University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Manuel Krämer and Thaisa Guio

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www.thp.uni-koeln.de/gravitation/courses/rci13.html

8th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 12th December 2013 during the exercise class.

Exercise 22 (6 credit points): *Integration*

Let f be a strictly positive smooth function on $[0, a)$ with $f(a) = f'(0) = 0$ and $f'(a) = -\infty$. By means of this function, we define a surface of revolution $\mathcal{A}$ via

$$z^2 = [f(r)]^2 , \quad r^2 = x^2 + y^2 .$$

Determine the metric g_{ij} which is induced by the line element $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2$ on this surface.

Calculate the corresponding Ricci scalar and show by explicit calculation that the integral

$$\int_{\mathcal{A}} \sqrt{g}\, R\, \mathrm{d}^2 x$$

does not depend on the choice of the function f.

Exercise 23 (14 credit points): *Killing vector fields*

23.1 Show that the Killing equation $\nabla_{(\mu}\, v_{\nu)} = 0$ can also be written as $\mathcal{L}_v\, g_{\mu\nu} = 0$.
What does this mean from a physical point of view?

23.2 Give a proof for the following integrability condition for a Killing vector field v^μ:

$$v_{\lambda;\kappa\nu} = -\, v_\mu\, R^\mu{}_{\nu\lambda\kappa} .$$

23.3 Consider a timelike Killing vector field ξ^μ. Show that there is a coordinate system in which the metric does not depend on time, i.e. $\frac{\partial g_{\mu\nu}}{\partial t} = 0$ holds.

23.4 Find all Killing vector fields for Minkowski spacetime.

23.5 Let $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ be the tangent vector of a geodesic, i. e. $u^\nu\, \nabla_\nu\, u^\mu = 0$, and let ξ^μ be a Killing vector field. Show that $u_\mu\, \xi^\mu$ is constant along the geodesic. Use these conserved quantities to illustrate the physical meaning of the Killing vector fields from part **23.4**.

23.6 Let $T^{\mu\nu}$ be a symmetric tensor field with vanishing divergence, i. e. $\nabla_\mu\, T^{\mu\nu} = 0$, and let ξ^μ be a Killing vector field. Calculate $\left(\xi^\mu\, T_\mu{}^\nu\right)_{;\nu}$.
The result is of great importance for the construction of conserved integral quantities.