss: Thu, Jul. 1st

Exercise 53 (16 credit points): *Kerr metric I*

In this exercise we are going to consider the line element of the Kerr metric

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mra \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2,$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$.

53.1 Calculate the surface area of the event horizon of a rotating black hole ($|a| \leq M$) for $t = \text{const}$.

Furthermore, show that the circumference around the poles is always smaller than the circumference around the equator, i.e. that the geometry of the event horizon is not spherical. Give an estimate for the ratio of the polar and equatorial circumference for the extremal case $|a| = M$.

53.2 Consider the limit $r \gg M$ and $r \gg |a|$ for the Kerr metric.

Show that the non-diagonal term $\propto d\phi dt$ is equal to the non-diagonal term of a slowly and rigidly rotating spherical mass distribution, which has been given in the lecture as

$$-\frac{4I}{r^3} \varepsilon^{\alpha\beta\gamma} dx^\alpha \omega^\beta x^\gamma dt, \quad I : \text{moment of inertia},$$

if one chooses the z -axis to be parallel to the direction of rotation.

53.3 Consider the Kerr metric in the limit $M \rightarrow 0$, $a = \text{const}$.

Show – preferably by means of an explicit coordinate transformation to Cartesian coordinates – that this limit describes flat spacetime. What is the geometric meaning of the coordinate r in this limit?

Hint: Try the ansatz $x = f(r) \sin(\theta) \cos(\phi)$, $y = f(r) \sin(\theta) \sin(\phi)$, $z = g(r) \cos(\theta)$ to transform the metric to Cartesian coordinates.

Exercise 54 (4 credit points): *Kerr metric II*

Give a rough estimate for the parameter a/M – as given in the Kerr metric – for Sun and Earth. Take the required physical parameters – like the radius of the Sun etc. – from the literature.

What do you notice?