Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism

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Motivation & Outline

Formal analogies between general relativity and electrodynamics: Matte 1953, Bel 1962
Physical analogies in gravitoelectromagnetism (GEM): Mashhoon et al. 1984
Question: Are there physical analogies that go beyond the linearized level?

- The Plebański–Demiański solution in brief
- Conventions, exterior calculus
- Curvature invariants and their similarities to electromagnetism
- The Bel–Robinson tensor and its 3-form
- The Kummer tensor
- Conclusions & Summary

The Plebański–Demiański solution in brief

Found by Plebański & Demiański in 1976.

It has **seven** free parameters and is of **Petrov type D** (Szekeres: “Coulomb-like”).

It can describe a massive,
rotating,
electrically & magnetically charged,
uniformly accelerating

black hole in a de Sitter background
with an additional NUT parameter.

Various subclasses: Schwarzschild, **Kerr**, C-metric, Taub–NUT, …

Physical interpretation: Griffiths & Podolský (2006)

Conventions, Exterior Calculus

We use exterior calculus. We have a **frame** e_μ and a dual **coframe** ϑ^ν:

$$e_\mu = e^b{}_\mu \partial_b, \quad \vartheta^\nu = e_a{}^\nu \, dx^a, \quad e_\mu \rfloor \vartheta^\nu = \delta^\nu_\mu$$

By means of the **metric**, we choose a pseudo-orthogonal frame and coframe:

$$\Big(g(e_\mu, e_\nu)\Big) = \Big(g_{\mu\nu}\Big) = \mathrm{diag}\Big(-1, 1, 1, 1\Big)$$

Expand **exterior forms** — say, the curvature 2-form — in terms of their components:

$$\mathrm{Riem}^\mu{}_\nu \;=\; \frac{1}{2!}\mathrm{Riem}_{\alpha\beta}{}^\mu{}_\nu \, \vartheta^\alpha \wedge \vartheta^\beta \;=\; \frac{1}{2!}\mathrm{Riem}_{ij}{}^\mu{}_\nu \, dx^i \wedge dx^j$$

anholonomic	holonomic
coframe	coordinate cobasis

Curvature invariants (for any type D spacetime)

1. **Kretschmann** scalar:

$$\mathsf{K} := -\star\left[\mathsf{Weyl}_{\alpha\beta}\wedge\left(\star\mathsf{Weyl}^{\alpha\beta}\right)\right] \;=\; \frac{1}{2}\mathsf{Weyl}_{\alpha\beta\gamma\delta}\mathsf{Weyl}^{\alpha\beta\gamma\delta} \;=\; -24\left(\mathbb{B}^2-\mathbb{E}^2\right)$$

2. **Chern–Pontryagin** pseudo-scalar:

$$\mathcal{P} := \star\left(\mathsf{Weyl}_{\alpha\beta}\wedge\mathsf{Weyl}^{\alpha\beta}\right) \;=\; \frac{1}{2}\left(\ast\mathsf{Weyl}_{\alpha\beta\gamma\delta}\right)\mathsf{Weyl}^{\alpha\beta\gamma\delta} \;=\; -48\,\mathbb{E}\,\mathbb{B}$$

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$$\mathsf{I}_1 := -\star\left[\mathsf{F}\wedge(\star\mathsf{F})\right] = \frac{1}{2}\mathsf{F}_{\alpha\beta}\mathsf{F}^{\alpha\beta} = \mathbf{B}^2-\mathbf{E}^2$$

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$$\mathsf{I}_2 := \star\left(\mathsf{F}\wedge\mathsf{F}\right) = \frac{1}{2}\left(\star\mathsf{F}_{\alpha\beta}\right)\mathsf{F}^{\alpha\beta} = 2\,\mathbf{E}\cdot\mathbf{B}$$

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Physical analogy between general relativity and electrodynamics (for **Kerr** solution):

$\mathbb{E}\propto$ mass (gravitational charge), $\mathbb{B}\propto$ angular momentum (gravitational current)
$\mathbf{E}\propto$ electric charge, $\mathbf{B}\propto$ electric current

The Bel–Robinson tensor

Related to **super-energy**. Traditionally defined as a $\binom{0}{4}$ tensor:

$$\tilde{\mathsf{B}}_{\mu\nu\rho\sigma} := \mathsf{Weyl}_{\mu\alpha\beta\rho}\mathsf{Weyl}_{\nu}{}^{\alpha\beta}{}_{\sigma} + (*\mathsf{Weyl}_{\mu\alpha\beta\rho})\left(*\mathsf{Weyl}_{\nu}{}^{\alpha\beta}{}_{\sigma}\right)$$

Physical dimension:

$$\left[\widetilde{\mathsf{B}}_{\mu\nu\rho\sigma}\right] = \left(\frac{\text{energy}}{\text{3-volume}}\right)^2$$

Algebraic properties:

$$\widetilde{\mathsf{B}}_{\mu\nu\rho\sigma} = \widetilde{\mathsf{B}}_{(\mu\nu\rho\sigma)}, \quad \widetilde{\mathsf{B}}^{\alpha}{}_{\nu\alpha\sigma} = 0$$

Similar to the electromagnetic energy momentum (tracefree and symmetric).

The Bel–Robinson 3-form

Geometrically, energy-momentum is a current 3-form.

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Analogous expressions for vacuum electrodynamics and general relativity:

$$2\,\Sigma_\mu := \quad F \wedge (e_\mu \rfloor \star F) \quad - \quad (\star F) \wedge (e_\mu \rfloor F)$$

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2\,\Sigma_\mu &:= \mathsf{F} \wedge (e_\mu \rfloor \star\mathsf{F}) - (\star\mathsf{F}) \wedge (e_\mu \rfloor \mathsf{F}) \\
\tilde{\Sigma}_{\nu\rho\sigma} &:= \mathsf{Weyl}_{\rho\alpha} \wedge (e_\nu \rfloor \star\mathsf{Weyl}^{\alpha}{}_{\sigma}) - (\star\mathsf{Weyl}_{\rho\alpha}) \wedge (e_\nu \rfloor \mathsf{Weyl}^{\alpha}{}_{\sigma})
\end{aligned}
$$

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\end{aligned}
$$

The symmetric energy-momentum tensors are derived analogously:

$$
\begin{aligned}
T_{\mu\nu} &:= e_\mu \rfloor (\star \Sigma_\nu) \\
\tilde{B}_{\mu\nu\rho\sigma} &:= e_\mu \rfloor \left(\star \tilde{\Sigma}_{\nu\rho\sigma}\right)
\end{aligned}
$$

Both yield "energy density" squared:

$$
\begin{aligned}
\star\left[\Sigma_\alpha \wedge (\star\Sigma^\alpha)\right] &= \left(\mathbf{E}^2 + \mathbf{B}^2\right)^2 \\
\star\left[\tilde{\Sigma}_{\alpha\beta\gamma} \wedge \left(\star\tilde{\Sigma}^{\alpha\beta\gamma}\right)\right] &= 24^2 \left(\mathbb{E}^2 + \mathbb{B}^2\right)^2
\end{aligned}
$$

Both are traceless:

$$
\begin{aligned}
\vartheta^\alpha \wedge \Sigma_\alpha &= 0 \\
\vartheta^\alpha \wedge \tilde{\Sigma}_{\alpha\rho\sigma} &= 0
\end{aligned}
$$

Kummer tensor

Cubic in curvature, can be defined for Riemann as well as Weyl.
Systematic introduction to electrodynamics and gravity by Baekler et al. (2014)

$$\mathsf{K}^{\mu\nu\rho\sigma} := \mathsf{Weyl}^{\alpha\mu\beta\nu}\,{}^{*}\mathsf{Weyl}^{*}{}_{\alpha\gamma\beta\delta}\mathsf{Weyl}^{\gamma\rho\delta\sigma}$$

It is related to so-called Kummer surfaces (propagation of waves),
and principal null directions of curvature.

Can be irreducibly decomposed into six pieces. There are two invariants:

$$\begin{aligned} \mathsf{S} &:= {}^{(1)}\mathsf{K}^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} &&= 24\mathbb{E}\left(3\mathbb{B}^2 - \mathbb{E}^2\right) \\ \mathcal{A} &:= \eta_{\alpha\beta\gamma\delta}{}^{(6)}\mathsf{K}^{\alpha\beta\gamma\delta} &&= 24\mathbb{B}\left(3\mathbb{E}^2 - \mathbb{B}^2\right) \end{aligned}$$

They are Bel's fundamental vacuum scalars.

Conclusions & Summary

Physical analogy between general relativity and electrodynamics (Kerr solution):

$\mathbb{E} \propto$ mass (gravitational charge), $\mathbb{B} \propto$ angular momentum (gravitational current)
$\mathbf{E} \propto$ electric charge, $\mathbf{B} \propto$ electric current

Result 1) can be generalized to the Plebański–Demiański solution.

Analogous expressions for vacuum electrodynamics and general relativity:

$$2\,\Sigma_\mu := \mathsf{F} \wedge (\mathsf{e}_\mu \rfloor \star\mathsf{F}) - (\star\mathsf{F}) \wedge (\mathsf{e}_\mu \rfloor \mathsf{F})$$

$$\tilde{\Sigma}_{\nu\rho\sigma} := \mathsf{Weyl}_{\rho\alpha} \wedge (\mathsf{e}_\nu \rfloor \star\mathsf{Weyl}^\alpha{}_\sigma) - (\star\mathsf{Weyl}_{\rho\alpha}) \wedge (\mathsf{e}_\nu \rfloor \mathsf{Weyl}^\alpha{}_\sigma)$$

Bel-Robinson 3-form $\tilde{\Sigma}_{\nu\rho\sigma}$: needs more attention.

More details in J.B., "Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism", arXiv:1412.1958 [gr-qc], submitted to Gen. Rel. Grav.

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“Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism” arXiv:1412.1958

Abstract:

Analogies between gravitation and electromagnetism have been known since the 1950s. Here, we examine the exact seven parameter solution of Plebański–Demiański (PD) to demonstrate these analogies for a physically meaningful spacetime:

The two quadratic curvature invariants are shown to take the form $\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E} \cdot \mathbf{B}$, and we show that $\mathbf{E}$ and $\mathbf{B}$ are related to the mass and angular momentum parameters of the PD solution, respectively.

Furthermore, the square of the Bel–Robinson tensor reads $\left(\mathbf{B}^2 + \mathbf{E}^2\right)^2$ for the PD solution, reminiscent of the square of the energy density in electrodynamics. Analogously to the energy-momentum 3-form of the electromagnetic field, we introduce the Bel–Robinson 3-form, from which the Bel–Robinson tensor can be derived.

We also determine the Kummer tensor, a tensor cubic in curvature, for the PD solution for the first time, and calculate the pieces of its irreducible decomposition.

The calculations are carried out in two coordinate systems: in the original polynomial PD coordinates, and in a modified Boyer–Lindquist-like version introduced by Griffiths and Podolský (GP) allowing for a more straightforward physical interpretation of the free parameters.

The Plebański–Demiański solution (1/2)

Found by Plebański & Demiański in 1976, expressed in the coordinates $\{\tau, \mathsf{q}, \mathsf{p}, \sigma\}$.
It has **seven** free parameters and is of **Petrov type D** (Szekeres: "Coulomb-like").
The pseudo-orthogonal coframe 1-forms read:

$$\vartheta^{\hat{0}} := \frac{1}{1-\mathsf{pq}}\sqrt{\frac{\mathscr{Q}(\mathsf{q})}{\mathsf{p}^2+\mathsf{q}^2}}\left(d\tau - \mathsf{p}^2 d\sigma\right),$$

$$\vartheta^{\hat{1}} := \frac{1}{1-\mathsf{pq}}\sqrt{\frac{\mathsf{p}^2+\mathsf{q}^2}{\mathscr{Q}(\mathsf{q})}}\, d\mathsf{q},$$

$$\vartheta^{\hat{2}} := \ominus\frac{1}{1-\mathsf{pq}}\sqrt{\frac{\mathsf{p}^2+\mathsf{q}^2}{\mathscr{P}(\mathsf{p})}}\, d\mathsf{p},$$

$$\vartheta^{\hat{3}} := \ominus\frac{1}{1-\mathsf{pq}}\sqrt{\frac{\mathscr{P}(\mathsf{p})}{\mathsf{p}^2+\mathsf{q}^2}}\left(d\tau + \mathsf{q}^2 d\sigma\right)$$

The metric is given by $g = g_{\alpha\beta}\,\vartheta^{\alpha}\otimes\vartheta^{\beta}$, with $(g_{\mu\nu}) = \mathrm{diag}\,(-1,1,1,1)$.
$\mathscr{Q}(\mathsf{q})$ and $\mathscr{P}(\mathsf{p})$ are fourth-order polynomials, prescribed by the Einstein–Maxwell equations: $\mathscr{Q}(\mathsf{q})'''' = \mathscr{P}(\mathsf{p})'''' = -8\Lambda$, where Λ is the cosmological constant.

The Plebański–Demiański solution (2/2)

The polynomials are given by

$$\mathscr{P}(\mathsf{p}) := \hat{\mathsf{k}} + 2\hat{\mathsf{n}}\mathsf{p} - \hat{\epsilon}\mathsf{p}^2 + 2\hat{\mathsf{m}}\mathsf{p}^3 + \left(\hat{\mathsf{k}} + \hat{\mathsf{e}}^2 + \hat{\mathsf{g}}^2 - \frac{\Lambda}{3}\right)\mathsf{p}^4,$$

$$\mathscr{Q}(\mathsf{q}) := \hat{\mathsf{k}} + \hat{\mathsf{e}}^2 + \hat{\mathsf{g}}^2 - 2\hat{\mathsf{m}}\mathsf{q} + \hat{\epsilon}\mathsf{q}^2 - 2\hat{\mathsf{n}}\mathsf{q}^3 + \left(\hat{\mathsf{k}} - \frac{\Lambda}{3}\right)\mathsf{q}^4.$$

The vector potential 1-form is

$$\mathsf{A} := \frac{1 - \mathsf{pq}}{\sqrt{\mathsf{p}^2 + \mathsf{q}^2}}\left(\frac{\hat{\mathsf{e}}\mathsf{q}}{\sqrt{\mathscr{Q}(\mathsf{q})}}\vartheta^{\hat{0}} + \frac{\hat{\mathsf{g}}\mathsf{p}}{\sqrt{\mathscr{P}(\mathsf{p})}}\vartheta^{\hat{3}}\right).$$

The free parameters are thus $\{\hat{\mathsf{m}}, \hat{\mathsf{n}}, \hat{\mathsf{e}}, \hat{\mathsf{g}}, \hat{\epsilon}, \hat{\mathsf{k}}, \Lambda\}$.

Griffiths & Podolský (1/2)

Physical interpretation of polynomial coordinates problematic.

New coordinates $\{t,\ r,\ \theta, \phi\}$ replace the polynomial $\{\tau, \mathsf{q}, \mathsf{p}, \sigma\}$.

$$
\begin{aligned}
\vartheta^{\hat{0}} &:= \frac{\sqrt{\Delta}}{\Omega\rho}\left[\mathrm{d}t - \left(a\sin^2\theta + 4\ell^2\sin^2\frac{\theta}{2}\right)\mathrm{d}\phi\right]\\
\vartheta^{\hat{1}} &:= \frac{\rho}{\Omega\sqrt{\Delta}}\mathrm{d}r\\
\vartheta^{\hat{2}} &:= \frac{\rho}{\Omega\sqrt{\chi}}\mathrm{d}\theta\\
\vartheta^{\hat{3}} &:= \frac{\sqrt{\chi}\sin\theta}{\Omega\rho}\left\{a\,\mathrm{d}t - \left[r^2 + (a+\ell)^2\right]\mathrm{d}\phi\right\}
\end{aligned}
$$

The metric is given by $g = g_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta$, with $(g_{\mu\nu}) = \mathrm{diag}\,(-1,1,1,1)$.

The new parameters are $\{m, \ell, a, \alpha, e, g, \Lambda\}$.

Griffiths & Podolský (2/2)

The auxiliary functions are given by

$$\Delta := \omega^2 k + e^2 + g^2 - 2mr + \epsilon r^2 - 2\frac{\alpha}{\omega} n r^3 - \left(\alpha^2 k - \frac{\Lambda}{3}\right) r^4 \; = \; \left(\frac{\omega}{\alpha}\right)^2 \mathscr{Q},$$

$$\chi := 1 - \alpha_3 \cos\theta - \alpha_4 \cos^2\theta \; = \; \frac{\omega^2}{\alpha^2 a^2 \sin^2\theta} \mathscr{P},$$

$$\Omega := 1 - \frac{\alpha}{\omega} r(\ell + a\cos\theta) \; = \; 1 - pq,$$

$$\rho^2 := r^2 + (\ell + a\cos\theta)^2 \; = \; \frac{\omega}{\alpha}\left(p^2 + q^2\right).$$

The vector potential 1-form is

$$A := \frac{\Omega}{\rho}\left[\frac{er}{\sqrt{\Delta}}\,\vartheta^{\hat{0}} \; + \; \frac{g(\ell/a + \cos\theta)}{\sin\theta\sqrt{\chi}}\,\vartheta^{\hat{3}}\right].$$