University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Tatevik Vardanyan and Mauricio Ortiz Torres

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ver. 1.01

Second exercise sheet on Relativity and Cosmology II

Summer term 2023

Release: Thu, Apr. 20th **Submit**: Thu, Apr. 27th **Discuss**: Thu, May 4th

In the following exercises, consider the coordinates (t, r, θ, ϕ). The 2-*dimensional equatorial spatial slice* Σ is defined by $t = $ const., $\theta = \pi/2$, and its induced metric reads $\mathrm{d}\sigma_\Sigma^2$. Moreover, the Schwarzschild metric in such coordinates reads

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega^2\,, \qquad f(r) := 1 - \frac{R_\mathrm{S}}{r}\,, \quad R_\mathrm{S} := 2GM\,, \quad \mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

Furthermore, denote the Euclidean metric on $\mathbb{R}^n$ by $\mathrm{d}\vec{x}^2_{\mathbb{R}^n}$.

Exercise 38 (10 points): *Schwarzschild metric in isotropic coordinates*

Consider a coordinate transformation

$$r = \left(1 + \frac{R_\mathrm{S}}{4\bar{r}}\right)^2 \bar{r}\,, \qquad t = \bar{t}\,, \quad \theta = \bar{\theta}\,, \quad \phi = \bar{\phi}\,.$$

38.1 Express the Schwarzschild metric in the new cooordinates $\left(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}\right)$, which are called *isotropic*.

38.2 Compare briefly the metric components in old and new coordinates at the event horizon $R_S = 2GM$.

38.3 In the *standard* coordinates, consider a radial range $R_\mathrm{S} < r < R$.

Use the *isotropic* coordinates to calculate

- the surface area on the equatorial spatial slice between these radii, and
- the volume of a spherical shell with $t = $ const. within the range.

38.4 Compare your results in **38.3** to the corresponding quantities in the Euclidean space.

Exercise 39 (4 points): *Isometric embedding I: the Schwarzschild space*

39.1 Consider the cylindrical coordinates (ρ, ψ, z) of $\mathbb{R}^3$. Set

$$\mathrm{d}\vec{x}^2_{\mathbb{R}^3}\Big|_{z=z(r)} \equiv \mathrm{d}\sigma_\Sigma^2$$

and integrate the resulting equation (*Flamm's paraboloid*).

39.2 In the current case, if $\mathrm{d}\sigma_\Sigma^2 \to \mathrm{d}\vec{x}^2_{\mathbb{R}^2}$ as $r \to +\infty$, Σ is called *asymptotically flat*.

Analytically extend the embedding in **39.1**, and show that there can be *two* distinct regions, which are both asymptotically flat (*Einstein and Rosen*).

Hint: Sketching the embedding may help.

Exercise 40 (6 points): *Isometric embedding II: a wormhole*

Consider a spacetime $\mathcal{M}_{\mathrm{W}}$ with the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2 + r^2\right)\mathrm{d}\Omega^2$$

where b is a constant of dimension length.

40.1 Argue briefly that the equatorial spatial slice Σ of $\mathcal{M}_{\mathrm{W}}$ is representative for the latter.

40.2 Consider the cylindrical coordinates (ρ, ψ, z) of $\mathbb{R}^3$. Set

$$\mathrm{d}\vec{x}^2_{\mathbb{R}^3}\Big|_{\rho=\rho(r),\, z=z(r)} \equiv \mathrm{d}\sigma^2_{\Sigma}$$

and integrate the resulting equation for z and ρ.

40.3 Argue briefly that there is a hole-like structure in $\mathcal{M}_{\mathrm{W}}$.

Hint: Sketching the embedding may help.

Remark. The Einstein tensor of the metric reads

$$G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{b^2}{\left(b^2+r^2\right)^2}\left(-\mathrm{d}t^2 - \mathrm{d}r^2 + \left(b^2+r^2\right)\mathrm{d}\Omega^2\right).$$

If the matter were modelled by an ideal fluid, its energy density would turn out to be negative,

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{1}{\varkappa}\frac{b^2}{\left(b^2+r^2\right)^2} < 0\,.$$

In other words, matter with negative energy density is needed to source this spacetime.

Inversely, one may also ask, if it is possible to have a viable wormhole spacetime sourced by matter with positive energy density. Under precise and additional assumptions, this has been excluded by the so-called *topological censorship theorem*.