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http://www.thp.uni-koeln.de/gravitation/courses/rcii23.html
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Seventh exercise sheet on Relativity and Cosmology II

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Exercise 49 (15 credit points): *Interior solution for spherically symmetric stars*

On the fifth sheet, in exercise **45** you have started from the general line element of a spherically symmetric spacetime,

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2a(r,t)}\,\mathrm{d}t^2 + \mathrm{e}^{2b(r,t)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{1}$$

and solved the Einstein equations for vacuum, resulting in the Schwarzschild spacetime. In this exercise, consider instead of vacuum a stationary ideal fluid

$$T^{\mu}{}_{\nu} = -\,(\rho(r) + p(r))\,\delta^{\mu}_{0}\delta^{0}_{\nu} + p(r)\,\delta^{\mu}_{\nu}\,, \tag{2}$$

with mass density $\rho(r)$ and pressure $p(r)$.

49.1 Starting from the Einstein tensor for a general spherically symmetric metric derived for exercise **45**, show that the Einstein equations reduce to

$$\mathrm{e}^{2b(r)} = \left(1 - \frac{2\mathrm{G}M(r)}{r}\right)^{-1}\,, \quad \text{with} \quad M(r) = M_0 + 4\pi\int_0^r \mathrm{d}\tilde{r}\;\rho(\tilde{r})\,\tilde{r}^2\,, \tag{3}$$

$$a(r) = -b(r) - 4\pi\mathrm{G}\int_r^\infty \mathrm{d}\tilde{r}\;\mathrm{e}^{2b(\tilde{r})}\,\tilde{r}\,\left(\rho(\tilde{r}) + p(\tilde{r})\right)\,. \tag{4}$$

You do not need to consider $G^2{}_2 = 8\pi\mathrm{G}\,T^2{}_2$ and $G^3{}_3 = 8\pi\mathrm{G}\,T^3{}_3$. They can be reduced to the Tolman–Oppenheimer–Volkoff equation, which will be derived below in a different way.

49.2 Show that one can recover the Schwarzschild solution by setting $\rho = p = 0$. To describe a star in equilibrium, what value should be chosen for the constant M_0, and why?

49.3 Finally, derive the Tolman–Oppenheimer–Volkoff equation from the covariant conservation of the energy-momentum tensor,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\mathrm{G}\left(M(r) + 4\pi r^3\,p(r)\right)}{r^2\left(1 - \frac{2\mathrm{G}M(r)}{r}\right)}\left(\rho(r) + p(r)\right)\,. \tag{5}$$

Is it possible to have a pressureless spherically symmetric star in equilibrium?

Exercise 50 (5 credit points): *Misner–Sharp mass*

A notion of mass useful for spherically symmetric stars is the Misner–Sharp mass. For spacetimes with line elements of the form

$$\mathrm{d}s^2 = \gamma_{IJ}\mathrm{d}x^I\mathrm{d}x^J + r^2\,\mathrm{d}\Omega^2\,, \tag{6}$$

where I, J do not run over the angular variables (θ, ϕ), it is defined as

$$M_{\mathrm{MS}} = \frac{r}{2\mathrm{G}}\left(1 - \gamma^{IJ}r_{,I}\,r_{,J}\right)\,. \tag{7}$$

50.1 Evaluate the Misner–Sharp mass for line elements of the form (1). What values does it take for the solution from exercise **49** and for the Schwarzschild solution?

50.2 From the Einstein equations for (1) you computed for exercise **45**, show that under the assumption $T_{00} \geqslant 0$ the Misner–Sharp mass is everywhere non-decreasing with r.

Note that due to the possible occurrence of so-called trapped surfaces*, this is not sufficient to prove that the Misner–Sharp mass is always positive. The full proof is much more involved, and also requires a stronger restriction on the energy-momentum tensor.*