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http://www.thp.uni-koeln.de/gravitation/courses/rcii23.html
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First exercise sheet on Relativity and Cosmology II

Summer term 2023

Release: Thu, Apr. 13th **Submit**: Thu, Apr. 20th **Discuss**: Thu, Apr. 27th

Exercise 37 ($20 = 10 + 6 + 4$ points): *Effective Schwarzschild potential*

The aim of this exercise is to analyse certain properties of the movement of massive test particles in the Schwarzschild space-time.
For this purpose, consider the equation of motion on the equatorial plane $\theta = \pi/2$ with an effective potential V_{eff} that results from the geodesic equation

$$\frac{\dot{r}^2}{2} + V_{\text{eff}}(r) = E\,, \qquad V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}.$$

Here ℓ and E indicate constants of motion.

In the following, express radial distances in terms of the Schwarzschild radius $r_S = 2GM$.

37.1 **a)** Analyse and sketch the potential $V_{\text{eff}}(r)$ for all relevant cases (characterised by the values of M and ℓ).

b) In which cases do bound particle orbits exist? Analyse the stability of all orbits.

c) Which conditions does a test particle approaching from infinity ($r \to +\infty$) have to fulfil in order to fall into the centre of the effective potential?

Under which circumstances does a particle that starts from rest at infinity fall into the centre?

d) Compare the results obtained so far to the situation in Newtonian gravity.

e) Show the following statements:

i. For $\ell/GM < 2\sqrt{3}$ every in-falling particle falls towards the event horizon $r = 2GM$.

ii. The most strongly bound orbit is located at $r = 6GM$ with $\ell/GM = 2\sqrt{3}$ and it possesses a relative binding energy of $1 - \sqrt{8/9}$.

37.2 Consider a massive test particle initially being at rest at the radial coordinate $R > r_S$ that falls radially ($\ell = 0$) into the centre.

a) Find the solution of the resulting initial value problem.
Hint: The solution can be given in a parametrised form $r(\eta)$, $\tau(\eta)$ (where τ is the proper time of the particle) which describes a *cycloid* orbit.
At which proper time τ_0 does the particle reach the centre of the potential?

b) How long does it take for this particle to reach the Schwarzschild radius as measured by an observer at infinity?

37.3 In the lecture it was mentioned that Kepler's Third Law

$$GM = \omega^2 r^3\,, \quad \text{where} \quad \omega = \mathrm{d}\phi/\mathrm{d}t,$$

holds for circular orbits in the Schwarzschild space-time. Prove this statement.