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Tenth exercise sheet on Relativity and Cosmology II

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Exercise 55 (10 credit points): *Ideal fluids in Cosmology*

Consider an ideal fluid in a Friedmann–Lemaître model, with Robertson–Walker geometry governed by the Friedmann equations.

55.1 Show that comoving observers move on geodesics.

55.2 What is the energy-momentum tensor of the ideal fluid for this kind of observers? Under which conditions can it act like a cosmological constant in the Einstein equations?

55.3 Evaluate the covariant conservation of the energy-momentum tensor and show that this yields only *one* non-trivial equation, which on the other hand can also be deduced directly from the Friedmann equations.

55.4 Consider an equation of state of the form $p = w\rho$ with $w =$ const. Calculate the function $\rho(a)$. For which values of w does $\ddot{a} > 0$ hold? In which cases is $\rho + 3p > 0$ (the strong energy condition) fulfilled?

55.5 Calculate $\rho(a)$ for a so-called "Chaplygin gas" whose equation of state is $p = -A/\rho$ ($A =$ const. > 0) and discuss the extremal cases $a \to 0$ and $a \to \infty$.

Exercise 56 (4 credit points): *Friedmann I*

Consider a Friedmann–Lemaître model with $\mathcal{K} \neq 0$ and present density parameters $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$ and $\Omega_{\mathrm{v},0}$ as well as $\Omega := \Omega_{\mathrm{m},0} + \Omega_{\mathrm{r},0} + \Omega_{\mathrm{v},0}$. Furthermore, let

$$\rho_{\mathrm{c}}(a) = \frac{3\,\dot{a}^2}{8\pi \mathrm{G}\, a^2}$$

be the critical density at a time when the scale parameter had the value a, and let $\Omega_{\mathrm{m}}(a) = \rho_{\mathrm{m}}(a)/\rho_{\mathrm{c}}(a)$ etc. be the corresponding relative densities.

Determine the quantity $\Omega(a) - 1$ as a function of $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$, $\Omega_{\mathrm{v},0}$ and a. This quantity indicates how much the considered model "deviates" from a flat model at a certain time. What kind of problem with regard to the deviation from flatness at earlier times arises for a Friedmann–Lemaître model whose density parameter Ω differs only slightly from unity today?

Exercise 57 (6 credit points): *Friedmann II*

Solve the Friedmann equation for a universe that contains radiation as well as non-relativistic matter (dust). For this purpose, rewrite the Friedmann equation as a differential equation with respect to the conformal time η, solve this equation for the three possible values of $\mathcal{K}$ and write out the result in the form $(a(\eta), t(\eta))$.

Determine the t-dependence of $a(t)$ for early ($t \to 0$) and late ($t \to \infty$) times for all the possible values of $\mathcal{K}$.