

## 3<sup>rd</sup> exercise sheet on Relativity and Cosmology I

### Winter term 2017/18

**Deadline for delivery:** Thursday, 9<sup>th</sup> November 2017 at the beginning of exercise class.

#### Exercise 6: Covariant Maxwell Equations

Recall from classical electromagnetism the Maxwell equations:\*

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{j} \qquad (1a)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (1b)$$

In terms of the scalar ( $\Phi$ ) and vector ( $\vec{A}$ ) potentials, the electric and magnetic fields are  $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$  and  $\vec{B} = \vec{\nabla} \times \vec{A}$ . Let  $A^\mu = (\Phi, \vec{A})$  be the 4-potential,  $j^\mu = (\rho, \vec{j})$  the 4-current, and define the new covariant tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (2)$$

**6.1** Show that the expressions

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu \qquad (3a)$$

$$\partial_{[\rho} F_{\mu\nu]} = 0 \qquad (3b)$$

correspond to equations (1a) and (1b) respectively. The notation  $T_{[\rho\mu\nu]}$  stands for antisymmetrization, i.e.

$$T_{[\rho\mu\nu]} = \frac{1}{3!} (T_{\rho\mu\nu} - T_{\rho\nu\mu} + T_{\nu\rho\mu} - T_{\nu\mu\rho} + T_{\mu\nu\rho} - T_{\mu\rho\nu}). \qquad (4)$$

**6.2** Show that (3a) leads to continuity equation  $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}$ . How does the continuity equation look like in a Lorentz-boosted reference frame?

#### Exercise 7: Covariant Lorentz Force

Let  $P^\mu = (E, \vec{p})$  be the 4-momentum,  $u^\nu = (\gamma, \gamma\vec{v})$ , the 4-velocity,  $\tau$  proper time, and  $t$  coordinate time. Using (2), show that the space component of manifestly covariant 4-dimensional Lorentz force,

$$\frac{dP^\mu}{d\tau} = f^\mu = qF^{\mu\nu}u_\nu \qquad (5)$$

gives for small velocities the well-known Lorentz force:

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \qquad (6)$$

What is the physical meaning of the time component  $f^0$  of the covariant 4-force (5)?

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\*In units where  $c = \mu_0 = \epsilon_0 = 1$