Contents

Chapter 1

Thermodynamics of Black Holes and Hawking Radiation

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> *La filosofia è scritta in questo grandissimo libro, che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola.*
>
> Galileo Galilei, *Il Saggiatore*

Modern Science is built upon a mathematical description of Nature. Although its characters are no longer only triangles, circles and other geometrical figures, Galileo's characterisation of science is as valid as it was in his time.

Black holes – the subject of this school – provide an example par excellence for this mathematical description of Nature. Their very existence and simple geometrical properties were predicted by pure theory – Einstein's geometric theory of gravity, the general theory of relativity. By now, their existence has been proven by observations with almost certainty. It is expected, not least with the advent of powerful gravitational wave detectors, that observations of black holes will play a key role in 21st century astronomy.

The subject of my contribution is the thermodynamics of black holes and Hawking radiation. The main focus is therefore on quantum aspects, although I shall elaborate on all the classical aspects of black holes that are a necessary prerequisite for the understanding of their quantum behaviour. Apart from the physics of the early Universe, quantum aspects of black holes will provide the only key towards an understanding of a quantum theory of gravity.

I shall start in the next section with a general introduction into black hole physics. This serves both the purpose to recall the basic notions known from elementary courses as well as to give the material and set the notation needed for the main part. It may also be considered as an introduction to some of the material covered by other lectures.

Section 1.2 then enters the main theme and gives a review of the laws of black-hole mechanics. These laws have full analogies to the laws of thermodynamics, but at the classical level of section 1.2 this analogy is purely formal. A physical interpretation is only possible within quantum theory and given in section 1.3. I there discuss both the derivation of Hawking radiation and its physical interpretation. Section 1.4, then, is devoted to the interpretation of black-hole entropy and the problem of information loss. The last two sections enter the most speculative parts of black holes – the possible role they play in and for a quantum theory of gravity. Section 1.5 covers canonical gravity, while section 1.6 covers superstring theory.

To keep the references to a minimum, I put more emphasis on citing books and reviews than on original work. Two recent proceedings that cover all aspects are Hehl et al. (1998) and Wald (1998). A very recommendable general introduction is Thorne (1994); an excellent short introduction is Luminet (1998).

The metric convention is $\text{diag}(1, -1, -1, -1)$; I follow the abstract index notation of Wald (1984).

1.1 Black holes – A general introduction

Black holes have a long and fascinating story, see Israel (1987). Probably the first person who envisaged the existence of 'dark stars' was the Reverend John Michell in 1784. He wanted to develop a method to measure the distance of stars by taking into account the 'diminution of their light' escaping the star's gravitational field. (Recall that the first parallax was measured by Bessel in 1837, so in Michell's time this was an interesting proposal.) At that time, Newton's corpuscular theory of light was prevailing, so it was a sensible problem to study a star with an escape velocity greater than the speed of light. (Recall that the finiteness of the speed of light had already been noted in 1676.) Setting $v = c$ in the expression for the escape velocity v of a spherically symmetric body, $v^2 = 2GM/r$, one finds for the corresponding radius

$$r \equiv R_0 = \frac{2GM}{c^2} \approx 3\frac{M}{M_\odot}\text{km}. \tag{1.1}$$

This radius, R_0, is known as the *Schwarzschild radius* (see below). For a body with the mass of the earth, $R_0 \approx 0.9$cm only.

Michell asked: How big must a star with the density equal to the density of the Sun be, to have $v = c$? Since then the radius of the star is equal to its Schwarzschild radius R_0, one has ($R_\odot$ is the radius of the Sun)

$$\frac{R_0}{R_\odot} = \frac{2GM}{c^2 R_\odot} = \frac{2G}{c^2}\frac{M_\odot R_0^3}{R_\odot^4}$$

and therefore

$$\frac{R_0}{R_\odot} = \left(\frac{c^2}{2G}\frac{R_\odot}{M_\odot}\right)^{1/2} = \left(\frac{R_\odot}{R_{0\odot}}\right)^{1/2} \approx 483,$$

where $R_{0\odot}$ denotes the Schwarzschild radius of the Sun. Note that this corresponds to a mass of about $10^8 M_\odot$ – which coincides with the mass of some of the supermassive black holes that are now assumed to exist in the centre of the galaxies (see Treves' lectures)!

A similar discussion can be found in Laplace's famous work *Exposition du Système du Monde* (1796). It played also a role in Soldner's discussion (1801) about light deflection in the neighbourhood of stars and his speculation about a very massive dark object in the centre of our Galaxy.

1801 was also the year where Young discovered the interference properties of light. This gave rise to the advent of the wave theory of light, in which the presence of dark stars did not seem to make much sense. For this reason all reference to them was omitted in the 1808 edition of Laplace's *Exposition*.

The modern theory of black holes started with the advent of general relativity in 1915. Soon after the field equations had been published, Karl Schwarzschild discovered an exact solution to these equations, which describes the gravitational field outside a spherically-symmetric mass distribution. This solution is also of the utmost importance for black holes, and it is described below.

The first exact solution that describes the collapse of a body – a pressureless dust cloud – was obtained by Oppenheimer and Snyder in 1939. In this case the dust cloud crunches to a spacetime singularity at $r = 0$ and describes what is now known as a black hole, an object so dense that not even light can escape. However such objects were not taken seriously by most scientists, including Einstein, for a long time. This was in part due to the ill-understood nature of a singularity, that occurred at $r = R_0$, see Thorne (1994) for a fascinating account of this story. For this reason it was also not really accepted that black holes could result as the final stage in the life of a star.

This changed in the sixties, and one could call the period from 1960 to 1975 the *classical period* of black hole research. The nature of the singularity at $r = R_0$ was clarified through the introduction of Kruskal coordinates (see below). The singularity theorems showed that the singularity at $r = 0$ found by Oppenheimer and Snyder is not an artifact of the spherical symmetry assumed in their model, but occurs generally under well-defined conditions, see Hawking and Penrose (1996). The uniqueness theorems ('black holes have no hair') exhibit, very surprisingly, that stationary black holes are fully characterised by a small set of parameters (see below).

On the observational side, neutron stars were discovered in 1967. Since neutron stars are possible final states of a star collapse – the others being white dwarfs and black holes –, also the most exotic option, the black hole, was then taken more seriously. (1967 was also the year where the name *black hole* was coined by John Wheeler.) In retrospect, the year 1963 is also very important through the discovery of quasars, although this had not been recognised at that time.

Today, there is a general consensus that black holes do exist in Nature and that they have been observed. The best stellar black-hole candidate at present is probably the X-ray nova V404 Cygni with a mass $M > 6\,M_\odot$, and the best supermassive black-hole candidate is the black hole that lurks in the centre of the Milky Way and has a mass of about $2.6 \times 10^6\,M_\odot$, see the lectures by Treves.

The year 1974 saw the fascinating and surprising – theoretical – discovery that black holes are not really black when quantum theory is taken into account, but radiate with a thermal spectrum like an ordinary 'black body'. This so-called Hawking radiation is the cornerstone of all modern theoretical developments and will play a central role in my review. The year 1974 thus opened the *quantum period* of black hole research, a period where no end is yet in sight. The main open questions are the full quantum picture of black-hole evaporation and the interpretation of black-hole entropy. Candidates for a quantum theory of gravity, such as canonical quantum gravity or superstring theory, attempt to find a solution for these problems.

In the following I shall give a brief introduction into the Schwarzschild metric describing the gravitational field outside a spherically symmetric mass distribution.[1] This topic is covered in many excellent textbooks including Misner, Thorne, and Wheeler (1973), Sexl and Urbantke (1983), Straumann (1984), Wald (1984), and – on a more mathematical level – Hawking and Ellis (1973). The Schwarzschild metric is the unique spherically-symmetric solution to the vacuum Einstein equations

$$R_{ab} = 0. \tag{1.2}$$

In the standard coordinates, its line element reads

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{1.3}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the line element on the unit two-sphere. It is a direct consequence of spherical symmetry that this metric is also *static* (Birkhoff's theorem). The constant $2GM$ is determined by comparison with the Newtonian limit (this is why, in spite of the vacuum equations (1.2), G comes into play). One easily recognises the singularities in the metric (1.3) at $r = 0$ and $r = 2GM = R_0$. While the singularity at $r = 0$ is a real one (divergence of curvature invariants), the singularity at $r = R_0$ is a coordinate singularity, see below.

It is of interest to consider also the *interior* solution of a spherically-symmetric body whose exterior solution is given by (1.3). Making the ansatz of a static spherically-symmetric metric,

$$\mathrm{d}s^2 = \mathrm{e}^{\nu(r)}\,\mathrm{d}t^2 - \mathrm{e}^{\lambda(r)}\,\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{1.4}$$

and taking for matter an ideal fluid whose energy-momentum tensor reads

$$T_{ab} = (\rho + p)u_a u_b - g_{ab}p\,, \tag{1.5}$$

where ρ and p are respectively density and pressure, one arrives at the

[1] From now on $c = 1$.

TOV-solution

$$\begin{aligned} e^{\lambda(r)} &= \left(1 - \frac{2GM(r)}{r}\right)^{-1}, \\ \nu(r) &= -\lambda(r) + 8\pi G \int_\infty^r \mathrm{d}r'\, r'(p+\rho)e^{\lambda(r')}, \\ M(r) &\equiv 4\pi \int_0^r \mathrm{d}r'\, r'^2 \rho(r'). \end{aligned}$$

It is evident that outside the body this is equal to the Schwarzschild solution.

From the covariant conservation equation $T_{ab;}{}^{b} = 0$ one finds

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\left(M(r) + 4\pi r^3 p\right)}{r^2}\left(1 - \frac{2GM(r)}{r}\right)^{-1}(\rho + p)\,. \tag{1.6}$$

This is the general relativistic extension of the well-known hydrodynamical equilibrium equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho \tag{1.7}$$

found in Newtonian theory. The modifications due to general relativity are: In addition to $M(r)$ one has a term proportional to p, since also pressure generates gravity; one has $\rho + p$ instead of ρ, since gravity also acts on pressure; and one has $r^{-2} \longrightarrow r^{-2}(1 - 2GM(r)/r)^{-1}$, meaning that gravity increases faster than r^{-2}. As a consequence of this, there is an upper limit to the mass of a neutron star.

For the very important special case $\rho =$ constant, one finds from (1.6) that the pressure in the centre of the star diverges if its radius $R \leq \frac{9}{8}R_0$. This is a direct consequence of the general-relativistic feature of nonlinearity that pressure generates more pressure. This lower bound on the radius leads to an upper bound of the mass, $M \leq M_{\max} = 4[\,9(3G^3\pi\rho)^{1/2}\,]^{-1}$. One can show that the existence of such limits remains for $\rho \neq$ constant (Wald 1984). The above special case of constant density was already discussed by Schwarzschild in 1916; he was pleased by this result, since it suggested to him that the singularity at $R = R_0$ is of no relevance. However, this limit only shows that a static situation can no longer occur for $R \leq \frac{9}{8}R_0$; it does not say anything about a dynamical situation, see below.

Can one give an illustration for the geometry of the Schwarzschild metric (1.3)? Taking $t =$ constant (staticity) and $\theta =$ constant $= \pi/2$ (spherical symmetry), one obtains the spatial line element

$$\mathrm{d}\sigma^2 = \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\,. \tag{1.8}$$

This can be embedded in an auxiliary three-dimensional euclidean space with line element $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}z^2 + r^2\mathrm{d}\phi^2$ by looking for a rotation surface $z(r)$ that reproduces (1.8). One immediately finds the rotational paraboloid

$$z(r) = \pm\sqrt{8GM}\sqrt{r - 2GM}\,. \tag{1.9}$$

The corresponding embedding diagram is shown in figure 1.1 (schematically). One recognises that the region $r < R_0$ is not covered and that there

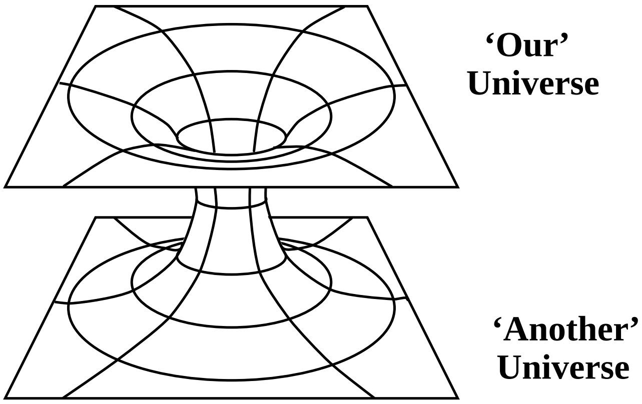

Figure 1.1. Einstein-Rosen bridge

exist two asymptotically flat regions, one corresponding to 'Our' Universe. Note also that each point on the surface represents in fact a two-sphere. The geometry in figure 1.1 is often called an *Einstein-Rosen bridge*.

The geometry of the TOV-solution with ρ = constant is, for t = constant, given by

$$\mathrm{d}\sigma^2 = \left(1 - \frac{8\pi G r^2 \rho}{3}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{1.10}$$

and describes a space with positive curvature, i.e. a three-sphere with radius $\mathcal{R} = [3/(8\pi G\rho)]^{1/2}$. Taking for the interior part of a star this solution and for the exterior part the Schwarzschild solution, the corresponding embedding diagram looks like figure 1.2. Since the Einstein field equations only determine the local spacetime geometry, the global *topology* is not fixed by them. Instead of figure 1.1, one can identify the two asymptotic regions and arrive at a so-called *wormhole*, as depicted in figure 1.3.

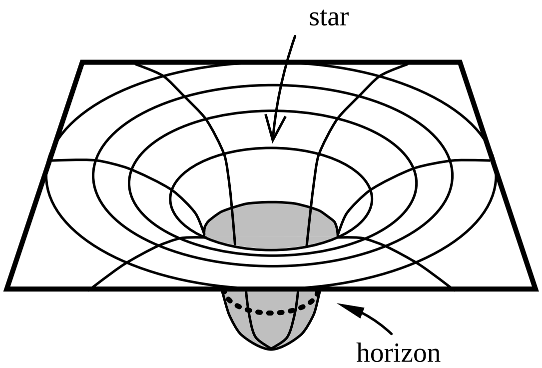

Figure 1.2. Geometry in the vicinity of a star

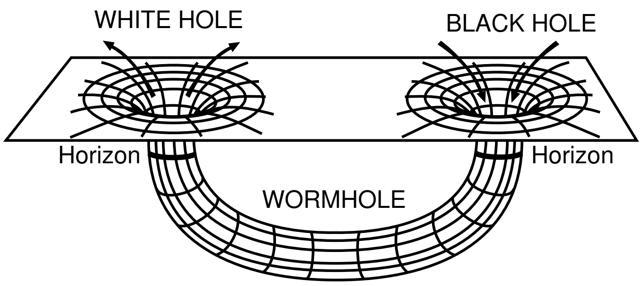

Figure 1.3. Wormhole

To classify the nature of the singularity at $r = R_0$, one looks for a coordinate system that is as nonsingular as possible. This proceeds in two parts:

First, the line element (1.3) is written in the form

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\left(\mathrm{d}t^2 - \mathrm{d}r_*^2\right) - r^2\mathrm{d}\Omega^2 \qquad (1.11)$$

through the introduction of the *tortoise coordinate*

$$r_* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|. \qquad (1.12)$$

This has the advantage that radial light rays propagate as in flat space since $\mathrm{d}t = \pm\,\mathrm{d}r_*$.

Second, one looks for a coordinate transformation $(t, r_*) \longmapsto (T, X)$ that preserves this properties, but avoids coordinate singularities. This is achieved by the *Kruskal coordinates*

$$\begin{aligned} X &= \exp\left(\frac{r_*}{4GM}\right)\cosh\frac{t}{4GM} \\ &= \sqrt{\frac{r}{2GM}-1}\,\exp\left(\frac{r}{4GM}\right)\cosh\frac{t}{4GM}\,, \end{aligned} \tag{1.13a}$$

$$\begin{aligned} T &= \exp\left(\frac{r_*}{4GM}\right)\sinh\frac{t}{4GM} \\ &= \sqrt{\frac{r}{2GM}-1}\,\exp\left(\frac{r}{4GM}\right)\sinh\frac{t}{4GM}\,. \end{aligned} \tag{1.13b}$$

The inverse transformation can be given only implicitly,

$$X^2 - T^2 = \left(\frac{r}{2GM}-1\right)\exp\left(\frac{r}{2GM}\right)\,, \tag{1.14a}$$

$$\frac{T}{X} = \tanh\frac{t}{4GM}\,. \tag{1.14b}$$

The line element (1.11) then reads

$$\mathrm{d}s^2 = \frac{32(GM)^3}{r}\exp\left(-\frac{r}{2GM}\right)\left(\mathrm{d}T^2 - \mathrm{d}X^2\right) - r^2(T,X)\mathrm{d}\Omega^2\,. \tag{1.15}$$

Note that there is no longer any coordinate singularity at $r = R_0 = 2GM$; only the curvature singularity at $r = 0$ remains. The coordinate transformation (1.13) can thus be extended in a straightforward manner to $r < R_0$, and one arrives at the Kruskal diagram figure 1.4. This is the maximal analytic extension of the Schwarzschild manifold, meaning that every geodesic can be extended either to the value ∞ of its affine parameter *or* encounters a singularity.

One recognises from figure 1.4 that the singularity at $r = 0$ is, in fact, spacelike and therefore distinguishes a certain time, not a space point. An observer present in *II* cannot 'see' the singularity. Since radial light rays propagate on straight lines inclined by $\pm 45°$ to the axes, the *causal properties* of the Kruskal manifold are evident.

Note in particular the presence of *event horizons* that separate the various regions from each other: By no means whatsoever can any signal emitted in II reach the outside regions I or III. The opposite is true for region IV: no signal emitted from I or III can enter it. The region II never becomes part of the past of an outside observer, as old as he might become.

The Einstein-Rosen bridge shown in figure 1.1 is obtained from figure 1.4 as the cross section $t =$ constant through the origin. The two asymptotically flat regions of figure 1.1 are thus the regions I and III in figure

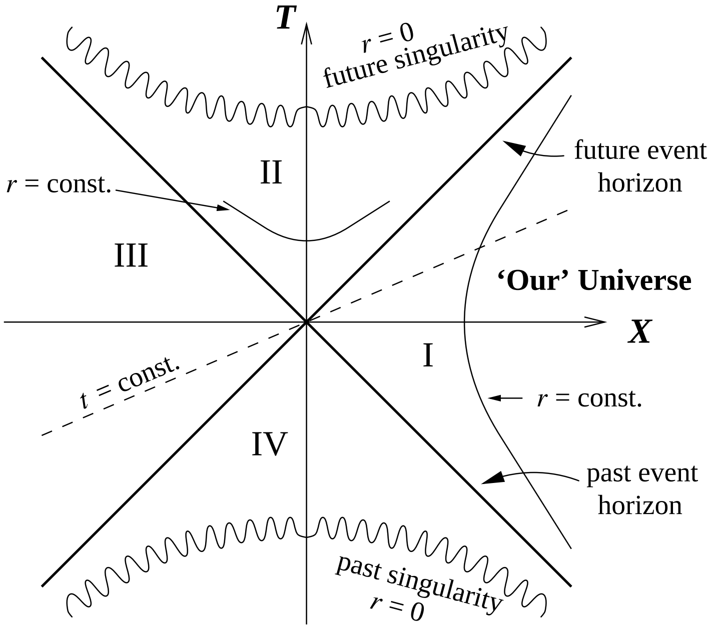

Figure 1.4. Kruskal diagram

1.4. If one investigated instead cross sections T = constant in figure 1.4, one would obtain changing embedding diagrams suggesting the picture of a dynamical collapse; this demonstrates that the Schwarzschild metric (1.3) is static only in its exterior part, not in its interior part $r < R_0$. The origin in figure 1.4 is often called the *bifurcation two-sphere.*

Instead of the Kruskal diagram (figure 1.4) it is often more convenient to consider a diagram of a conformally related spacetime in which the regions of infinity are mapped to a finite boundary. This so-called *Penrose diagram* is depicted in figure 1.5. It will also play a crucial role in the discussion of Hawking radiation in section 1.3. I want to note that with some topological identifications one can arrive from the Kruskal manifold shown in figures 1.4 and 1.5 at a manifold with only *one* exterior solution. This so-called *geon* is discussed at length by Louko and Marolf (1998), see also the brief discussion in section 1.5 below (figure 1.19).

It is straightforward to study the equations of motion for light rays and observers in the Kruskal spacetime (1.15). The solutions of the corresponding geodesic equation is found in the textbooks cited above. I only want to recall that a particle moving from region I to II reaches the singu-

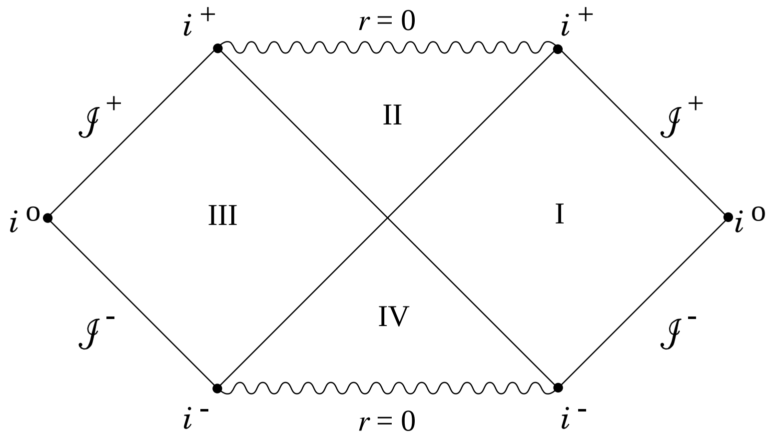

Figure 1.5. Penrose diagram of the Kruskal spacetime

larity at a finite proper time – in spite of the fact that the coordinate time $t \to \infty$ for $r \to R_0$. For example, the free-fall time s from the horizon to the singularity is given by

$$s = \pi GM \approx 1.54 \times 10^{-5} \frac{M}{M_\odot} \sec , \tag{1.16}$$

and the proper time for nongeodesic motion is even shorter. The time (1.16) roughly corresponds to the time it would take for light to propagate a distance of R_0 in flat space. We shall encounter this characteristic time at several occasions again.

I want to emphasise that the situation depicted in figures 1.4 and 1.5 is still time symmetric. The region IV (the 'white hole region') is the time reverse of region II (the 'black hole region'). This time-symmetric situation is sometimes called an *eternal hole*. A genuine *black hole* is obtained from a time *asymmetric* situation such as a star collapsing to a singularity (this is the case, for example, in the Oppenheimer-Snyder model). The time reverse of such a situation – a star expanding out of a singularity - is called a *white hole*. Both situations are shown in figure 1.6. It is evident that only two of the four regions of figure 1.4 are present. In particular, the second asymptotically flat region III is absent.

Coordinates that are often used to study a spherically-symmetric collapse situation are the ('in-going' version of the) *Eddington-Finkelstein coordinates*. Instead of the original Schwarzschild coordinates (t, r) one uses $(\tilde{v}, r)$ with $\tilde{v} = t + r_*$ and r_* according to (1.12). The line element (1.3)

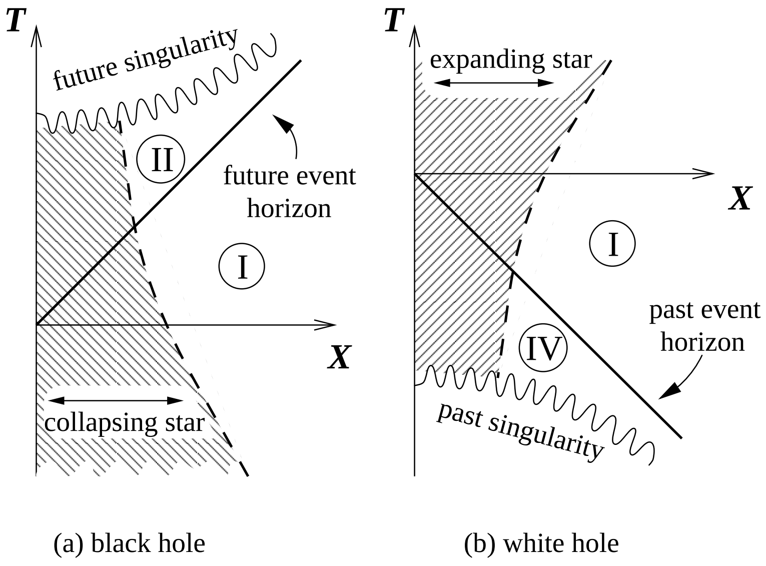

Figure 1.6.

then reads

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tilde{v}^2 - 2 d\tilde{v} dr - r^2 d\Omega^2 \,. \tag{1.17}$$

The spacetime diagram showing the collapse of a star to form a black hole is shown, using these coordinates, in figure 1.7. This diagram demonstrates in particular that light rays emitted from the surface of the collapsing star are received by the distant observer at later and later times. For the same reason, the emitted light becomes increasingly redshifted according to (for radial rays)

$$\Delta z \equiv \frac{\Delta\lambda}{\lambda} = \exp\left(\frac{t}{4GM}\right) \,. \tag{1.18}$$

For the total decrease of luminosity L one must also take into account non-radial rays, with the result

$$L(t) \approx L_0 \exp\left(-\frac{t}{3\sqrt{3}GM}\right) \,. \tag{1.19}$$

The characteristic time scale τ in this formula is

$$\tau \approx 3\sqrt{3}GM \approx 2.5 \times 10^{-5} \frac{M}{M_\odot} \text{ sec} \tag{1.20}$$

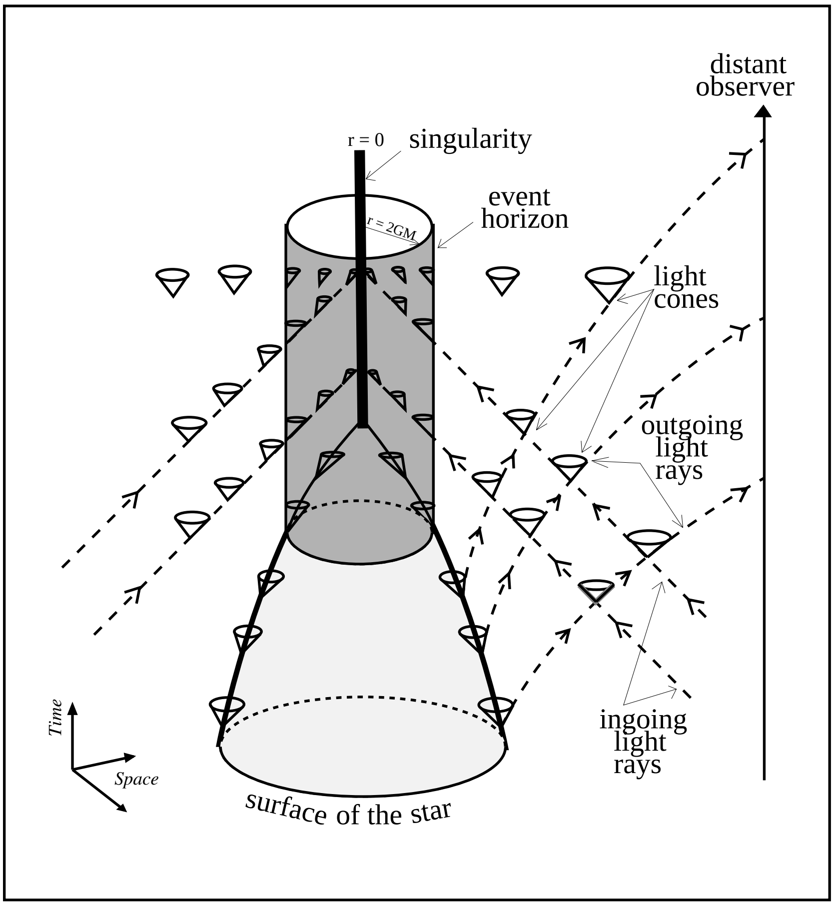

Figure 1.7. Star collapsing to form a black hole

and of the same order of magnitude as (1.16).

That a singularity occurs under a much wider range of conditions than under a spherically symmetric situation is demonstrated by the *singularity theorems* (Hawking and Penrose 1996). One needs a certain energy condition, a condition for the global structure (for example, that there are no closed timelike curves), and the existence of so-called *trapped surfaces*. A trapped surface is a surface where not only in-going light rays converge, but also *out*-going light rays. Such trapped surfaces are for example present in the shaded interior region of figure 1.7. Under these conditions, a singularity necessarily occurs. A spacetime is called *singular* if it is timelike or null geodesically incomplete and cannot be embedded in a bigger spacetime.

A characteristic feature of the collapse situation of figure 1.7 is the occurrence of an *event horizon* that prevents the singularity from being seen from outside. It is unknown whether in a realistic collapse an event horizon always forms; this is the content of the *cosmic censorship hypothesis*: Nature abhors naked singularities.

I emphasise that the cosmic censorship hypothesis does *not* exclude the occurrence of spacelike past singularities such as white holes. An important example of a spacelike past singularity is the Big Bang. One can always put a Cauchy surface at a position later than such a past singularity and predict all future evolution from initial data on this Cauchy hypersurface. This is not possible for a timelike naked singularity. The apparent non-occurrence of white holes is related to the Second Law of thermodynamics (Zeh 1992) and will be briefly discussed in section 1.5. Their presence is often excluded by the *Weyl tensor hypothesis* (Hawking and Penrose 1996).

Can one say anything about the nature of the singularity that occurs in a generic collapse? The general expectation is that upon approaching the singularity so-called BKL-oscillations occur – chaotic oscillations in tidal curvature that occur in random directions (Belinsky et al. 1982). The nature of the singularity itself can only be clarified in a quantum theory of gravity.

A most remarkable development in the mathematical study of black holes has been the proof of various *uniqueness theorems* for black holes (Heusler 1996, 1998). In the Einstein-Maxwell theory, stationary black holes (the asymptotic final stage after the collapse) are uniquely characterised by *three* parameters: mass M, angular momentum J and electric charge q. All other degrees of freedom ('multipoles') are radiated away during the collapse ('Black holes have no hair.') In the presence of other fields (e.g., non-abelian gauge fields) this theorem no longer necessarily holds, but the corresponding black-hole solutions are usually unstable.

I want to conclude this section with a brief description of the stationary black-hole solutions that have charge and/or angular momentum.

The spherically-symmetric black-hole solution with electric charge q is the *Reissner-Nordström* solution:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r} + \frac{Gq^2}{r^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r} + \frac{Gq^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \,. \tag{1.21}$$

It can be generated from the Schwarzschild metric (1.3) through the substitution

$$M \longrightarrow M - \frac{q^2}{2r} \,.$$

The metric (1.21) is a solution to the non-vacuum Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \tag{1.22}$$

with the energy-momentum tensor corresponding to a point charge, $4\pi T_{ab} = q^2/(2r^4)\,\mathrm{diag}(1,-1,-1,-1)$. In geometrical units, the unit of charge is given by $e\sqrt{G} \approx 1.38 \times 10^{-34}\,\mathrm{cm}$.

For charge $|q| < \sqrt{G}M$, the metric (1.21) has coordinate singularities at

$$r_\pm = GM \pm \sqrt{(GM)^2 - Gq^2}\,. \tag{1.23}$$

The variable r_+ denotes the coordinate radius of the event horizon ($r_+ \to R_0$ for $q \to 0$); r_- characterises the *Cauchy horizon* – the evolution of fields beyond the Cauchy horizon cannot be predicted from initial data on a spacelike hypersurface. All these features can be immediately recognised from the Penrose diagram of this solution, see figure 1.8

The interesting feature is that the singularity at $r = 0$ is now timelike – and therefore is a naked singularity for some regions, e.g. regions III – and that there is an infinite repetition of this structure. It should be mentioned, however, that this structure is unstable with respect to small perturbations and that BKL-oscillations are expected to occur; they would produce a spacelike singularity. Recent years have seen a tremendous progress in understanding the internal structure of black holes (Israel 1998).

The special case $|q| = \sqrt{G}M$ is referred to as describing an *extremal* black hole; this case plays a crucial role in the quantum gravity sections 1.5 and 1.6. The two horizons now coincide, and the corresponding Penrose diagram is shown in figure 1.9. For $|q| > \sqrt{G}M$, there is no event horizon present: Instead of a black hole, there is now a naked singularity. The Penrose diagram is shown in figure 1.10. From an astrophysical point of view, charged black holes are not expected to play any role since they would rapidly attract opposite charges and discharge. They play, however, a useful role as a toy model for the realistic case of rotating black holes to which I now turn.

The solution for a rotating stationary black hole, the so-called *Kerr* solution, is no longer spherically symmetric and static: It is axisymmetric and only stationary. This is characterised by the presence ot the two Killing vectors $\xi^a = (\frac{\partial}{\partial t})^a$ and $\psi^a = (\frac{\partial}{\partial \phi})^a$. Apart from its mass M, the solution is characterised by its angular momentum J. It is sometimes more convenient to use the parameter $a = J/(GM)$ that has unit of length. In particular coordinates, so-called Boyer-Lindquist coordinates, the Kerr line element

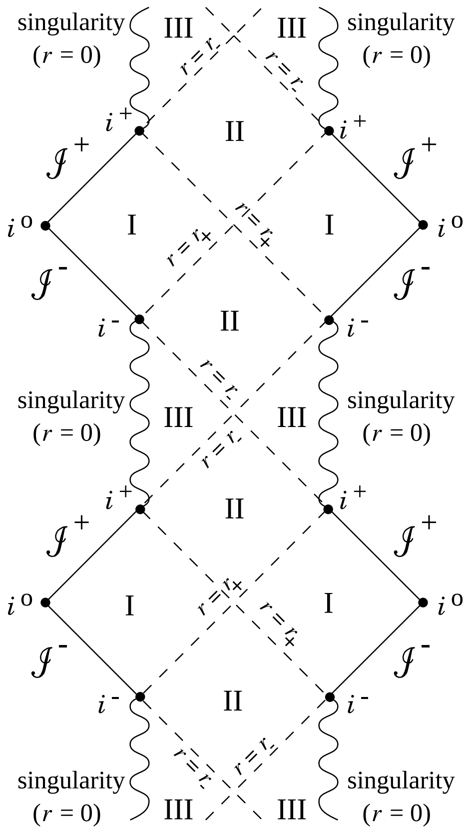

Figure 1.8. Penrose diagram for the Reissner-Nordström solution ($|q| < \sqrt{G}M$)

reads

$$\begin{aligned} ds^2 = & \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - a dt\right]^2 \\ & - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \end{aligned} \tag{1.24}$$

with

$$\begin{aligned} \rho^2 &\equiv r^2 + a^2\cos^2\theta\,, \\ \Delta &\equiv r^2 - 2GMr + a^2\,. \end{aligned} \tag{1.25}$$

The most general solution for a stationary black hole is the Kerr-Newman solution: It possesses in addition an electric charge q and can be

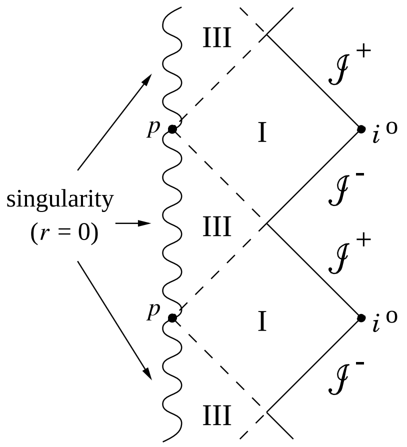

Figure 1.9. Penrose diagram for the extremal Reissner-Nordström solution ($|q| = \sqrt{G}M$). The points p don't belong to the singularity

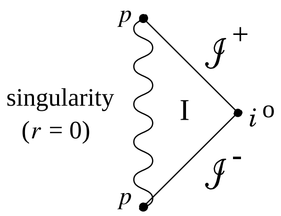

Figure 1.10. Penrose diagram for the Reissner-Nordström solution corresponding to a naked singularity ($|q| > \sqrt{G}M$)

obtained from (1.24) through the substitution

$$M \longrightarrow M - \frac{q^2}{2r}\,.$$

The Penrose diagrams for the Kerr solution show similar features as in figures 1.8, 1.9 and 1.10. (The singularity at $\rho^2 = 0$ is now a ring singularity through which one can 'escape' to a strange anti-gravity universe.) For $|a| < GM$ one has coordinate singularities at $r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$, r_+ referring to the event horizon, and r_- referring to the Cauchy horizon. Again, $|a| = GM$ is the extremal case and $|a| > GM$ describes a naked singularity.

In the next section I shall discuss in which sense these stationary black-hole solutions obey formally the laws of thermodynamics.

1.2 The laws of black-hole mechanics

In this section I shall discuss the laws of black-hole mechanics, laws obeyed by the stationary black-hole solutions, that formally mimic the laws of phenomenological thermodynamics. A general reference is Wald (1994), a very readable short introduction is Bekenstein (1980).

For this purpose it is illustrative to follow the historical route and start with some special properties of rotating black holes. The Kerr solution (1.24) exhibits the feature that the Killing field $\xi^a = (\frac{\partial}{\partial t})^a$ becomes spacelike ($\xi^a \xi_a < 0$) in a certain region *outside* the black hole. (For the spherically-symmetric black holes, this happens only inside the event horizon.) This region is the so-called *ergosphere*, characterised by

$$r_+ < r < GM + \sqrt{(GM)^2 - a^2 \cos^2 \theta}\,. \tag{1.26}$$

The ergosphere is shown (for $a \leq GM$) in figure 1.11.

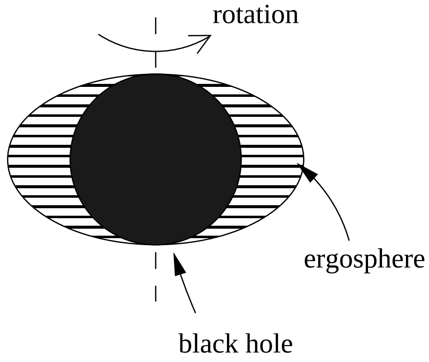

Figure 1.11. Ergosphere of a rotating black hole

Since the Killing field ξ^a generates time translations at asymptotic infinity, its spacelike nature in the ergosphere means that an observer there would have to travel with more than the speed of light to follow an orbit of ξ^a – he thus cannot remain static and he is forced to rotate with the hole. This is the extreme version of the Lense-Thirring effect that is very weak in the vicinity of the Earth and that only recently has been observed. Quite generally, black holes are characterised by general relativistic effects that under ordinary circumstances are only minor corrections.

For $r \to r_+$, the coordinate angular velocity of this rotation becomes

$$\Omega_{\mathrm{H}} = \frac{a}{r_+^2 + a^2} = \frac{a}{2GMr_+}\,. \tag{1.27}$$

This can be interpreted as the angular velocity of the hole itself. The Killing field

$$\chi^a \equiv \xi^a + \Omega_{\rm H}\psi^a \tag{1.28}$$

is tangential to the horizon.

Since $\chi^a\chi_a(r_+) = 0$, χ^a is null (and future-directed) on the horizon; since the horizon is itself a null surface, χ^a is at the same time tangential and normal to the horizon – the horizon thus also constitutes a so-called *Killing horizon*.

We note that the area of a closed spacelike section through the event horizon is

$$A = \int\limits_{r=r_+} \sqrt{g_{\theta\theta}g_{\phi\phi}}\, {\rm d}\theta {\rm d}\phi = 4\pi(r_+^2 + a^2) \equiv 16\pi(GM_{\rm irr})^2\,, \tag{1.29}$$

where the *irreducible mass* $M_{\rm irr}$ has been introduced. Its significance will be explained below.

As Penrose noted in 1969, the presence of the ergosphere leads to the possibility that energy can be extracted from the black hole. Because ξ^a is spacelike in the ergosphere, the energy of a test particle, defined by $E = p^a\xi_a$, need not be positive. Suppose a projectile with energy E_0 is sent into the ergosphere. Let the projectile disintegrate while being in the ergosphere, with one part (with energy E_1) falling down the hole and the other part (with energy E_2) escaping to infinity. If $E_1 < 0$, the recovered fragment has an energy *bigger* than E_0 ($E_0 = E_1 + E_2$). This energy must, of course, have been extracted from the mass of the black hole; $M \to M - |E_1|$. One also finds for the angular momentum L of the lost fragment that $L < E_1/\Omega_{\rm H}$. So a negative-energy particle has negative angular momentum.

How much energy can be extracted in this way from the black hole? It is clear that the black hole parameters should obey $\delta J < \delta M/\Omega_{\rm H}$ after the above fragment has been swallowed. Christodoulou was able in 1970 to rewrite this condition as a condition on the irreversible mass introduced in (1.29),

$$\delta M_{\rm irr}^2 = \frac{Mr_+}{\sqrt{(GM)^2 - a^2}}\,(\delta M - \Omega_{\rm H}\delta J) > 0\,. \tag{1.30}$$

Since the irreducible mass of the black hole can thus not be lowered by the Penrose process, the mass itself cannot be reduced below $M_{\rm irr}(M_0, J_0)$, where M_0, J_0 denote the initial mass and angular momentum. Since $M_{\rm irr}(M_0, J_0) \geq M_0/\sqrt{2}$, at most 29 % of the mass can be extracted by the Penrose process. The parameter space allowed by the Penrose process is shown in figure 1.12.

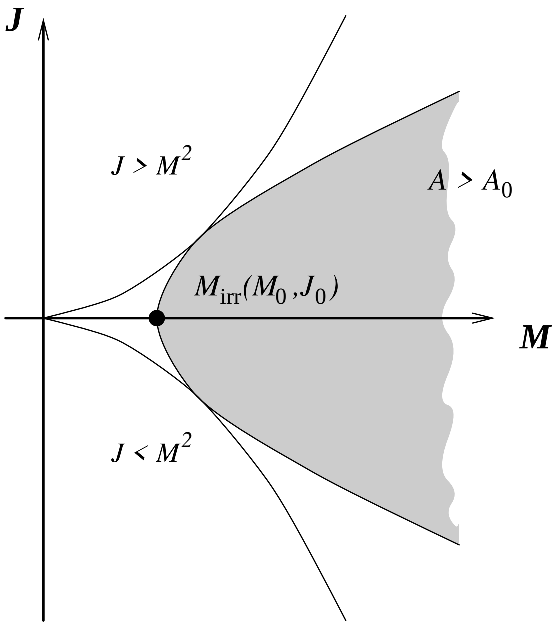

Figure 1.12. The region $A > A_0$ reachable via the Penrose process

I should note that there exists also a wave analogue of the Penrose process, the so-called *superradiance*. Consider an incident wave of the form $\phi_0(r, \theta)\exp(-\mathrm{i}\omega t)\exp(\mathrm{i}m\phi)$: For $0 < \omega < m\Omega_{\mathrm{H}}$, the reflected wave has an energy greater than the incident wave, as has been noted by Starobinsky and Zel'dovich in the seventies.

The Penrose process could become of utmost importance in astrophysics: rotational energy of a black hole immersed in a magnetic field can be transferred to an escaping jet ('Blandford-Znajek process') as observed in active galaxies.

From M_{irr} one also finds, see (1.29),

$$\delta A \geq 0\,, \tag{1.31}$$

i.e. the area of the horizon cannot decrease.

Surprisingly, this *area law* holds much more generally than just for the Penrose process. This generalisation is the content of Hawking's famous theorem that he proved in 1971:

For a predictable black hole satisfying $R_{ab}k^a k^b \geq 0$ for all null k^a, the surface area of the *future* event horizon *never* decreases with time.

A 'predictable' black hole is one for which the cosmic censorship hypothesis holds (see Wald 1984) – this is thus a major assumption for the area law. I emphasise that the time asymmetry comes into play because a statement is made about the future horizon, not the past horizon; the analogous statement for white holes would then be that the past event horizon never increases. I also emphasise that the area law only holds in the classical theory, not in the quantum theory (see section 1.3).

The area law seems to exhibit a close formal analogy to the Second Law of thermodynamics – there the *entropy* can *never* decrease with time (for a closed system). However, the conceptual difference could not be more drastic: while the Second Law is related to statistical behaviour, the area law is just a theorem in differential geometry. That the area law is in fact directly related to the Second Law will become clear in the course of this section.

Further support for this analogy is given by the existence of analogies to the other laws of thermodynamics. The Zeroth Law states that there is a quantity, the temperature, that is constant on a body in thermal equilibrium. Does there exist an analogous quantity for a black hole?

We introduced in (1.28) the Killing field χ^a that is null and future-directed on the horizon. Since then in particular $\chi^a\chi_a$ is constant (zero) on the horizon, its gradient is normal to the horizon and therefore parallel to χ^a. One has

$$\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a\,, \tag{1.32}$$

where κ is the so-called *surface gravity* that plays an important role in both classical and quantum black-hole physics. For a Kerr black hole, κ is given by

$$\kappa = \frac{\sqrt{(GM)^2 - a^2}}{2GMr_+} \quad \stackrel{a\to 0}{\longrightarrow} \quad \frac{1}{4GM} = \frac{GM}{R_0^2}\,, \tag{1.33}$$

and one recognises in the Schwarzschild limit the well-known expression for the Newtonian gravitational acceleration. Note that κ also sets the time scale for the redshift during collapse, see (1.18). One can show that κ is the limiting force that must be exerted at infinity to hold a unit test mass in place when approaching the horizon. This justifies the name surface gravity.

It is now possible to prove (see Wald 1984) that κ is in fact *constant* on the event horizon. This is the desired formal analogy to the constancy of temperature on a body in thermal equilibrium.

The surface gravity also enters the relation between two important parameters: The horizon is generated by null geodesics that are conveniently parametrised by their affine parameter λ. On the other hand, there exists the so-called Killing parameter, v, that parametrises the orbits of the Killing field (1.28). Both parameters are related by

$$\lambda \propto \mathrm{e}^{\kappa v}\,, \tag{1.34}$$

a relation that will be of crucial relevance in the discussion of the Hawking effect in section 1.3.

With a tentative formal relation between surface gravity and temperature, and between area and entropy, the question arises whether a First Law of thermodynamics can be proved. This can in fact be done and the result for a Kerr-Newman black hole is

$$\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \Omega_{\mathrm{H}}\mathrm{d}J + \Phi\mathrm{d}q\,, \tag{1.35}$$

where Φ denotes the electrostatic potential. As Wald (1994) emphasises, this relation can be obtained by conceptually different methods: A *physical process version* whereby a stationary black hole is altered by infinitesimal physical processes, and an *equilibrium state version* whereby the areas of two infinitesimally nearby stationary black-hole solutions to Einstein's equations are compared. Both methods lead to the same result (1.35).

Since M is the energy of the black hole, (1.35) is the analogue to the First Law of thermodynamics given by

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N\,. \tag{1.36}$$

'Modern' derivations of (1.35) make use of both Hamiltonian and Lagrangian methods of general relativity (Wald 1998). For example, a First Law follows from an arbitrary diffeomorphism invariant theory of gravity whose field equations can be derived from a Lagrangian.

What about the Third Law of thermodynamics? A 'physical process version' was proved by Israel – it is impossible to reach $\kappa = 0$ in a finite number of steps. This corresponds to the 'Nernst version' of the Third Law. Whether the stronger 'Planck version' holds, is a matter of dispute and will be discussed at some length in section 1.5. The 'Planck version' states that the entropy goes to zero (or a material-dependent constant) if the temperature approaches zero. The above analogies are summarised in table 1.1.

The identification of the horizon area with an entropy can be obtained from a conceptually different point of view. If a box with, say, thermal radiation of entropy S is thrown into the black hole, it seems as if the Second Law could be violated, since the black hole is characterised only by mass, angular momentum, and charge, and nothing else. The rescue of the Second Law immediately leads to the concept of a black-hole entropy, as will be discussed now (Bekenstein 1980; Sexl and Urbantke 1983).

Law	Thermodynamics	Stationary Black Holes
Zeroth	T constant on a body in thermal equilibrium	κ constant on the horizon of a black hole
First	$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$	$\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \Omega_{\mathrm{H}}\mathrm{d}J + \Phi\mathrm{d}q$
Second	$\mathrm{d}S \geq 0$	$\mathrm{d}A \geq 0$
Third	$T = 0$ cannot be reached	$\kappa = 0$ cannot be reached

Table 1.1.

Consider a box with thermal radiation of mass m and temperature T lowered from a spaceship far away from a spherically-symmetric black hole towards the hole (figure 1.13). As an idealisation, both the rope and the walls are assumed to have negligible mass. At a coordinate distance r from the black hole, the energy of the box is given by

$$E_r = m\sqrt{1 - \frac{2GM}{r}} \quad \stackrel{r \to R_0}{\longrightarrow} \quad 0\,. \tag{1.37}$$

If the box is lowered down to the horizon, the energy gain is thus given by m. The box is then opened and thermal radiation of mass δm escapes into the hole. It the box is then closed and heightened again to the spaceship, the energy loss is $m - \delta m$. In total the energy δm of the thermal radiation can be transformed into work with a degree of efficiency $\eta = 1$. This looks as if one possessed a perpetuum mobile of the second kind.

The key to the resolution of this apparent paradoxe lies in the observation that the box must be big enough to contain the wavelength of the enclosed radiation. This, in turn, leads to a lower limit on the distance that the box can approach the horizon. Therefore, only part of δm can be transformed into work, as I shall show now.

According to Wien's law, one must have a linear extension of the box of at least

$$\lambda_{\mathrm{max}} \approx \frac{\hbar}{k_{\mathrm{B}}T}\,. \tag{1.38}$$

I emphasise that at this stage Planck's constant $\hbar$ comes into play. The box can then be lowered down to the coordinate distance δr (assumed to be $\ll 2GM$) from the black hole, where according to the Schwarzschild metric (1.3) the relation between δr and λ_{max} is

$$\lambda_{\mathrm{max}} \approx \int\limits_{2GM}^{2GM+\delta r} \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}} \mathrm{d}r \ \approx 2\sqrt{2GM\delta r} \quad \Longrightarrow \quad \delta r \approx \frac{\lambda_{\mathrm{max}}^2}{8GM}\,.$$

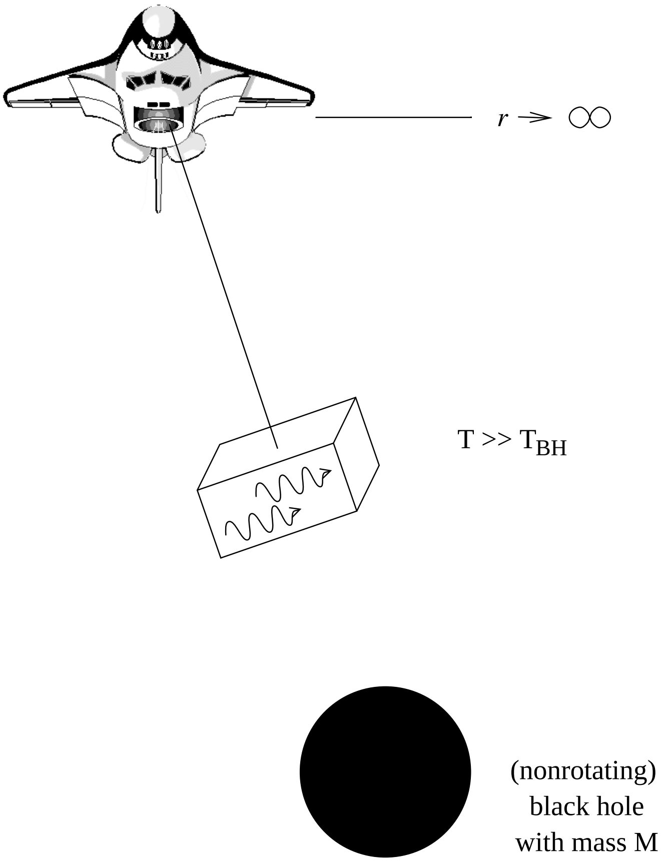

Figure 1.13. Gedankenexperiment to demonstrate the Second Law of thermodynamics for black holes

According to (1.37), the energy of the box at $r = 2GM + \delta r$ is

$$E_{2GM+\delta r} = m\sqrt{1 - \frac{2GM}{2GM + \delta r}} \approx \frac{m\lambda_{\max}}{4GM} \approx \frac{m\hbar}{4Gk_{\mathrm{B}}TM} \,.$$

Recalling that according to (1.35) the formal temperature of the black hole, T_{BH}, is proportional to the surface gravity $\kappa = 1/(4GM)$, the energy of the box before opening is

$$E^{(\mathrm{before})}_{2GM+\delta r} \approx m\frac{T_{\mathrm{BH}}}{T} \,,$$

while after opening it is

$$E^{(\mathrm{after})}_{2GM+\delta r} \approx (m - \delta m)\frac{T_{\mathrm{BH}}}{T} .$$

The degree of efficiency of transforming thermal radiation into work is thus given by

$$\eta \approx \left(\delta m - \delta m \frac{T_{\mathrm{BH}}}{T}\right) \Big/ \delta m = 1 - \frac{T_{\mathrm{BH}}}{T} < 1 ,$$

which is just the well-known Carnot limit for the efficiency of heat engines. From the First Law (1.35) one then finds for the entropy of the black hole $S_{\mathrm{BH}} \propto A = 16\pi(GM)^2$. It is this agreement of conceptually different approaches to black-hole thermodynamics that gives rise to the confidence into the results. In the next section I shall show how all these formal results can be physically interpreted in the context of quantum theory.

1.3 Hawking radiation

We have already seen in the gedankenexperiment discussed in the last section that $\hbar$ enters the scene, see (1.38). That Planck's constant has to play a role, can be seen also from the First Law (1.35). Since $T_{\rm BH}{\rm d}S_{\rm BH} = \kappa/(8\pi G)\,{\rm d}A$, one must have

$$T_{\rm BH} = \frac{\kappa}{G\zeta}\,, \quad S_{\rm BH} = \frac{\zeta A}{8\pi}$$

with an undetermined factor ζ. What is the dimension of ζ? Since $S_{\rm BH}$ has the dimension of Boltzmann's constant $k_{\rm B}$, $k_{\rm B}/\zeta$ must have the dimension of a length squared. There is, however, only one fundamental length available, the Planck length

$$l_{\rm p} = \sqrt{G\hbar} \quad \approx \quad 10^{-33}\,{\rm cm}. \tag{1.39}$$

(We do not yet consider string theory, see section 6).[2] Therefore,

$$T_{\rm BH} \propto \frac{\hbar\kappa}{k_{\rm B}}\,, \quad S_{\rm BH} \propto \frac{k_{\rm B}A}{G\hbar}\,. \tag{1.40}$$

The determination of the precise factors in (1.40) is the content of this section. To this purpose, it is necessary to briefly introduce the framework of quantum theory in curved spacetime. A standard reference for this is Birrell and Davies (1982), see also Wald (1994). An excellent brief introduction is Wipf (1998). Before entering this topic, I want to briefly recapitulate some basic notions of quantum field theory in flat Minkowski space. There, the Poincaré symmetry – together with Wightman axioms – select an invariant *vacuum state* and therefore a well-defined notion of *particles*. All inertial observers agree on these notions.

Consider for simplicity a free massive scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box + (m/\hbar)^2)\phi(x) = 0$. It can be decomposed into positive and negative frequencies with respect to the distinguished Killing time t according to

$$\begin{aligned}\phi(x) &= \frac{1}{(2\pi)^{3/2}}\int\frac{{\rm d}^3k}{\sqrt{2\omega_{\boldsymbol k}}}\left[a_{\boldsymbol k}\exp({\rm i}\boldsymbol{kx} - {\rm i}\omega_{\boldsymbol k}t) + a^\dagger_{\boldsymbol k}\exp(-{\rm i}\boldsymbol{kx} + {\rm i}\omega_{\boldsymbol k}t)\right]\\ &\equiv \phi^{(+)}(x) + \phi^{(-)}(x)\,,\end{aligned} \tag{1.41}$$

where $\omega_{\boldsymbol k} = \sqrt{\boldsymbol k^2 + (m/\hbar)^2}$. In quantum theory, $\phi(x)$ is a field operator that satisfies

$$[\phi(\boldsymbol x, t), \pi_\phi(\boldsymbol y, t)] = {\rm i}\hbar\,\delta^{(3)}(\boldsymbol x - \boldsymbol y)\,, \tag{1.42}$$

[2] It is amusing to note that Planck found this length before the 'invention' of $\hbar$, since $\hbar$ is implicitly contained in Wien's law, cf. (1.38).

where π_ϕ is the canonical momentum operator. From (1.41) and (1.42) one has

$$[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\,, \tag{1.43}$$

the well-known relations for the annihilation and creation operators of harmonic oscillators. From these relations the standard *Fock space* can be constructed; the vacuum state $|0\rangle$ is defined by

$$a_{\boldsymbol{k}}|0\rangle = 0, \quad \forall_{\boldsymbol{k}}\,, \tag{1.44}$$

and excited states are found through application of the creation operators $a^\dagger_{\boldsymbol{k}}$. In the presence of interactions, these concepts only hold in asymptotic 'free' regions. Since I shall be concerned with linear quantum fields only, it will not be necessary to elaborate on this.

I shall now consider quantum field theory on a globally hyperbolic spacetime with metric g_{ab} $(\det g_{ab} \equiv g)$. Restricting again attention to (minimally coupled) scalar fields, the curved-space version of the Klein-Gordon equation reads

$$\Box_g \phi + (m/\hbar)^2 \phi = 0 \tag{1.45a}$$

with

$$\Box_g = \frac{1}{\sqrt{-g}} \partial_a [\sqrt{-g} g^{ab} \partial_b]\,. \tag{1.45b}$$

Consider now two solutions, u_1 and u_2, of (1.45a,1.45b). Their conjugate momenta are $\pi_1 = n^a \nabla_a u_1$ and $\pi_2 = n^a \nabla_a u_2$, where n^a denotes the normal vector with respect to some spacelike hypersurface Σ. One can define the following inner product for such solutions:

$$(u_1, u_2) \equiv \mathrm{i} \int_\Sigma (u_1^* \pi_2 - \pi_1^* u_2) \mathrm{d}^3 x \; = (u_2, u_1)^*\,. \tag{1.46}$$

The Klein-Gordon equation guarantees that this inner product is independent of the choice of Σ ('independent of time'), but it is not positiv definite.

Choose now a complete set of solutions $\{u_k, u_k^*\}$ (k can stand here for an arbitrary index, not necessarily $\boldsymbol{k}$) normalised according to

$$(u_k, u_{k'}) = \delta(k, k') \quad \Rightarrow \quad (u_k^*, u_{k'}^*) = -\delta(k, k') \tag{1.47}$$

and

$$(u_k, u_{k'}^*) = 0\,. \tag{1.48}$$

The $\{u_k\}$ are the generalisation of the plane-wave solutions

$$u_{\boldsymbol{k}} = \frac{1}{(2\pi)^{3/2}} \frac{\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x} - \mathrm{i}\omega_{\boldsymbol{k}} t)}{\sqrt{2\omega_{\boldsymbol{k}}}}$$

used in (1.41) to curved spacetime. Since $\{u_k, u_k^*\}$ are a complete set of solutions, any field operator ϕ can – analogous to the flat-space case (1.41) – be expanded with respect to them,

$$\phi(x) = \int \mathrm{d}\mu(k) \left(a_k u_k + a_k^\dagger u_k^* \right) . \tag{1.49}$$

Here, $\mathrm{d}\mu(k)$ is an abbreviation for the used measure. From (1.49) follows $a_k = (u_k, \phi)$ and $a_k^\dagger = -(u_k^*, \phi)$. As in the flat case, a Fock space can be constructed from the vacuum $|0\rangle_u$, where

$$a_k |0\rangle_u = 0, \quad \underset{k}{\forall} . \tag{1.50}$$

The vacuum state is supposed to be normalised according to ${}_u\langle 0|0\rangle_u$, where $\langle\,|\,\rangle$ denotes the positive definite inner product in the constructed Hilbert space (not to be confused with the inner product (1.46)).

The crucial point is now that in a general spacetime – in contrast to inertial coordinates for flat space – there is no distinguished set of coordinates, in particular no distinguished time, with respect to which (1.49) can be uniquely defined. This is of course a consequence of the 'general covariance' of general relativity. The definition of the vacuum therefore depends on the chosen set of solutions – this fact has already been taken into account by adjoining the index 'u' to $|0\rangle$ in (1.50).

One can therefore expand the field into a *different* set of complete solutions $\{v_p, v_p^*\}$,

$$\phi(x) = \int \mathrm{d}\mu(p) \left(b_p v_p + b_p^\dagger v_p^* \right) . \tag{1.51}$$

One can also expand one basis with respect to the other,

$$v_p = \int \mathrm{d}\mu(k) \left(\alpha(p,k) u_k + \beta(p,k) u_k^* \right) , \tag{1.52}$$

where α and β are the so-called *Bogolubov coefficients*:

$$\alpha(p,k) = (u_k, v_p), \qquad \beta(p,k) = -(u_k^*, v_p) . \tag{1.53}$$

In an obvious matrix notation (suppressing the indices (p,k)), the Bogolubov coefficients obey the following conditions

$$\alpha\alpha^\dagger - \beta\beta^\dagger = 1 , \tag{1.54}$$

$$\beta\alpha^{\mathrm{T}} - \alpha\beta^{\mathrm{T}} = 0 . \tag{1.55}$$

Comparing the alternative expressions (1.49) and (1.51), one can also express the 'old' creation and annihilation operators with respect to the 'new' ones,

$$\begin{pmatrix} a & a^\dagger \end{pmatrix} = \begin{pmatrix} b & b^\dagger \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} . \tag{1.56}$$

For a given Fock space, the operator $a_k^\dagger a_k$ 'measures' the particle content of type k in a given state and is therefore called the *particle number operator*. Its expectation value with respect to the vacuum is of course zero. If, however, the expectation value of the 'new' particle number operator $b_p^\dagger b_p$ with respect to the 'old' vacuum $|0\rangle_u$ is calculated, the result does *not* vanish in general:

$${}_u\langle 0|b_p^\dagger b_p|0\rangle_u = \int \mathrm{d}\mu(k)|\beta(p,k)|^2 . \tag{1.57}$$

The 'old' vacuum thus contains 'new' particles! Note that the integral in (1.57) may even be divergent, in which case both Fock spaces cannot be related by a unitary transformation (this is a possibility that exists in the case of infinitely many degrees of freedom).

For quantum field theory on a curved spacetime (this is also true for general external fields) the definition of a vacuum – and therefore the whole particle concept – is ambiguous if $\beta(p,k)$ is non-vanishing; as Paul Davies once noted: 'Particles don't exist.'

How can one define a sensible vacuum state? In general, no set of solutions to (1.45a,1.45b) is distinguished. An exception holds if the external spacetime exhibits certain symmetries. For a *stationary* spacetime – a spacetime that has a timelike Killing vector $\xi^a \equiv (\partial/\partial t)^a$ – there are distinguished modes of *positive frequency* that obey

$$\frac{\partial u_k}{\partial t} = -\mathrm{i}\omega_k u_k . \tag{1.58}$$

(Strictly speaking, the left-hand side is the Lie derivative with respect to ξ^a.) Such solutions are a natural generalisation ot the plane waves in Minkowski space, see (1.41). If a different set of solutions $\{v_p\}$ is a linear combination of the $\{u_k\}$ only, i.e. independent of the $\{u_k^*\}$, the Bogolubov coefficient $\beta(p,k)$ is zero and both set of modes share a common vacuum state. For $\beta(p,k) \neq 0$, $\{v_p\}$ contains a mixture of positive frequencies $\{u_k\}$ and negative frequencies $\{u_k^*\}$ and the 'v-vacuum' contains 'u-particles' (and vice versa).[3]

[3] Even if there are no Killing fields present, there exists for a globally hyperbolic spacetime a distinguished vacuum state if the two-point functions obey the so-called *Hadamard condition* (Wipf 1998).

Before discussing the Hawking effect, I want to address briefly an analogous effect discovered by Unruh (1976) that already exists for *non-inertial* observers in flat space. As remarked above, only inertial observers share the standard Minkowski vacuum. What happens for non-inertial observers?

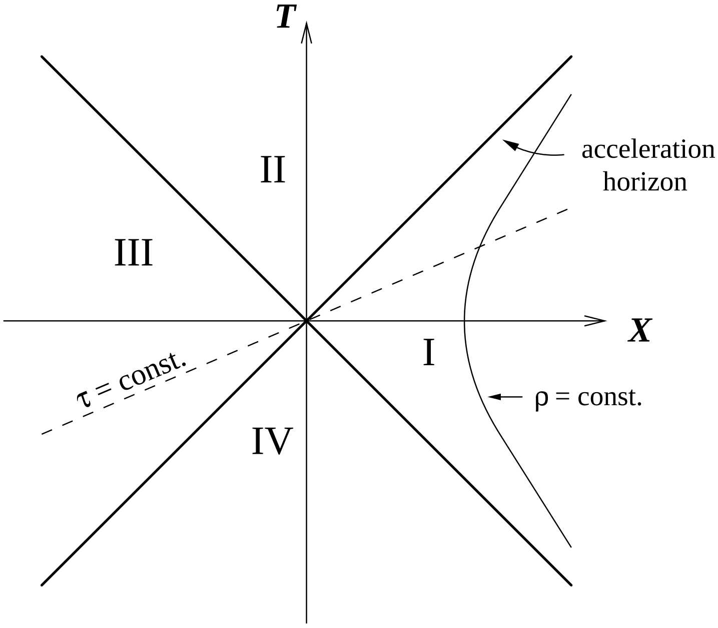

Figure 1.14. Uniformly accelerated observer in Minkowski space

Consider an observer that is uniformly accelerating along the X-direction in (1+1)-dimensional Minkowski spacetime (figure 1.14). To emphasise the analogy with the Kruskal situation (figure 1.4), the Minkowski cartesian coordinates are labelled *here* by upper-case letters. The orbit of this observer is the hyperbola shown in figure 1.14. One recognises that, as in the Kruskal situation, the observer encounters a horizon; there is, however, no singularity behind this horizon. The region I is a globally hyperbolic spacetime on its own – the so-called *Rindler spacetime*. This spacetime can be described by coordinates (τ, ρ) that are connected to the cartesian coordinates via the coordinate transformation

$$\begin{pmatrix} T \\ X \end{pmatrix} = \rho \begin{pmatrix} \sinh a\tau \\ \cosh a\tau \end{pmatrix} , \tag{1.59}$$

where a is a constant (the orbit in figure 1.14 describes an observer with acceleration a, who has $\rho = 1/a$).

Since

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}X^2 = a^2\rho^2\mathrm{d}\tau^2 - \mathrm{d}\rho^2 \,, \tag{1.60}$$

the orbits ρ = constant are also orbits of a timelike Killing field $\partial/\partial\tau$. It is clear that τ corresponds to the external Schwarzschild coordinate t and that ρ corresponds to r in figure 1.4. Like in the Kruskal case, $\partial/\partial\tau$ becomes spacelike in regions II and IV.

The analogy with Kruskal becomes even more transparent if the Schwarzschild metric (1.3) is expanded around the horizon at $r = 2GM$. Introducing $\rho^2/(8GM) = r - 2GM$ and recalling (1.33), one has

$$\mathrm{d}s^2 \approx \kappa^2\rho^2\mathrm{d}t^2 - \mathrm{d}\rho^2 - \frac{1}{4\kappa^2}\mathrm{d}\Omega^2 \,. \tag{1.61}$$

Comparison with (1.60) shows that the first two terms on the right-hand side of (1.61) correspond exactly to the Rindler spacetime (1.60) with the acceleration a replaced by the surface gravity κ. The last term in (1.61) describes a two-sphere with radius $(2\kappa)^{-1}$.[4]

An inertial observer in Minkowski spacetime would of course employ the quantisation of a massless scalar field according to (1.41). In (1+1)-dimensions one has, with $\omega_k = |k|$,

$$\phi(T, X) = \int \mathrm{d}k(a_k u_k + a_k^\dagger u_k^*) \,, \tag{1.62a}$$

$$u_k(T, X) = \frac{1}{\sqrt{4\pi|k|}}\mathrm{e}^{-\mathrm{i}|k|T + \mathrm{i}kX} \,. \tag{1.62b}$$

The accelerated observer is restricted to region I and employs a quantisation scheme that is adapted to the 'Rindler coordinates' τ and ρ. Instead of the plane waves (1.62b) one has to use the corresponding set of solutions to (1.45a,1.45b) rewritten in terms of τ and ρ.

This leads to (Birrell and Davies 1982)

$$\phi(\tau, \rho) = \int \mathrm{d}p(b_p v_p + b_p^\dagger v_p^*) \,, \tag{1.63a}$$

$$v_p(\tau, \rho) = \frac{1}{\sqrt{4\pi|p|}}\,\mathrm{e}^{-\mathrm{i}|p|\tau}\rho^{\mathrm{i}p/a} \,. \tag{1.63b}$$

[4] It is this term that is responsible for the non-vanishing curvature of (1.61) compared to the flat-space metric (1.60) whose extension into the (neglected) other dimensions would be just $-\mathrm{d}Y^2 - \mathrm{d}Z^2$.

Calculating the Bogolubov coefficient $\beta(p,k) = -(u_k^*, v_p)$, see (1.53), one finds for the expectation value of the particle number operator $b_p^\dagger b_p$ with respect to the standard Minkowski vacuum $|0\rangle_{\rm M}$ the expression

$$\begin{aligned} {}_{\rm M}\langle 0|b_p^\dagger b_p|0\rangle_{\rm M} &= \int {\rm d}k|\beta(p,k)|^2 \\ &= (\text{volume}) \times \frac{1}{{\rm e}^{2\pi|p|/a} - 1}\,. \end{aligned} \tag{1.64}$$

(The volume term becomes infinite if the orbit is infinitely long.) Equation (1.64) describes a *Planckian distribution* at a temperature

$$T_{\rm U} = \frac{\hbar a}{2\pi k_{\rm B}} \quad \approx 4 \times 10^{-23} a \left[\frac{\rm cm}{\rm sec^2}\right] K\,. \tag{1.65}$$

An observer that is accelerating uniformly through Minkowski space thus sees a *thermal* distribution of particles. This is an important manifestation of the non-uniqueness of the vacuum state in quantum field theory, even for flat spacetime.

A more detailed investigation of an accelerated detector makes use of the so-called *response function* $\mathcal{F}(E)$ as evaluated along the spacetime path $x(\tau) \equiv (T(\tau), X(\tau))$,

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} {\rm d}\tau \int_{-\infty}^{\infty} {\rm d}\tau' {\rm e}^{-{\rm i}E(\tau-\tau')/\hbar}\, {}_{\rm M}\langle 0|\phi(x(\tau))\phi(x(\tau'))|0\rangle_{\rm M}\,, \tag{1.66}$$

where E denotes the detector's energy.

For the situation of the uniformly accelerating observer, the two-point function appearing in (1.66) again contains a Planck factor with the temperature (1.65), see Birrell and Davies (1982). The vacuum two-point function for a uniformly accelerated detector corresponds to the thermal two-point function for an inertial detector.

Can the Unruh temperature (1.65) be observed? Although $T_{\rm U}$ is tiny for most accelerations, it might be noticeable for electrons in accelerators where spin precession is used as 'detector'. Unruh (1998) has argued that the well-known depolarisation effect of an electron in a storage ring can be interpreted as a vacuum effect of this kind; however, due to the *circular* nature of the acceleration, the effect is not a thermal one.[5]

I shall now turn to the case of black holes. From the form of the line element near the horizon, (1.61), one can already anticipate that – according to the equivalence principle – there is a black hole radiation with

[5] A different, but related, effect is the radiation produced by an accelerating mirror through Minkowski space. It has been argued that the observed sonoluminescence is a manifestation of this effect, see e.g. Liberati et al. (1998).

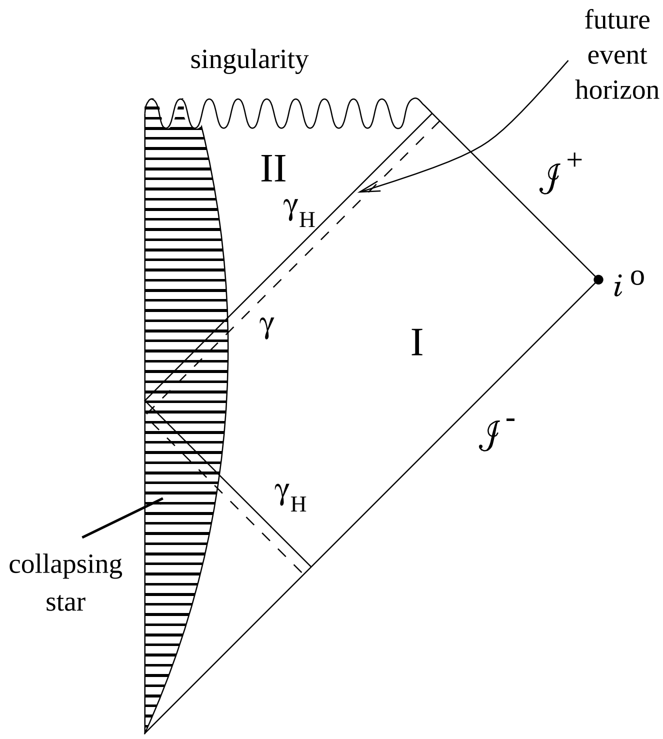

Figure 1.15. Penrose diagram showing the collapse of a star to form a black hole; γ denotes a light ray that is traced back from $\mathcal{I}^+$ through the collapsing star to $\mathcal{I}^-$.

temperature (1.65) in which a is replaced by κ. This is in fact what we shall find.

The following discussion follows the original calculation performed by Hawking (1975). We consider a spherically-symmetric star that collapses to form a black hole, see figure 1.15. I shall again treat the case of a scalar field, see (1.45). Because the background is spherically symmetric, a solution of the Klein-Gordon equation may be separated according to

$$\phi(x) = \frac{f(t,r)}{r} Y_{lm}(\theta,\phi)\,. \tag{1.67}$$

Inserting this ansatz into (1.45) and using the coordinate r_*, see (1.12), one finds

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} + V(r_*)f = 0\,. \tag{1.68}$$

This is just a two-dimensional wave equation with a potential

$$V(r_*) = \left(1 - \frac{2GM}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2GM}{r^3} + \left(\frac{m}{\hbar}\right)^2\right]\,. \tag{1.69}$$

The task now is to solve this equation with appropriate boundary conditions. For r approaching the horizon, $r \to 2GM$, the potential vanishes and (1.68) becomes a free wave equation. For $r \to \infty$, the potential approaches $(m/\hbar)^2$ for $m \neq 0$ and zero for $m = 0$. For simplicity, I shall restrict myself to $m = 0$, but this qualitatively also reflects the case $m \neq 0$.

It is convenient to use the null coordinates

$$u = t - r_*\,, \qquad v = t + r_*\,, \tag{1.70}$$

that play the role of a retarded and an advanced time, respectively. In figure 1.15, v runs along $\mathcal{J}^-$ from $-\infty$ to $+\infty$, and u runs along $\mathcal{J}^+$ from $-\infty$ to $+\infty$.

Considering for the moment the full Kruskal spacetime, figure 1.5, the solution to (1.68) is for $m = 0$ uniquely fixed by either specifying $f(t, r)$ on the union of future horizon and $\mathcal{J}^+$ or on the union of past horizon and $\mathcal{J}^-$.

On $\mathcal{J}^+$ and $\mathcal{J}^-$ there is a well-defined notion of positive frequency in the sense of (1.58) ($f \propto \exp(-\mathrm{i}\omega t)$), while there is some ambiguity in this definition on the horizon. One can there, for example, use either the affine parameter or the Killing parameter, which are related by (1.34). The exact definition is, however, irrelevant as long as one restricts attention only to 'measurements' in region I, since the definition of creation and annihilation operators of particles escaping to $\mathcal{J}^+$ is independent of the notion used (Wald 1984).

We now consider solutions in region I of the full Kruskal manifold that have positive frequency, $f_\omega \propto \exp(-\mathrm{i}\omega t)$. Since for $r \to 2GM (r_* \to -\infty)$ the potential (1.69) vanishes, such solutions approach close to the horizon 'plane waves' in (t, r_*)-coordinates,

$$\begin{aligned} f_\omega(t, r_*) &= a\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega r_*} + b\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\omega r_*} \\ &= a\mathrm{e}^{-\mathrm{i}\omega u} + b\mathrm{e}^{-\mathrm{i}\omega v} \\ &\equiv f_\omega^{(\mathrm{out})} + f_\omega^{(\mathrm{in})}\,. \end{aligned} \tag{1.71}$$

The solution $f_\omega^{(\mathrm{out})}$ is referred to as 'outgoing', since $u =$ constant corresponds to 'lightrays' escaping tp $\mathcal{J}^+$; analogously, the solution $f_\omega^{(\mathrm{in})}$ is 'ingoing', since v = constant corresponds to 'light rays' penetrating the horizon.

It is now important to notice that close to the horizon $u \to \infty$ and that therefore $f_\omega^{(\mathrm{out})}$ rapidly oscillates. This becomes especially transparent by using Kruskal coordinates: defining from the standard coordinates (T, X) in (1.13) the corresponding null coordinates

$$U = T - X, \qquad V = T + X\,, \tag{1.72}$$

one has

$$U = \ - \exp\left(-\frac{u}{4GM}\right) = -\mathrm{e}^{-\kappa u}\,, \tag{1.73a}$$

$$V = \ \exp\left(\frac{v}{4GM}\right) = \mathrm{e}^{\kappa v}\,, \tag{1.73b}$$

and therefore

$$f_\omega^{(\mathrm{out})} = a\,\mathrm{e}^{\mathrm{i}\omega\kappa^{-1}\ln(-U)}\,. \tag{1.74}$$

Note that $U \to 0$ as the horizon is approached. Because of the rapid oscillation of $f_\omega^{(\mathrm{out})}$, the approximation of geometric optics should be excellent for $r \to 2GM$; this is why it is justified to talk of 'light rays' (more precisely, rays corresponding to the scalar field).

Consider a null geodesic that is entering the black-hole region II from region I; be λ its affine parameter ($\lambda = 0$ corresponding to the crossing point with the horizon). Then $U = -\lambda$ (since $\mathrm{d}s^2 \sim \mathrm{d}U\mathrm{d}V + \ldots$, U is the affine parameter, not u) and one has

$$f_\omega^{(\mathrm{out})} = a\,\mathrm{e}^{\mathrm{i}\omega\kappa^{-1}\ln\lambda}\,. \tag{1.75}$$

The frequency of each mode thus *diverges* at the horizon – an extreme manifestation of the gravitational redshift (compare (1.18)) that is also responsible for the behaviour of the modes of a quantum field. In fact, this redshift lies at the heart of the Hawking effect.

Hawking (1975) now noticed that it is most convenient to consider in the collapse diagram (figure 1.15) an outgoing ray at $\mathcal{J}^+$ and trace it *back* to $\mathcal{J}^-$; one part is directly scattered back to $\mathcal{J}^-$, the other part passes *through* the collapsing matter and reaches $\mathcal{J}^-$ – it is this part that is of interest for our analysis. In figure 1.15, γ is an example of such a ray; the passage through the collapsing star corresponds in the diagram to a reflection at the origin. The ray γ_{H} denotes a limiting ray that stays on the future horizon and is traced back to $\mathcal{J}^-$. Since the considered rays (such as γ) are close to γ_{H}, the potential V in (1.68) is negligible.

The propagator of γ back to $\mathcal{J}^-$ reaches $\mathcal{J}^-$ at a distance λ in the affine parameter along null geodesics on $\mathcal{J}^-$ (this is because $\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \ldots$

along $\mathcal{J}^-$, so that v is there the affine parameter). If the crossing of γ_H with $\mathcal{J}^-$ is at $v = 0$, γ reaches $\mathcal{J}^-$ at $v = -\lambda$.

The propagation of (1.75) therefore leads to the following solution in the vicinity of $v = 0$ on $\mathcal{J}^-$:

$$f_\omega(v) = \begin{cases} a\,\mathrm{e}^{\mathrm{i}\omega\kappa^{-1}\ln(-v)} & , \quad v < 0 \\ 0 & , \quad v > 0 \end{cases} . \tag{1.76}$$

Since γ_H is the limiting ray, $f_\omega(v)$ vanishes for $v > 0$. To obtain the frequency content of (1.76), its Fourier transform is calculated:

$$\begin{aligned} \tilde{f}_\omega(\omega') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}\omega' v} f_\omega(v) \\ &\sim \int_{-\infty}^{0} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}\omega' v} \mathrm{e}^{\mathrm{i}\omega\kappa^{-1}\ln(-v)} = \int_{0}^{\infty} \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}\omega' v} v^{\mathrm{i}\omega\kappa^{-1}} . \end{aligned} \tag{1.77}$$

(Note the similarity of the integrand to (1.63b) for the Unruh effect.) The integral (1.77) can be evaluated if use is made of the integral formula

$$\begin{aligned} &\int_0^\infty \mathrm{d}x x^{\nu-1} \exp[-(A + \mathrm{i}B)x] = \\ &\qquad \Gamma(\nu)(A^2 + B^2)^{-\nu/2} \exp\left(-\mathrm{i}\nu \arctan \frac{B}{A}\right) . \end{aligned} \tag{1.78}$$

One then easily recognises that, taking $\omega' > 0$,

$$\tilde{f}_\omega(-\omega') = -\mathrm{e}^{\omega\pi\kappa^{-1}} \tilde{f}_\omega(\omega') \neq 0, \tag{1.79}$$

so that a mode of positive frequency on $\mathcal{J}^+$ is a *mixture of positive and negative frequency* on $\mathcal{J}^-$! To find the exact amount of particle creation, the Bogolubov-coefficient β has to be calculated. Decomposing in the manner of (1.52) the solution $f_\omega(v)$, (1.76), which is of positive frequency on $\mathcal{J}^+$, into positive and negative frequencies on $\mathcal{J}^-$, one has up to numerical factors

$$f_\omega(v) \sim \int_0^\infty \mathrm{d}\omega \left(\alpha_{\omega\omega'} \frac{\mathrm{e}^{-\mathrm{i}\omega' v}}{\sqrt{\omega'}} + \beta_{\omega\omega'} \frac{\mathrm{e}^{\mathrm{i}\omega' v}}{\sqrt{\omega'}} \right) . \tag{1.80}$$

On the other hand, using (1.79),

$$\begin{aligned} f_\omega(v) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega'\, \mathrm{e}^{-\mathrm{i}\omega' v} \tilde{f}_\omega(\omega') \\ &\sim \int_0^\infty \mathrm{d}\omega' \left(\tilde{f}_\omega(\omega')\mathrm{e}^{-\mathrm{i}\omega' v} - \mathrm{e}^{-\omega\pi\kappa^{-1}} \tilde{f}_\omega(\omega')\mathrm{e}^{\mathrm{i}\omega' v} \right) . \end{aligned} \tag{1.81}$$

Comparing (1.80) with (1.81), one finds

$$\beta_{\omega\omega'} = -\mathrm{e}^{-\mathrm{i}\omega\pi\kappa^{-1}}\alpha_{\omega\omega'}\,. \tag{1.82}$$

Using (1.76) and (1.80) one can then evaluate α and β in terms of Γ-functions. If one now attempted to calculate the particle number expectation value according to (1.57), one would find a diverging result. This is due to the fact that the collapsing body produces particles infinitely long, as can be recognised from figure 1.15 (again an effect of the infinite redshift at the horizon). One can instead calculate the number of emitted particles per unit time (Birrell and Davies 1982) – either by considering wave packets or confining the system into a box – to obtain for the number of particles per unit time in the frequency range ω to $\omega + \mathrm{d}\omega$ the expression

$$\frac{\mathrm{d}\omega}{2\pi}\frac{1}{e^{2\omega\pi\kappa^{-1}}-1}\,. \tag{1.83}$$

This is a Planck distribution with the temperature

$$T_{\mathrm{BH}} = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}}\,. \tag{1.84}$$

One immediately notes that this 'Hawking temperature' follows from the Unruh temperature in (1.65) through the substitution $a \to \kappa$, as anticipated.

An alternative derivation of this result can be made through the use of the energy-momentum tensor (DeWitt 1975). This treatment also allows a straightforward implementation of the back-scattering effect: If the potential (1.69) is fully taken into account, some of the modes are scattered back into the black hole instead of escaping to $\mathcal{I}^+$. This leads to the following expression for the total luminosity of the black hole:

$$L = -\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{1}{2\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty}\mathrm{d}\omega\,\omega\frac{\Gamma_{\omega l}}{e^{2\omega\pi\kappa^{-1}}-1}\,. \tag{1.85}$$

The term $\Gamma_{\omega l}$ – called 'greybody factor' because it encodes a deviation from the black-body spectrum – is the fraction of the incoming mode that enters the black hole; it depends explicitly on ω and l. Its calculation requires a detailed discussion of (1.68,1.69) (and similar equations for higher spins).

For the special case of the Schwarzschild metric, $\kappa = (4GM)^{-1}$, and (1.84) becomes

$$T_{\mathrm{BH}} = \frac{\hbar}{8\pi G k_{\mathrm{B}} M} \approx 10^{-6}\frac{M_{\odot}}{M}\mathrm{K}\,. \tag{1.86}$$

For solar-mass black holes, this is of course utterly negligible – the black hole absorbs much more from the ubiquitous 3K-microwave background radiation than it radiates.

One can, however, estimate the lifetime of a black hole by making the plausible assumption that the decrease in mass is equal to the energy radiated to infinity and using Stefan-Boltzmann's law:

$$\frac{\mathrm{d}M}{\mathrm{d}t} \propto -AT_{\mathrm{BH}}^4 \propto -M^2 \times \left(\frac{1}{M}\right)^4 = -\frac{1}{M^2}\,,$$

which integrated yields

$$t(M) \propto (M_0^3 - M^3) \approx M_0^3\,, \tag{1.87}$$

where M_0 is the initial mass, and it has been assumed that after the evaporation $M \ll M_0$. Very roughly, the lifetime of a black hole is thus given by

$$\tau_{\mathrm{BH}} \approx \left(\frac{M_0}{m_{\mathrm{p}}}\right)^3 t_{\mathrm{p}} \approx 10^{65} \left(\frac{M_0}{M_\odot}\right)^3 \text{years} \tag{1.88}$$

(m_{p} and t_{p} denote Planck mass and Planck time: $m_{\mathrm{p}} = \hbar/l_{\mathrm{p}}$, $t_{\mathrm{p}} = l_{\mathrm{p}}$.) If in the early universe primordial black holes with $M_0 \approx 5 \times 10^{14}$g were created, they would evaporate at the present age of the universe.

A very detailed investigation into black-hole evaporation was made by Page (1977). He found that for $M \gg 10^{17}$g the power emitted from an (uncharged, non-rotating) black hole is

$$P \approx 2.28 \times 10^{-54} L_\odot \left(\frac{M}{M_\odot}\right)^{-2}\,,$$

of which 81.4% is in neutrinos (he considered only electron- and muon-neutrinos), 16.7% in photons, and 1.9% in gravitons, assuming of course that there are no other massless particles. Since a black hole evaporates *all* existing particles in Nature, this result would of course be changed by the existence of massless supersymmetric or other particles. In the range $5 \times 10^{14}\mathrm{g} \ll M \ll 10^{17}\mathrm{g}$, Page found

$$P \approx 6.3 \times 10^{16} \left(\frac{M}{10^{15}\mathrm{g}}\right)^{-2} \frac{\mathrm{erg}}{\mathrm{sec}}\,,$$

of which 45% is in electrons and positrons, 45% in neutrinos, 9% in photons, and 1% in gravitons. Massive particles with mass m are only suppressed if $k_{\mathrm{B}}T_{\mathrm{BH}} < m$. For $M < 5 \times 10^{14} g$, also higher-mass particles are emitted.

All of the above derivations use the approximation where the spacetime background remains classical.[6] In a theory of quantum gravity, however, such a picture cannot be maintained, see sections 1.5 and 1.6. Since the black hole becomes hotter while radiating, see (1.86), its mass will eventually enter the quantum-gravity domain $M \approx m_{\rm p}$, where the semiclassical approximation breaks down. The evaporation then enters the realm of speculation, see the following sections. As an intermediate step one might consider the heuristic 'semiclassical' Einstein equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G \langle T_{ab} \rangle \,, \qquad (1.89)$$

where on the right-hand side the quantum expectation value of the energy-momentum tensor appears. The evaluation of $\langle T_{ab} \rangle$ – which requires regularisation and renormalisation – is a difficult subject on its own (Birrell and Davies 1982). The renormalised $\langle T_{ab} \rangle$ is essentially unique (its ambiguities can be absorbed in coupling constants) if certain sensible requirements are imposed (Wald 1984). Technically, it is most convenient to handle $\langle T_{ab} \rangle$ through the use of the *effective action* (Wipf 1998). Evaluating the components of the renormalised $\langle T_{ab} \rangle$ near the horizon, one finds that there is a flux of *negative energy* into the hole. Clearly this leads to a decrease in the black hole's mass. These negative energies are a typical quantum effect and are well-known from the – accurately measured – Casimir effect. This occurrence of negative energies is also responsible for the breakdown of the classical area law discussed in the last sections.

The negative flux near the horizon lies also at the heart of the 'pictorial' representation of Hawking radiation that is often used (see figure 1.16): In vacuum, virtual pairs of particles are created and destroyed. However, close to the horizon, one partner of this virtual pair might fall into the black hole, thereby liberating the other partner to become a real particle and escaping to infinity as Hawking radiation.

It is interesting to note that the quantum fields exhibit entanglement ('EPR-correlations') between the interior and exterior of the event horizon (Wald 1986). This was shown for both the 'eternal hole' (figure 1.5) and the Rindler spacetime (figure 1.14), see Birrell and Davies (1982). The global vacuum state comprising the regions I and II in these diagrams can be written in the form

$$|0\rangle = \prod_{\omega} \sqrt{1 - {\rm e}^{-2\pi\omega\kappa^{-1}}} \sum_{n} {\rm e}^{-n\pi\omega\kappa^{-1}} |n_{\omega}^{\rm I}\rangle \otimes |n_{\omega}^{\rm II}\rangle \,, \qquad (1.90)$$

where $|n_{\omega}^{\rm I}\rangle$ and $|n_{\omega}^{\rm II}\rangle$ are n-particle states with frequency ω in regions I and II, respectively; in the situation of figure 1.14, κ has to be replaced by

[6] This limit is referred to as the semiclassical approximation to quantum gravity (see e.g. Kiefer 1994).

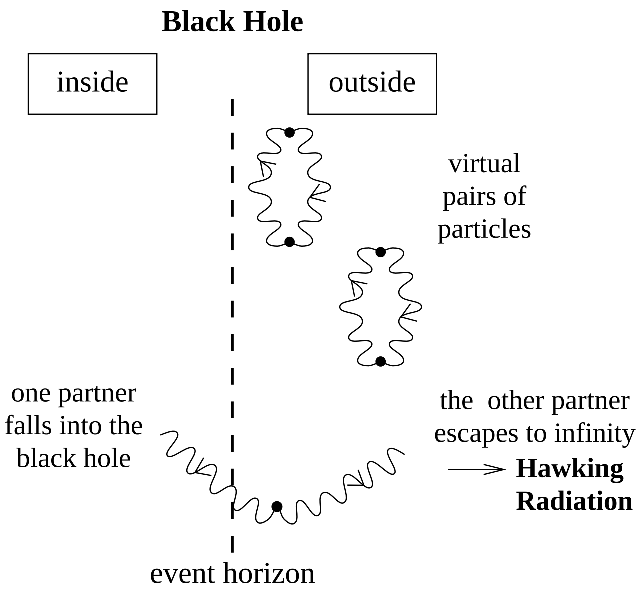

Figure 1.16. Heuristic 'visualisation' of the Hawking effect

a. The expression (1.90) is just the Schmidt expansion for two entangled quantum states, see e.g. Giulini et al. (1996); note the analogy of (1.90) to a BCS-state in the theory of superconductivity.

Since in the presence of an event horizon, observations are restricted to the outside region, the state (1.90) cannot be distinguished by operators with support in I only from a density matrix that is found from (1.90) by tracing out all degrees of freedom in region II,

$$\begin{aligned}\rho_{\rm I} &\equiv {\rm Tr}_{\rm II}\,|0\rangle\langle 0| \\ &= \prod_\omega \left(1-{\rm e}^{-2\pi\omega\kappa^{-1}}\right)\sum_n {\rm e}^{-2\pi n\omega\kappa^{-1}}|n_\omega^{\rm I}\rangle\langle n_\omega^{\rm I}|\,. \end{aligned} \qquad (1.91)$$

Note that $\rho_{\rm I}$ describes a canonical ensemble with the temperature (1.84). The thermal nature of Hawking radiation is thus a consequence of the fact that observations are restricted to region I – and this is a consequence of the presence of an event horizon!

I want to end this section by giving the explicit expressions for the Hawking temperature (1.84) in the case of rotating and charged black holes.

For the Kerr solution (1.24), one has

$$k_{\mathrm{B}}T_{\mathrm{BH}} = \frac{\hbar\kappa}{2\pi} = 2\left(1+\frac{M}{\sqrt{M^2-a^2}}\right)^{-1}\frac{\hbar}{8\pi M} \quad < \frac{\hbar}{8\pi M}\,. \tag{1.92}$$

Rotation thus reduces the Hawking temperature. The integrand in (1.85) then becomes

$$\frac{\omega\Gamma_{\omega l}}{\mathrm{e}^{2\pi\kappa^{-1}(\omega-m\Omega_{\mathrm{H}})}-1}\,, \tag{1.93}$$

where Ω_{H} is given by (1.27) and m is here the azimuthal number of the incident wave. For $\omega - m\Omega_{\mathrm{H}} < 0$ and $\kappa \to 0$ (i.e., $T_{\mathrm{BH}} \to 0$), (1.93) goes to $-\omega\Gamma_{\omega l}$: This is just the classical phenomenon of superradiance mentioned in section 1.2 (see the paragraph above (1.31)), see also DeWitt (1975).

For the Reissner-Nordström solution (1.21) one has

$$k_{\mathrm{B}}T_{\mathrm{BH}} = \frac{\hbar}{8\pi M}\left(1-\frac{(Gq)^4}{r_+^4}\right) \quad < \frac{\hbar}{8\pi M}\,. \tag{1.94}$$

Thus, also electric charge reduces the Hawking temperature. For an extremal black hole, $r_+ = GM = \sqrt{G}|q|$, and thus $T_{\mathrm{BH}} = 0$. The question whether its entropy is also zero or proportional to $A \neq 0$ plays a crucial role in the quantisation of black holes, see sections 1.5 and 1.6.

1.4 Interpretation of Entropy and the Problem of Information Loss

We have seen in the last section that – if quantum theory is taken into account – black holes emit thermal radiation with the temperature (1.84). Consequently, the laws of black-hole mechanics discussed in section 1.2 have indeed a physical interpretation as thermodynamical laws – black holes *are* thermodynamical systems.

One can therefore from the First Law (1.35) also infer the expression for the black-hole entropy. From $\mathrm{d}M = T_{\rm BH}\mathrm{d}S_{\rm BH}$ one finds the 'Bekenstein-Hawking entropy'

$$S_{\rm BH} = \frac{k_{\rm B}A}{4G\hbar}\,, \tag{1.95}$$

in which the unknown factor in (1.40) has now been fixed. For the special case of a Schwarzschild black hole, this yields

$$S_{\rm BH} = \frac{k_{\rm B}\pi R_0^2}{G\hbar}\,. \tag{1.96}$$

It can easily be estimated that $S_{\rm BH}$ is much bigger than the entropy of the star that collapsed to form the black hole. A physical interpretation of $S_{\rm BH}$ must therefore be based on other principles – but on which? Certainly, up to now the laws of black-hole mechanics are only phenomenological thermodynamical laws. The central open question is: Can $S_{\rm BH}$ be derived from quantum-statistical considerations? This would mean that $S_{\rm BH}$ could be calculated from a Gibbs-type formula according to

$$S_{\rm BH} \stackrel{?}{=} -k_{\rm B}\,{\rm Tr}(\rho\ln\rho) \equiv S_{\rm SM}\,, \tag{1.97}$$

where ρ denotes an appropriate density matrix; $S_{\rm BH}$ would then somehow correspond to the number of quantum microstates that are consistent with the macrostate of the black hole that is – according to the no-hair theorem – uniquely characterised by mass, angular momentum, and charge. Some important questions are:

- Does $S_{\rm BH}$ correspond to states hidden behind the horizon?
- Does $S_{\rm BH}$ corresponds to the number of possible initial states?
- Where is $S_{\rm BH}$ located (if it is located at all)?
- What happens with $S_{\rm BH}$ after the black hole has evaporated?

There have been some attempts to calculate $S_{\rm BH}$ by counting internal states of freedom (see the review in Kiefer 1998). However, although one could derive $S_{\rm BH} \propto A$ in this way, the factor of proportionality was divergent and needed some regularisation. At least these derivations seem to indicate that the entropy is somehow located at or near to the event horizon. Preliminary results in the theory of induced gravity show that a regularisation can indeed be invoked such that the desired result (1.95) can be obtained, although at the price of introducing non-minimally coupled fields. The only clear-cut microscopic derivation of $S_{\rm BH}$ was done in string theory (see section 1.6), although the applicability is as yet restricted to extremal (or near-extremal) black holes.

The attempts to calculate $S_{\rm BH}$ by state counting are usually done in the 'one-loop limit' of quantum field theory in curved spacetime – this is the limit where gravity is classical but non-gravitational fields are fully quantum, and it is the limit where the Hawking radiation (1.84) has been derived. Surprisingly, however, the expression (1.95) can already be calculated from the so-called 'tree level' of the theory, where only the gravitational degrees of freedom are taken into account (Hawking 1979; Hawking and Penrose 1996). This is already reviewed in Kiefer (1998), and I shall be brief on this in the following.

The derivations employ the analogy of euclidean path integrals and partition sums. Calculating the euclidean path integral to quantum gravity at highest order (tree-level or saddle-point approximation), one finds for the partition sum in the Schwarzschild case

$$Z \approx \exp\left(-\frac{\hbar\beta^2}{16\pi G}\right) , \tag{1.98}$$

with $\beta = (k_{\rm B}T)^{-1}$. One can then calculate the standard thermodynamical quantities in the usual manner. In this way one arrives at the mean energy

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\hbar\beta^2}{8\pi G} , \tag{1.99}$$

from which (since E must be equal to the mass M of the black hole) the Hawking temperature (1.86) can be found, $T = T_{\rm BH}$. For the entropy one then finds

$$S = k_{\rm B}(\ln Z + \beta M) = \frac{\hbar\beta^2}{16\pi G} = \frac{k_{\rm B}A}{4G\hbar} = S_{\rm BH} , \tag{1.100}$$

and for the specific heat

$$C = -\beta\frac{\partial S}{\partial \beta} = -\frac{\hbar\beta^2}{8\pi G} < 0 , \tag{1.101}$$

whose negativity signals instability; this just expresses the fact that the black hole in asymptotically flat space becomes hotter by radiating – a typical thermodynamical feature of gravitational systems (Zeh 1992). One can try to stabilise the black hole by putting it into a box or embedding it into an asymptotically anti-de Sitter space, but I shall not elaborate on this here.

It is also instructive to see how the Hawking temperature in the Schwarzschild case can be found from the *euclidean* line element of the Schwarzschild metric (1.3). Writing $\tau = \mathrm{i}t$, one obtains

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}\tau^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,. \qquad (1.102)$$

Introducing the new radial coordinate

$$R = 4GM\sqrt{1 - \frac{2GM}{r}}\,, \qquad (1.103)$$

this assumes the form

$$\mathrm{d}s^2 = R^2\mathrm{d}\left(\frac{\tau}{4GM}\right)^2 + \left(\frac{r(R)}{2GM}\right)^4\mathrm{d}R^2 + r^2(R)\mathrm{d}\Omega^2 \,. \qquad (1.104)$$

This metric has a coordinate singularity at $R = 0$ (corresponding to $r = 2GM$). *Regularity* is obtained if $\tau/(4GM)$ is interpreted as an angular coordinate with periodicity 2π; τ itself has then periodicity $8\pi GM$ (figure 1.17) which, when set equal to $\beta\hbar$, yields the Hawking temperature (1.86). This result suggests the existence of a thermal equilibrium state on the eternal black-hole spacetime (figure 1.5) at a temperature $T = T_{\mathrm{BH}}$; this state is the so-called *Hartle-Hawking state* (a general formulation is achieved in terms of KMS-states).

Since R in (1.103) is only defined outside the horizon ($r > 2GM$), the euclidean Schwarzschild line element (1.102) does not penetrate into the horizon – this is an expression of the fact that the interior is never classically forbidden, see section 1.5.

Hawking put forward the conjecture that gravitational entropy is connected with a non-trivial *topology* of the euclideanised spacetime (see e.g. his article in Wald 1998). The euclidean line element (1.104) has the boundary$S^2 \times S^1$ at infinity, where S^2 is a large two-sphere at $r \to \infty$ and S^1 corresponds to the periodically identified imaginary time coordinate. In the euclidean Schwarzschild case, the topology is $\mathbb{R}^2 \times S^2$ (figure 1.17), while in the case of filling this boundary with flat euclidian space, one would obtain the topology $\mathbb{R}^3 \times S^1$. A non-vanishing entropy is obtained only in the first case. Other examples suggest the same connection between entropy and topology. Obviously, in this interpretation gravitational entropy would be a truly *global* concept and cannot be assumed to be localised near the horizon. The important question, however, is: How fundamental are *euclidean* concepts? An answer can probably only be found in a quantum theory of gravity, see sections 1.5 and 1.6.

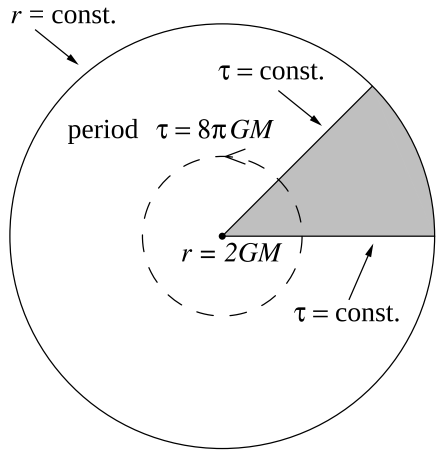

Figure 1.17. Coordinates for the euclidean Schwarzschild solution; the euclidean time is identified with period $\tau = 8\pi GM$. All slices τ = constant meet at the origin.

If the entropy (1.95) is to make sense, there should be a generalised Second Law of thermodynamics in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}(S_{\mathrm{BH}} + S_{\mathrm{M}}) \geq 0\,, \tag{1.105}$$

where S_{M} denotes all contributions to non-gravitational entropy. The validity of (1.105), although far from being proven in general, has been shown in a variety of gedankenexperiments. One of the most instructive of such experiments has been devised by Unruh and Wald (see Wald 1994). It makes use of the box shown in figure 1.13 that is adiabatically lowered towards a (spherically symmetric) black hole.

At asymptotic infinity $r \to \infty$, the black-hole radiation is given by (1.84). However, for finite r the temperature is modified by the occurrence of a redshift factor $\chi(r) \equiv (1 - 2GM/r)^{1/2}$ in the denominator. Since the box is not in free fall, it is accelerated with an acceleration a. From the relation (Wald 1984)

$$\kappa = \lim_{r \to R_0} (a\chi)\,, \tag{1.106}$$

one has

$$T_{\mathrm{BH}}(r) = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}\chi(r)} \quad \stackrel{r \to R_0}{\longrightarrow} \quad \frac{\hbar a}{2\pi k_{\mathrm{B}}}\,, \tag{1.107}$$

which is just the Unruh temperature (1.65)! This means that a freely falling observer near the horizon observes no radiation at all, and the whole effect (1.107) comes from the observer (or box) being non-inertial with acceleration a.

The analysis of Unruh and Wald, which is a generalisation of the gedankenexperiment discussed at the end of section 1.2, shows that the entropy of the black hole increases at least by the entropy of the Unruh radiation displaced at the floating point – this is the point where the gravitational force (pointing downwards) and the buoyancy force from the Unruh radiation (1.107) are in equilibrium. Amusingly, it is just the application of 'Archimedes'principle' to this situation that rescues the generalised Second Law (1.105).

An inertial, i.e. free-falling, observer does not see any Unruh radiation. How does he interpret the above result? For him the box is accelerated and therefore the interior of the box fills up with negative energy and pressure – a typical quantum effect that occurs if a 'mirror' is accelerated through the vacuum, cf. footnote 5. The 'floating point' is then reached after this negative energy is so large that the total energy of the box is zero.

I want to conclude this section with some speculations about the final stages of black-hole evolution and the information loss problem. The point is that – in the semiclassical approximation used by Hawking – the radiation of a black hole is purely thermal. If the black hole evaporates completely and leaves only thermal radiation behind, one would have a conflict with established principles in quantum theory: Any initial state (in particular a pure state) would evolve into a mixed state. In ordinary quantum theory, because of the unitary evolution of the total system,[7] this cannot happen. Formally, $\mathrm{Tr}\,\rho^2$ remains *constant* under the von Neumann equation; the same is true for the entropy $S_{\mathrm{SM}} = -k_{\mathrm{B}}\,\mathrm{Tr}(\rho\ln\rho)$: For a unitarily evolving system, there is no increase in entropy. If these laws are violated during black-hole evaporation, information would be destroyed. This is indeed the speculation that Hawking made after his discovery of black-hole radiation. The attitudes towards this *information loss problem* can be roughly divided into the following classes (see e.g. Page 1994 and Preskill 1993 for reviews):

- The information is indeed lost during black-hole evaporation, and the quantum-mechanical Liouville equation is replaced by an equation of the form
$$\rho \longrightarrow \$\,\rho \neq S\rho S^{\dagger}\,. \tag{1.108}$$
- The full evolution is in fact unitary; the black-hole radiation contains subtle quantum correlations that cannot be seen in the semiclassical approximation.

[7] For an *open* quantum system, a state will in general become more mixed and its entropy will increase (Giulini et al. 1996).

- The black hole does not evaporate completely, but leaves a 'remnant' with mass in the order of the Planck mass that carries the whole information.

All of these options have advantages and disadvantages that are discussed at length in the above-cited literature.[8] From a conservative point of view, it looks reasonable to stick to the second option as long as possible. In fact, as long as the black hole has not evaporated and there exists a horizon, it is always possible to assume that the information is 'hidden' behind the horizon. As (1.90) and (1.91) demonstrate, although the global vacuum state is pure, it appears outside the black hole as a thermal state, since there are nonlocal correlations with inaccessible states behind the horizon. The entropy (1.95) would then be just an expression of this lack of information about the global quantum state. Its appearance would signal the presence of nonlocal entanglement, and $S_{\rm BH}$ would thus be itself a *global* quantity.

The relations (1.90) and (1.91) have only been shown for the 'eternal hole' (figure 1.5) and the Rindler spacetime (figure 1.14), but not for a collapse situation. One might, however, expect that – independently of the unknown details of the final stage – all quantum correlations would become accessible again, demonstrating that the full evolution is unitary.[9]

One argument in favour of the first option above is the observation that in the euclideanised spacetime constant time surfaces meet at a common point, leading to a zero point of the time translation vector field (see Hawking's article in Wald 1998). An example of this can be seen in the geometry of the euclidean Schwarzschild geometry, see figure 1.17. In such a case one cannot use a Hamiltonian to get a unitary evolution from the initial to the final state, and a transition from a pure to a mixed state can be expected to occur. It is, however, by no means evident that such a euclidean viewpoint is more fundamental than the lorentzian viewpoint. In particular, the constant-time surfaces in figure 1.17 all meet at the origin – the place where the horizon sits. Their crossing at this point might be only an indication for the presence of quantum entanglement behind the horizon in the lorentzian framework.

[8] A recent vote among experts at a meeting in Utrecht showed an overwhelming majority for the second option and a practical exclusion of the third option.

[9] The accelerated observer in figure 1.14 would only have to go over to inertial motion in order to gain all quantum correlations again.

1.5 Black holes in Canonical Quantum Gravity

The framework for the discussion in the last two sections is the limit where the gravitational background is classical, but all non-gravitational fields are quantum. The central results are that black holes radiate with a thermal spectrum and that they possess an intrinsic gravitational entropy.

According to (1.86), black holes become hotter through emitting radiation, while losing mass. The framework of the fixed gravitational background is expected to break down if the hole's mass approaches the *Planck mass*

$$m_{\rm p} = \sqrt{\frac{\hbar}{G}} \approx 10^{-5}{\rm g}\,, \tag{1.109}$$

that is related to the Planck length (1.39) by $m_{\rm p} = \hbar/l_{\rm p}$. The reason for this expectation is that $m_{\rm p}$ should set the scale for the occurrence of quantum gravitational effects – effects where the classical picture of spacetime breaks down. (That the approximation of section 1.3 must break down somewhere, is by itself evident, since otherwise $T_{\rm BH} \to \infty$.)

The final stages of black-hole evaporation can thus only be understood within a *quantum theory of gravity.* Unfortunately, such a theory is not yet available. Nevertheless, there exist quite advanced approaches towards such a theory, within which sensible questions can be asked and partially answered. The two most popular approaches at present are *canonical quantum gravity* and *superstring theory.* I shall devote the last two sections to them.

Canonical quantum gravity, the topic of this section, is the framework that is found if standard quantisation rules are applied to the general theory of relativity. This is a rather conservative approach, since no unification of fundamental forces is attempted. However, this approach exhibits in a most transparent way the basic conceptual features that a quantum theory of gravity should contain. Starting point for canonical quantisation is a Hamiltonian formulation on a classical spacetime $\mathcal{M}$ that is globally hyperbolic, i.e. that can be written in the form

$$\mathcal{M} = \Sigma \times \mathbb{R}\,, \tag{1.110}$$

where Σ is a three-dimensional manifold (Wald 1984). Depending on the topological structure of Σ one obtains different configuration spaces. In addition, the canonical framework can be further subdivided into the following classes that are characterised by the configuration variable that is used:

- quantum geometrodynamics: three-metric h_{ab} on Σ;
- quantum connection dynamics: SU(2)-connection ${A_a}^i$ on Σ;

- quantum loop dynamics: the trace of the holonomy of the connection along a loop γ, $\mathrm{tr}\,\mathcal{P}\exp\oint_\gamma \mathcal{A}$.

Although much progress has been made in the last two approaches (Ashtekar 1997), I restrict myself to the first approach that uses the three-dimensional metric as the fundamental variable. The main feature is the existence of *constraints*, one Hamiltonian constraint (per space point),

$$\mathcal{H} \approx 0\,, \tag{1.111}$$

and three diffeomorphism constraints (per space point),

$$\mathcal{D}_a \approx 0\,. \tag{1.112}$$

The sign $\approx$ denotes *weak equality* in the sense of Dirac: The constraints define a subspace in the full phase space.

If Σ is compact without a boundary (a situation often used in cosmological models), the total Hamiltonian is a combination of these constraints only and thus (weakly) vanishes. If Σ is asymptotically flat (this is the case relevant for black holes), the total Hamiltonian has in addition boundary terms. A comprehensive introduction into all aspects of canonical gravity can be found in Ehlers and Friedrich (1994).

The crucial feature is the treatment of the constraints (1.111, 1.112) in the quantum theory. Here I shall follow Dirac's approach and implement the constraints – at least formally – as constraints on physically allowed wave functionals:

$$\hat{\mathcal{H}}\Psi = 0\,, \tag{1.113}$$

$$\hat{\mathcal{D}}_a\Psi = 0\,. \tag{1.114}$$

The wave functional Ψ depends (apart from non-gravitational fields) on the three-metric $h_{ab}(\boldsymbol{x})$; (1.114) guarantees that Ψ remains invariant under coordinate transformations, so it de facto depends only on the three-dimensional geometry – this is often emphasised by writing $\Psi[^{(3)}\mathcal{G}]$, but one must remember that Ψ is always given as $\Psi[h_{ab}(\boldsymbol{x})]$.

On the fundamental level, there is only a collection of spaces (of three-dimensional geometries), but no spacetime. The latter has only meaning in a semiclassical approximation (Kiefer 1994). This lack of spacetime on a fundamental level is often referred to as the 'problem of time in quantum gravity' (see e.g. Kiefer 1997). Since the event horizon of a black hole plays a crucial role in the derivation of Hawking radiation, and since the horizon is a genuine *classical* spacetime concept, this drastically demonstrates that the semiclassical picture of section 1.3 *must* be modified in quantum gravity.[10]

[10] Figure 21.4 in Misner, Thorne and Wheeler (1973) shows a foliation of the Kruskal diagram (figure 1.4) into three-geometries. In quantum gravity, Ψ depends on both this set of three-geometries (classically allowed ones for this case) and other three-geometries (that are classically not allowed).

In the following I shall discuss the quantisation of spherically-symmetric 'eternal' black holes. This can either be interpreted as an exact quantisation of the matter-free case or as the first step in the semiclassical approximation to the case where also matter is present (see Kiefer 1998 for details and references).

Starting point is the general spherically-symmetric metric,

$$\mathrm{d}s^2 = N^2(r,t)\mathrm{d}t^2 - \Lambda^2(r,t)(\mathrm{d}r + N^r\mathrm{d}t)^2 - R^2(r,t)\mathrm{d}\Omega^2\,, \qquad (1.115)$$

where $\mathrm{d}\Omega^2$ is the metric on the unit two-sphere, Λ and R are the dynamical variables (the only components that are left in $h_{ab}(\boldsymbol{x})$ after spherical symmetry is imposed), and N and N^r are Lagrange multipliers (the so-called lapse and shift functions). To encompass also the case of charged black holes, I shall include a spherically-symmetric electromagnetic one-form

$$A = \Phi(r,t)\mathrm{d}t + \Gamma(r,t)\mathrm{d}r\,. \qquad (1.116)$$

The classical constraints (1.111) and (1.112) then read

$$\begin{aligned}\mathcal{H} &= \frac{G}{2}\frac{\Lambda P_\Lambda^2}{R^2} - G\frac{P_\Lambda P_R}{R} + \frac{\Lambda P_\Gamma^2}{2R^2} \\ &\quad + G^{-1}\left(\frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}\right) \approx 0\,, \qquad (1.117)\\ \mathcal{D}_r &= P_R R' - \Lambda P_\Lambda' - \Gamma P_\Gamma' \approx 0\,. \qquad (1.118)\end{aligned}$$

In addition, we have Gauß' law,

$$\mathcal{G} = -P_\Gamma' \approx 0\,. \qquad (1.119)$$

(The explicit form of $\mathcal{D}_r$ is different from Kiefer (1998), since we have redefined in the action $\Phi \to \Phi - N^r\Gamma$.) The variables P_Λ,P_R, P_Γ are the momenta canonically conjugate to Λ, R, Γ, respectively.

I consider for Σ a three-space that in the classical picture of figure 1.4 would correspond to a hypersurface that starts at the bifurcation point of the horizons (taken to be $r \to 0$) and spatial infinity ($r \to \infty$). A crucial role in the whole procedure is played by the careful discussion of *boundary conditions* at $r \to 0$ and $r \to \infty$.

I shall first consider boundary conditions at $r \to \infty$. To avoid the unwanted conclusion that the Lagrange multipliers N and Φ vanish there one has to add the boundary term (Kiefer 1998)

$$-G\int \mathrm{d}t\, N_+ M - \int \mathrm{d}t\, \Phi_+ q \qquad (1.120)$$

in the action; N_+ (Φ_+) is the limiting value of N (Φ); M is the ADM mass and q is the electric charge. A canonical transformation then exhibits that (except for variables at $r \to 0$, see below) the only dynamical variables are M and q.

What about boundary conditions at the bifurcation point $r \to 0$? I wish to adopt boundary conditions that enforce every classical solution to be (part of) an exterior region of a Reissner-Nordström black hole, see (1.21); the constant t hypersurfaces are asymptotic to the constant Killing time hypersurfaces as $r \to 0$. It turns out that the extremal case ($\sqrt{G}M = |q|$) has boundary conditions *different* from the non-extremal case ($\sqrt{G}M > |q|$), see Kiefer and Louko (1998).

Consider first the non-extremal case. The variables N, Λ, R then exhibit the following asymptotic behaviour at $r \to 0$:

$$N(r,t) = N_1(t)r + \mathcal{O}(r^3)\,, \tag{1.121a}$$
$$\Lambda(r,t) = \Lambda_0(t) + \mathcal{O}(r^2)\,, \tag{1.121b}$$
$$R(r,t) = R_0(t) + R_2(t)r^2 + \mathcal{O}(r^4)\,. \tag{1.121c}$$

To avoid the unwanted conclusion that $N_1 = 0$, a boundary term similar to (1.120) must be added at $r \to 0$,

$$(2G)^{-1}\int \mathrm{d}t\, N_0 R_0^2\,, \tag{1.122}$$

where $N_0 \equiv N_1/\Lambda_0$. The quantity

$$\alpha \equiv \int_{t_1}^{t} \mathrm{d}t\, N_0(t) \tag{1.123}$$

plays the role of a 'rapidity' because it boosts the normal vector to the constant t hypersurfaces at $r \to 0$.

For the extremal case, one has instead of (1.121) the boundary conditions

$$N(r,t) = \Lambda^{-1}R'(\tilde{N}_0(t) + \mathcal{O}(r))\,, \tag{1.124a}$$
$$\Lambda(r,t) = \Lambda_{-1}(t)r^{-1} + \mathcal{O}(1)\,, \tag{1.124b}$$
$$R(r,t) = R_0(t) + R_1(t)r + \mathcal{O}(r^2)\,. \tag{1.124c}$$

The falloff (1.124b), in particular, encodes the fact that in the extremal case the point at $r \to 0$ is infinitely far away, $\int_0^{r_1} g_{rr}\mathrm{d}r \to \infty$. It then turns out from the action that *no* term of the form (1.122) has to be added. The geometrical reason for this lies in the fact that the boundary term is proportional to the surface gravity κ, and one has $\kappa = 0$ for the extremal case, cf. (1.94).

For the canonical formalism, one needs canonical *pairs* of variables at the boundaries. This is achieved by the introduction of the variables τ, λ, and α, and *parametrisation* (Kiefer 1998),

$$N_+(t) = \dot{\tau}(t)\,, \tag{1.125a}$$

$$\Phi_+(t) = \dot{\lambda}(t)\,, \tag{1.125b}$$

$$N_0(t) = \dot{\alpha}(t)\,. \tag{1.125c}$$

(In the extremal case, the last condition is absent.) The physical interpretation of these variables is as follows: τ is the proper time at $r \to \infty$, λ the gauge parameter, and α is the rapidity (1.123). While τ and λ are conjugate to mass and charge, respectively, α is conjugate to the area A of the event horizon. The remaining quantum constraints can than be solved, and a plane-wave-like solution reads

$$\Psi(\alpha,\tau,\lambda) = \chi(M,q)\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{A(M,q)\alpha}{8\pi G} - M\tau - q\lambda\right)\right]\,. \tag{1.126}$$

$\chi(M,q)$ is an arbitrary function of M and q; one can construct superpositions of the solutions (1.126) in the standard way by integrating over M and q.

Varying the phase in (1.126) with respect to M and q yields the classical equations

$$\alpha = 8\pi G\left(\frac{\partial A}{\partial M}\right)^{-1}\tau = \kappa\tau\,, \tag{1.127}$$

$$\lambda = \frac{\kappa}{8\pi G}\frac{\partial A}{\partial q}\tau = \Phi\tau\,. \tag{1.128}$$

The solution (1.126) holds for non-extremal holes. If one made a similar quantisation for extremal holes on their own, the first term in the exponent of (1.126) would be absent.

An interesting analogy with (1.126) is the plane-wave solution for a free nonrelativistic particle,

$$\exp(\mathrm{i}kx - \omega(k)t)\,. \tag{1.129}$$

As in (1.126), the number of parameters is one less than the number of arguments, since $\omega(k) = k^2/2m$. A quantisation for extremal holes on their own would correspond to choosing a particular value for the momentum, say p_0, at the classical level, and demanding that no dynamical variables (x,p) exist for $p = p_0$. This is, however, not the usual way to find classical correspondence – this is gained not from the plane-wave solution (1.129) but from *wave packets* that are obtained by superposing different wave numbers k. This then yields quantum states that are sufficiently concentrated around classical trajectories such as $x = p_0 t/m$.

It seems therefore appropriate to proceed similarly for black holes – construct wave packets for non-extremal holes that are concentrated around the classical values (1.127, 1.128) and then *extend* them by hand to the extremal limit. This would correspond to 'extremisation after quantisation', in contrast to the 'quantisation after extremisation' made above. Expressing in (1.126) M as a function of A and q and using Gaussian weight functions, one has

$$\begin{aligned} \Psi(\alpha,\tau,\lambda) = \int\limits_{A>4\pi q^2} & \mathrm{d}A\mathrm{d}q \, \exp\left[-\frac{(A-A_0)^2}{2(\Delta A)^2} - \frac{(q-q_0)^2}{2(\Delta q)^2}\right] \\ & \times \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{A\alpha}{8\pi G} - M(A,q)\tau - q\lambda\right)\right] . \end{aligned} \tag{1.130}$$

The result of this calculation is given and discussed in Kiefer and Louko (1998). As expected, one finds Gaussian packets that are concentrated around the classical values (1.127, 1.128). As for the free particle, the wave packets exhibit dispersion with respect to Killing time τ. Using for ΔA the Planck-length squared, $\Delta A \propto G\hbar \approx 2.6 \times 10^{-66}\mathrm{cm}^2$, one finds for the typical dispersion time in the Schwarzschild case

$$\tau_* = \frac{128\pi^2 R_0^3}{G\hbar} \approx 10^{65}\left(\frac{M}{M_\odot}\right)^3 \text{years} . \tag{1.131}$$

Note that this is just of the order of the black-hole evaporation time (1.88)! The dispersion of the wave packet just gives the time scale after which the semiclassical approximation breaks down.

Coming back to the charged case, and approaching the extremal limit $\sqrt{G}M = |q|$, one finds that the widths of the wave packet (1.130) are *independent* of τ for large τ. This is due to the fact that for the extremal black hole $\kappa = 0$ and therefore no evaporation takes place. If one takes, for example, $\Delta A \propto G\hbar$ and $\Delta q \propto \sqrt{G\hbar}$, one finds for the α-dependence of (1.130) for $\tau \to \infty$ the factor

$$\exp\left(-\frac{\alpha^2}{128\pi^2}\right) , \tag{1.132}$$

which is independent of both τ and $\hbar$. It is clear that this packet, although concentrated at the value $\alpha = 0$ for extremal holes, has support also for $\alpha \neq 0$ and is qualitatively not different from a wave packet that is concentrated at a value $\alpha \neq 0$ close to extremality.

An interesting question is the possible occurrence of a naked singularity (cf. figure 1.10) for which $\sqrt{G}M < |q|$. Certainly, both the boundary conditions (1.121) and (1.124) do not comprise the case of a singular three-geometry. However, the wave packets discussed above also contain parameter values that would correspond to the 'naked' case. Such geometries could be avoided if one imposed the boundary condition that the wave function vanishes for such values. But then continuity would enforce the wave function also to vanish on the boundary, i.e. at $\sqrt{G}M = |q|$. This would mean that extremal black holes could not exist at all in quantum gravity – an interesting speculation.

A possible thermodynamical interpretation of (1.126) can only be obtained if – analogous to section 1.4 – an appropriate transition into the euclidean regime is performed. This transition is achieved by the 'Wick rotations' $\tau \to -\mathrm{i}\beta\hbar$, $\alpha \to -\mathrm{i}\alpha_{\mathrm{E}}$ (from (1.123) it is clear that α is connected to the lapse function and must be treated similarly to τ), and $\lambda \to -\mathrm{i}\beta\hbar\Phi$. With an argument analogous to the one used below (1.103) – regularity of the euclidean line element – one arrives at the conclusion that $\alpha_{\mathrm{E}} = 2\pi$. But this means that the euclidean version of (1.127) just reads $2\pi = \kappa\beta\hbar$, which with $\beta = (k_{\mathrm{B}}T_{\mathrm{BH}})^{-1}$, is just the expression for the Hawking temperature (1.84)! Alternatively, one could use (1.84) to derive $\alpha_{\mathrm{E}} = 2\pi$.

The euclidean version of the state (1.126) then reads

$$\Psi_{\mathrm{E}}(\alpha,\tau,\lambda) = \chi(M,q)\exp\left(\frac{A}{4G\hbar} - \beta M - \beta\Phi q\right) . \tag{1.133}$$

One recognises in the exponent of (1.133) the occurrence of the Bekenstein-Hawking entropy (1.95). Of course, (1.133) is still a pure state and should not be confused with a partition sum. However, the transition to a partition sum is straightforward, and one then finds indeed from the standard thermodynamical relations in section 1.4 the expression for S_{BH}. Moreover, the factor $\exp[A/(4G\hbar)]$ in (1.133) directly gives the enhancement factor for the rate of black-hole pair creation relative to ordinary pair creation. It must be emphasised that S_{BH} fully arises form a boundary term at the horizon ($r \to 0$). This is similar to the path-integral approaches discussed in section 1.4 (The entropy arises there from a boundary term at the center of the disk in figure 1.17).

It is now clear that a quantisation scheme that treats extremal black holes as a limiting case gives $S_{\mathrm{BH}} = A/(4G\hbar)$ also for the extremal case. On the other hand, quantising extremal holes on their own would yield $S_{\mathrm{BH}} = 0$. From this point of view it is also clear why the extremal (Kerr) black hole that occurs in the transition from the disk-of-dust solution to the Kerr-solution has entropy $A/(4G\hbar)$, see Neugebauer's article in Hehl et al. (1998). If $S_{\mathrm{BH}} \neq 0$ for the extremal hole (that has temperature zero), the stronger version of the Third Law of thermodynamics (that would require $S \to 0$ for $T \to 0$) mentioned in section 1.2 apparently does not hold. This is not particularly disturbing, since many systems in ordinary

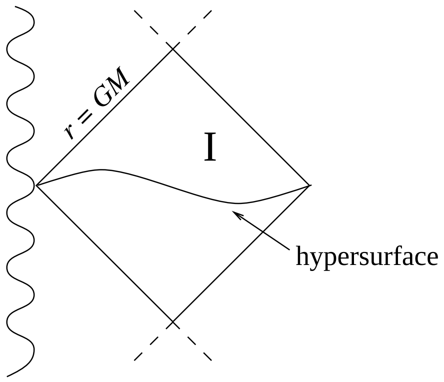

Figure 1.18. Spacelike hypersurface in the extremal Reissner-Nordström solution

thermodynamics violate the strong form of the Third Law; it just means that the system does not approach a unique state for $T \to 0$.

The topological difference of the classical charged black-hole solutions between extremal and non-extremal cases can immediately be inferred from figures 1.8 and 1.9. If the extremal case were quantised on its own, the reason for its vanishing entropy could be understood as follows. Consider first the non-extremal case (figure 1.8) and a spacelike hypersurface that starts at one of the bifurcation two-spheres and extends through the right part of region I up to spatial infinity i^0. Initial data on such hypersurfaces can be evolved only in the right part of region I; one could thus interpret the occurrence of the entropy $A/(4G\hbar)$ as signalling a 'lack of information' about the left part of region I and region II. (In this interpretation,'full' knowledge would refer to the evolution up to the Cauchy-horizon at $r = r_-$.) From a hypersurface that extends from the left i^0 up to the right i^0 one could infer the whole evolution up to r_-; in fact, no boundary term appears in the canonical analysis that would give rise to a term $A/(4G\hbar)$.

On the other hand, for a hypersurface that passed from point p in the extremal case (figure 1.9) through region I up to i^0 one could infer the whole evolution in region I from initial data on this hypersurface (see figure 1.18). Consequently, its entropy should be zero since there is no 'lack of information'. However, the situation shown in figure 1.18 cannot be reached by any continuous limit from figure 1.8. Moreover, it must be emphasised again that spacetime is a *classical* concept and that in particular a singular case such as the extremal hole may play no role in quantum gravity.

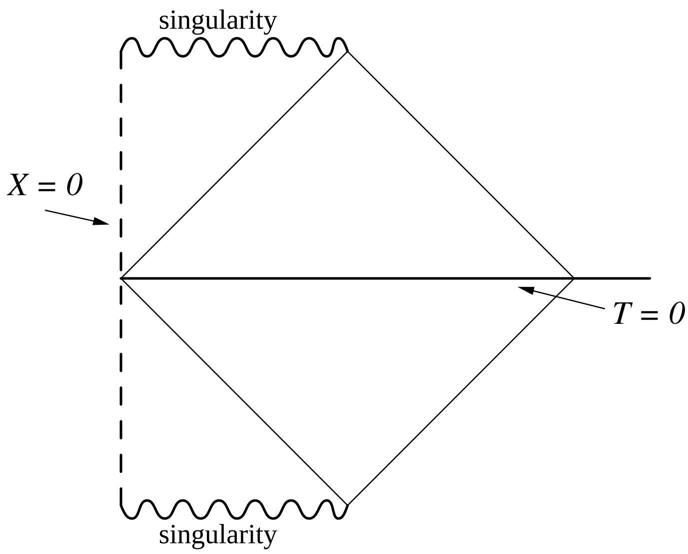

Figure 1.19. Spacelike hypersurface in the $\mathbb{RP}^3$ geon

An interesting example that I want to mention only in passing is the $\mathbb{RP}^3$ geon (Louko and Marolf 1998). This is an eternal black hole that is locally isometric to the Kruskal spacetime (figure 1.4), but contains only one exterior region. For this spacetime it was found that it possesses an entropy that is half of the Bekenstein-Hawking value, i.e. $A/(8G\hbar)$. The Penrose diagram for the $\mathbb{RP}^3$ geon is shown in figure 1.19; in which sense the result $S_{\rm BH}/2$ is related to some 'lack of information' is not yet clear.

In the case where additional fields are present, the above-discussed quantisation of black holes is only valid at the highest order of a semiclassical approximation (Kiefer 1998). Even at that order, the solution (1.126) is augmented by a factor $\exp({\rm i}S_0/(G\hbar))$, where S_0 is a solution to the functional Hamilton-Jacobi equation that follows from the constraints (1.113, 1.114) in this limit. From a discussion of S_0 one can also infer that the interior of the black hole horizon is always a classically allowed region – this is why the horizon shrinks to a point in the euclidean version that exhibits classically forbidden regions, see figure 1.17.

The next order of the semiclassical approximation makes it clear why the system can no longer be reduced to a system with finitely many degrees of freedom – field theoretic aspects play an important role. It is the level where Hawking radiation becomes manifest in quantum gravity. Unfortunately, the equations are much too complicated (in particular a viable regularisation is needed) for being solved. Therefore, the full evolution of black holes in canonical gravity remains unknown.

Even at the semiclassical level, non-classical black-hole quantum states can easily be constructed by using the superposition principle. However, most of such states *decohere*, i.e. become indistinguishable from a classical stochastic ensemble, through their own Hawking radiation (Giulini et al. 1996). Such a decoherence only follows for macroscopic (semiclassical) black holes; it does not occur for microscopic (virtual) black holes (more properly called black-and-white holes). The time symmetry of such microscopic states remains thus unbroken, and the loss of quantum coherence that is claimed to happen by scattering off vacuum fluctuations in which virtual black-and-white holes appear and disappear (see Hawking's article in Wald 1998) is spurious: As for the corresponding situation in QED (Giulini et al. 1996), no loss of quantum coherence should occur.

As far as the quantum state (1.126) is concerned, all variables and parameters are of a continuous nature, like for the free particle. It is, however, often speculated that mass and area are quantised (Bekenstein 1998). This can be found heuristically from (1.126) if it is assumed that the range of τ(or α) is compact – in a similar way one can find momentum quantisation on finite spaces, e.g. on a circle. Since $\Psi(\alpha,\tau,\lambda) = \Psi(\alpha,\tau+\Delta\tau,\lambda)$, one arrives at (Kastrup 1996)

$$\frac{M\Delta\tau}{\hbar} = 2\pi n\,. \tag{1.134}$$

Restricting to vanishing charge, one can assume that $\Delta\tau$ is proportional to the Schwarzschild radius,

$$\Delta\tau = \gamma R_0 = 2\gamma GM \tag{1.135}$$

with an unknown constant γ that is probably of order unity. This then yields for mass and area, respectively,

$$M_n = \sqrt{\frac{\pi n}{\gamma}}m_{\rm p} \quad \Rightarrow \quad A = \frac{16\pi^2}{\gamma}nl_{\rm p}^2\,. \tag{1.136}$$

Since one would expect that then also α has a finite range, one is led to

$$\frac{A\Delta\alpha}{8\pi G\hbar} = 2\pi \quad \Rightarrow \quad \Delta\alpha = \gamma \tag{1.137}$$

and therefore $\Delta\tau = 2\Delta\alpha GM$ (which apart from a factor 2 would follow from the classical relation (1.127)).

A similar quantisation would follow if one imposed an ad hoc Bohr-Sommerfeld quantisation rule in the euclidean version (recall $\alpha_{\rm E} = 2\pi = \gamma$)

$$nh = \oint \pi_{\alpha_{\rm E}} {\rm d}\alpha_{\rm E} = \int_0^{2\pi} \frac{A{\rm d}\alpha_{\rm E}}{8\pi G} = \frac{A}{4G}\,. \tag{1.138}$$

Whether these results survive a rigorous derivation, remains at present open. If true, however, this area quantisation would modify the thermal spectrum of black hole radiation found in the semiclassical limit – even for black holes much bigger than the Planck length (Bekenstein 1998)!

I want to emphasise finally that canonical quantum gravity can also address the issue of black holes in quantum cosmology (Kiefer and Zeh 1995), but this goes beyond the scope of this review.

1.6 Black holes in string theory

A much more ambitious framework for a quantum theory of gravity is *superstring theory.* In fact, this theory is not constructed through the application of standard quantisation rules to well-known classical theories, but through the use of a very different route that leads directly to a fundamental unified quantum theory. The various interactions – such as gravity – can then only be distinguished within this theory in certain limits.

The role of black holes in string theory has been extensively discussed in the lectures by Dijkgraaf and D'Auria. For this reason I shall be rather brief in this section and try mainly to discuss how this topic fits into the general scheme outlined in my earlier sections; for details I refer to the other lectures as well as to reviews such as Horowitz (1998) and Peet (1998).

Superstring theory (or 'M-theory') on its most fundamental level does not know any notion of spacetime, although it contains one fundamental length scale, $l_{\rm s} = \sqrt{\alpha'\hbar}$ (α' being the inverse string tension). Spacetime only emerges in a semiclassical approximation, very similar to canonical quantum gravity (Kiefer 1994). This is most conveniently expressed through an *effective action* that is found by an expansion with respect to $l_{\rm s}$ (or $\sqrt{\hbar}$); it is also a low-energy expansion since higher orders in $l_{\rm s}$ lead to higher order spacetime derivatives in the effective action. Usually, only low-energy effective actions are considered in which terms of $l_{\rm s}$ and higher are neglected; their classical solutions correspond to a gravitational theory including novel fields such as a dilaton, axion fields, etc. The presence of the dilaton field, in particular, gives rise to the effective string coupling constant $g_{\rm s}$ that connects Planck length and string scale,

$$l_{\rm p} \propto g_{\rm s} l_{\rm s} \,. \tag{1.139}$$

In the low-energy approximation, one can in particular address either the weak-coupling ($g_{\rm s} \ll 1$) or the strong-coupling limits ($g_{\rm s} \gg 1$). As has been shown in recent years, *dualities* connect these limits in different (or even the same) string theories. For $g_{\rm s} \ll 1$, perturbation theory can be applied. For $g_{\rm s} \gg 1$ one can (still on the semi-classical level!) find nonperturbative classical solutions to the effective actions – in particular black holes with various charges that are generalisations of the Reissner-Nordström solution (1.21). For $g_{\rm s} \gg 1$ one has $l_{\rm p} \gg l_{\rm s}$ and one would thus expect that canonical quantum gravity should be a good approximation.

The presence of dualities, together with (1.139), allows to address the issue of black-hole entropy in string theory. A special role is played by BPS-states, states where mass is equal to charge (in a precise sense). Certain 'nonrenormalisation theorems' guarantee that, while the coupling $g_{\rm s}$ is varied, this relation and the degeneracy of the states remain unchanged. Now, in the weak-coupling limit $l_{\rm p} \to 0$ and one is effectively in a flat spacetime; the theory predicts the existence of bound states of certain extended objects ('D-branes') in flat space whose degeneracy can easily be calculated from standard considerations. A corresponding entropy is then defined as the logarithm of this degeneracy.

As g_s increases, the (effective) theory yields extremal black-hole solutions. They have nonvanishing horizon area, and the interesting result is that $S_{\rm BH} = A/(4G\hbar)$ is exactly equal to the D-brane entropy. It is in this sense that a 'microscopic derivation' of $S_{\rm BH}$ has been provided; one must, however, emphasise that this is merely a consistency check, since the D-brane state has no resemblance to a black hole (there is, in particular, no horizon), and the connection is established only via duality arguments. Perhaps more surprising is the fact that for non-extremal black holes (but close to extremality), the full spectrum including the greybody factor $\Gamma_{\omega l}$, see (1.85), is recovered for the corresponding D-brane states. Unfortunately, for the general case (black holes far from extremality) no rigorous calculation exists as yet; in particular, the case of the ordinary Schwarzschild black hole remains elusive.

In the string calculations, unitarity is always preserved. This seems to indicate that no information would be lost during black-hole evaporation. Since Hawking radiation corresponds, in the string picture, to the emission of closed strings from D-branes, the mixed character of Hawking radiation would solely result form decoherence (Giulini et al. 1996) – the closed strings are quantum entangled with the D-brane states and integrating out the latter would then lead to a mixed state. It would be an interesting exercise to find out how the *thermal* nature of Hawking radiation arises in this picture.

I must emphasise again that the string calculations are only made on the semiclassical level – the same level as the canonical treatment in section 1.5. Going beyond would need taking higher order corrections in l_s into account; this is, however, up to now as untractable as the treatment of the higher order l_p-corrections in section 1.5. In particular, the full black-hole evaporation and the final decision about the information-loss problem remains elusive.

Can anything of the quantum aspects of black holes be observed in the foreseeable future? As was mentioned after (1.88), Hawking radiation can only be measured for primordial black holes (PBH) that were created with initial mass $\approx 5 \times 10^{14}$g in the early Universe. Black holes that result from stellar collapse are much too heavy to show noticeable radiation. Still, attempts have been and are still being done to look for the existence from a contribution of PBH distribution to the diffuse γ-background and by looking directly for the final evaporation of a single PBH (Halzen et al. 1991). The first method yields an upper limit onto the density of PBH of $\mathcal{N} \lesssim 10^4\,{\rm pc}^{-3}$, while the second method yields a (conservative) evaporation rate ${\rm d}n/{\rm d}t < 4.4 \times 10^5 {\rm pc}^{-3}{\rm yr}^{-1}$ (Funk 1997). PBH explosions could in the future be observed, for example, by the MILAGRO project[11] that measures secondary showers arising from primary γ-rays. If the final evaporation of black holes could be observed, this would open the first window towards an experimental test of quantum gravity.

Acknowledgements: I would like to thank the organisers of the school for inviting me to give lectures in a very pleasant atmosphere. I am also grateful to Jorma Louko and Jörg Thorwart for discussions and a critical reading of this manuscript.

[11] http://hana-mana.lanl.gov/milagro/MGRO_Homepage.html

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