If spin is present, the energy-momentum tensor is asymmetric[1]

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To Nico Giulini's 60th birthday colloquium on 29 Oct. 2019 at Bremen University. I'd like to thank Claus Lämmerzahl and Volker Perlick for the invitation.

- Giulini: "Energy-mom. tensors and mot. in SR" (Puetzf.& Lämm.2015)
- Giulini: "Laue's theorem revisited: energy-momentum tensors,.." (2018)
- Obukhov: "Elmg. energy and mom..." Ann.Phys.(Berlin) **520**, 830 (2008)
- T.Y. Cao: "Conceptional Dev. of 20th Centuries Field Theories" (2019)
- Hehl: "On the energy tensor of spin. mass. matter..." (ROMP, 1976)

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1. Momentum current

1.1 (Force) stress σ_{ab} in continuum mechanics

Classical body: gas, fluid, solid (elastic, plastic, viscous,...). Turn to measurable quantities:

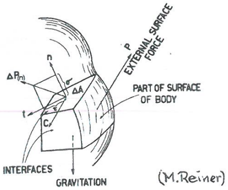
- Stress: Pascal, Euler, Navier, Cauchy,...stress tensor of 2nd rank, σ_{ab}, a,b,c,...=1,2,3, phys. dimension $f/l^2 \to$ pascal (Pa=N/m^2), $df^a = \sigma^a{}_b\, dA^b$, force df^a on the area element dA^b, σ_{ab} is asymmetric by definition. Exterior calculus $\boldsymbol{\sigma}^a = \frac{1}{2}\sigma^a{}_{bc} dx^b \wedge dx^c$, $\sigma^a{}_{(bc)} \equiv 0$. Bach parentheses $(ij) := \frac{1}{2}\{i+j\}$, $[ij] := \frac{1}{2}\{i-j\}$.
- Classical equilibrium conditions: $\partial_b\, \sigma_a{}^b = F_a\,, \quad \sigma_{[ab]} = 0\,.$
- Displacement gradient: $\partial_a u_b$. Strain $\varepsilon_{ab} = \frac{1}{2}\left(\overset{\text{def.}}{g_{ab}} - \overset{\text{undef.}}{g_{ab}}\right) = \partial_{(a} u_{b)}$
- Linear elasticity, Hooke's law:

$$\sigma_{ab} = c_{abcd}\, \varepsilon^{cd}, \qquad c_{[ab]cd} = c_{ab[cd]} = 0\,;\; c_{abcd} \overset{\textit{reversible}}{=} c_{cdab}$$

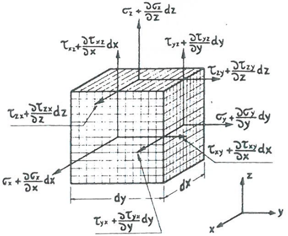
Irreducible decomp. under $GL(3,R)$: $\underbrace{c_{abcd}}_{21} = \underbrace{{}^{(1)}c_{abcd}}_{15} + \underbrace{{}^{(2)}c_{abcd}}_{6}$

${}^{(2)}c_{abcd} = 0$ is called Cauchy relations (special case, cf. Itin & H.)

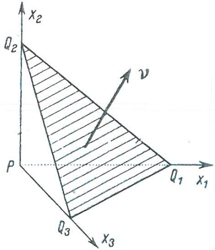
- Measurable quantities: stress σ_{ab}, strain ε_{ab}, elastic constants c_{abcd}



External and internal forces.



Components of tractions (M.Reiner)



(W.Prager)

$$\underbrace{df_a}_{\text{surface force}} = \underbrace{\sigma_{ab}}_{\text{(force) stress}} \cdot \underbrace{dA^b}_{\text{2d area element}}$$

$$\underbrace{dm_{ab}}_{\text{surface moment}} = \underbrace{\tau_{abc}}_{\text{moment stress}} \cdot \underbrace{dA^c}_{\text{2d area element}}$$

$$dm_{(ab)} = 0\,, \quad \tau_{(ab)c} = 0$$

1.2 Maxwell's stress tensor $\overset{\text{Max}}{\sigma}_{ab}$ in electrodynamics

• *Another theory of electricity, which I prefer, denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated.* James Clerk Maxwell (1870)

$$\overset{\text{Max}}{\sigma}_{ab} = D_a E_b + H_a B_b - \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{B})g_{ab}\,, \qquad \text{in vacuo, } \overset{\text{Max}}{\sigma}_{[ab]} = 0$$

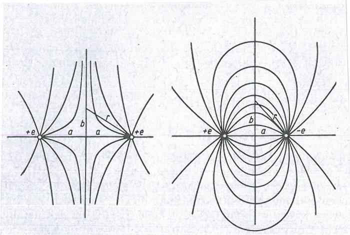
Exterior calculus: vector triad $\mathbf{e}_\alpha$, interior product $\rfloor$, exterior product $\wedge$

$$\overset{\text{Max}}{\boldsymbol{\sigma}}_a := \frac{1}{2}\left[(e_a\rfloor E)\wedge\mathcal{D} - (e_a\rfloor\mathcal{D})\wedge E \;+\; (e_a\rfloor\mathcal{H})\wedge B - (e_a\rfloor B)\wedge\mathcal{H}\right].$$

• Physical dimension of stress: $[\text{stress}] = [\underbrace{\text{mom.}}_{mv}\ \underbrace{\text{flux}}_{v}\ \underbrace{\text{density}}_{1/\ell^3}] = \frac{mv^2}{\ell^3} = \frac{f}{\ell^2}$

Lorentz' interpretation of the stress tensor as momentum flux density.
Light pressure as measured by Lebedew (1901), see Ashkin (1972) who moves small particles by laser light pressure.

Minkowski (1908) defines an asymmetric electromagnetic stress tensor (versus Abraham), see below; see the review paper of Yuri Obukhov, Annalen der Physik (2008)



Abstoßung gleicher und Anziehung entgegengesetzter gleicher Punktladungen, beschrieben durch die in der Symmetrieebene übertragenen Maxwellschen Spannungen (Becker - Sauter)

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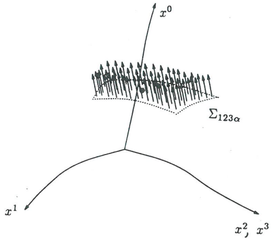

Stress as momentum current in 4-dimensional spacetime: The 3-dimensional (spacelike) hypersurface element carries a distribution of momenta.

1.3 Energy-momentum tensor $\mathfrak{T}_{ij}$ in electrodyn. and classical field theory

- In SR, coordinates x^i, with $i,j,k,...=0,1,2,3$, the stress $\overset{\text{Max}}{\sigma_{ab}}$ has to go along with momentum density, energy density, and energy flux density. The discovery in 4d of the energy-momentum tensor by Minkowski (1908):

$$(\mathfrak{T}_i{}^k) \sim \begin{pmatrix} \text{energy d.} & \text{energy flux d.} \\ \text{mom. d.} & \text{mom. flux d.} \end{pmatrix} \overset{\text{Max}}{\sim} \begin{pmatrix} u & s_a \\ p_a & \overset{\text{Max}}{\sigma_{ab}} \end{pmatrix}, \quad \partial \mathfrak{T}_i{}^k = f_i \,.$$

In a M_4, we can decompose $\mathfrak{T}_{ij}$ irreducible w.r.t. the Lorentz group:

$$\begin{aligned} \mathfrak{T}_{ij} &= \quad \mathfrak{\tilde T}_{ij} \quad + \quad \mathfrak{T}_{[ij]} \quad + \quad \frac{1}{4} g_{ij} \mathfrak{T}_k{}^k \\ 16 &= 9\,(\text{sym.tracefree}) \oplus 6(\text{antisym.}) \oplus 1\,(\text{trace})\,, \end{aligned}$$

- In electrodyn. in *vacuo*, only $\overset{\text{Max}}{\mathfrak{\tilde T}_{ij}}$ survives (9 comps.); it is massless, $\overset{\text{Max}}{\mathfrak{T}_k{}^k}=0$, and carries helicity, but no (Lorentz) spin, i.e., $\overset{\text{Max}}{\mathfrak{T}_{[ij]}}=0$. In *matter*, the Minkowksi energy-mom. tensor is asymm. (!), Abraham defined an alternative symm. one, Minkowski's result is consistent (see von Laue), as shown in the review of Obukhov (2008).

- Classical ideal (perfect, Euler) fluid of GR ($\rho =$ mass/energy density, $p =$ pressure, $u_i =$ velocity of fluid):

$$\mathfrak{T}_{ij} = (\rho + p)u_i u_j - p g_{ij}\,, \qquad \mathfrak{T}_{[ij]} = 0\,, \qquad \mathfrak{T}_k{}^k = \rho - 3p\,.$$

- Where took Einstein the symmetry $\mathfrak{T}_{[ij]} = 0$ of the energy-momentum tensor from? Einstein (*The Meaning of Relativity,* 1922, p.50) discussed the symmetry of the energy-momentum tensor of Maxwell's theory in vacuum. Subsequently, he argued: "*We can hardly avoid making the assumption that in all other cases, also, the space distribution of energy is given by a symmetrical tensor, $T_{\mu\nu}$, ...*" This is hardly a convincing argument if one recalls that the Maxwell field is massless and is of a "bosonic" type.

1.4 Action and canonical energy-momentum

- SR, Lorentz metric $g_{ij} \stackrel{*}{=} o_{ij} := \mathrm{diag}(+ - - -)$, Cart, coords. x^i, $i, j, k, ... = 0, 1, 2, 3$; matter field Ψ, could be a scalar, Weyl, Dirac, Maxwell, Proca, Rarita-Schwinger, Fierz-Pauli field etc.). Isolated material system with 1st order action (see Landau-Lifshitz, Corson):

$$W_{\mathrm{mat}} := \frac{1}{c} \int d\Omega \mathcal{L}(\Psi, \partial\Psi).$$

- Invariance under 4 transl.: $x'^{i} = x^i + a^i$. Noether theorem,

$$\boxed{\partial_k \mathfrak{T}_i{}^k = 0}, \quad \underbrace{\mathfrak{T}_i{}^k}_{4\times 4} := \mathcal{L}\delta_i^k - \frac{\partial \mathcal{L}}{\partial \partial_k \Psi} \partial_i \Psi\,, \qquad \text{if} \quad \frac{\delta \mathcal{L}}{\delta \Psi} = 0$$

canonical energy-momentum tensor of type $\binom{1}{1}$, Noether energy-momentum (or momentum current density), since asymmetric, 16 indep. comps., Whittaker: Minkowski's most important discovery.

[1.5 Translation and energy-momentum, premetric considserations, translation gauge theory

- Minkowski space is an affine space ("a vector space that lost its origin.") Weyl stressed this in his *Space, Time, Matter* as well as Kopczyński & Trautman in their textbook, see also Giulini, loc.cit.. Translations have nothing to do with a metric, i.e., the distance concept. Translations are premetric. Flat affine space (diff. manifold with a flat linear connection $\Gamma_\alpha{}^\beta$): Translations act therein. Thus, translation invariance plus Noether yield $\partial_k \mathfrak{T}_i{}^k = 0$. Gauging of the translations has a teleparallelism as a consequence.
- Transl. gauge theory is premetric. Transl. potential coframe ϑ^α, field strength torsion, $T^\alpha = D\vartheta^\alpha$, curv. vanishes $R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = 0$. Excitation or field momentum $H_\alpha \sim \partial V / \partial T^\alpha$, $H_\alpha = \chi_{\alpha\beta}(g) T^\beta$. Energy-momentum 3-form (premetric) Σ_α, energy-mom. tensor $\mathfrak{T}_\alpha{}^\beta = {}^\diamond(\vartheta^\beta \wedge \Sigma_\alpha)$. In analogy to electrodyn.:

$$\overset{\text{grav}}{\Sigma}_\alpha = \frac{1}{2}\left[T^\beta \wedge (e_\alpha \rfloor H_\beta) - H_\beta \wedge (e_\alpha \rfloor T^\beta)\right] \text{, prem. field eq. } \boxed{DH_\alpha - \overset{\text{grav}}{\Sigma}_\alpha = \overset{\text{mat}}{\Sigma}_\alpha},$$

see **Itin, Obukhov, Boos, H.** 2018 (in eldyn. $dH = J$, linear). If a Lorentz metric is assumed, which enters the const. law $H_\alpha = \chi_{\alpha\beta}(g) T^\beta$, we find the teleparallel equivalent $GR_{||}$ of GR.]

2. Spin current

2.1 Torque stress ${\tau_{ab}}^c$ in continuum mechanics

- Voigt 1887, $\sigma_{[ab]} \neq 0 \quad \longrightarrow \Sigma_{ab}$ (9 indep. comps.), on the surfaces of the Cauchy tetraeder, besides forces, there also act (spin) moment stresses $dm_{ab} = \tau_{abc} dA^c$, with $m_{(ab)} = \tau_{(ab)c} = 0$ (9 indep. comps., torque or couple stresses), on its volume also volume moments. Equilibrium of forces (3) and moments (3):

$$\partial_c {\Sigma_a}^c = F_a \,, \quad \partial_c {\tau_{ab}}^c + \Sigma_{[ab]} = M_{ab} \qquad [M_{(ab)} = 0] \,.$$

  It is a question to the material considered whether the stress is asymmetric. The classical body of continuum mechanics carry a symmetric stress, but not a spin fluid, e.g.

- E. & F. Cosserat 1908: They work out a classical field theory for continua with displacement u_a and rotation $\omega_{ab} = -\omega_{ba}$ and with force stress and moment stress: deformation (9) and contortion (9),

$$\beta_{ab} = \nabla_a u_b - \omega_{ab} \,, \; \kappa_{abc} = \nabla_a \omega_{bc} ; \quad \Sigma_{ab} \sim \beta_{ab} , \; \tau_{abc} \sim \kappa_{abc} .$$

- Cosserat 1d beams, 2d shells, 3d biaxial molecular fluids, spin fluids, liquid crystals, micomorphic media $\longrightarrow$ *media with microstructure,* see Capriz, Springer 1989.

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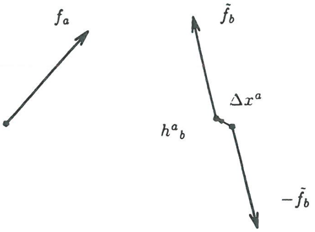
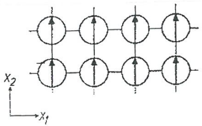
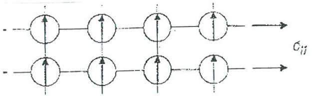


Force: Arrow symbolizing the force 1-form $f = f_a\, dx^a$.

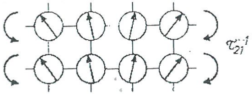
Hyperforce: Two opposite arrows displaced with respect to each other and symbolizing a hyperforce 1-form $h^a = h^a{}_b\, dx^b$ with $h^a{}_b = \lim \Delta x^a \tilde{f}_b$. Only after a suitable limiting transition with $\Delta x^a \to 0$ and $\tilde{f}_b \to \infty$, the double force becomes the hyperforce h^a.



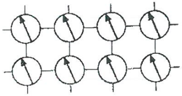
Schematic view on a two-dimensional Cosserat continuum: Unde-
nitial state.



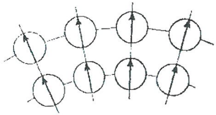
Conventional homogeneous strain ε_{11} of a Cosserat continuum: Dis-
anges of the "particles" caused by force stress σ_{11}.



Homogeneous contortion κ_{112} of a Cosserat continuum: Orientation
s of the "particles" caused by spin moment stress ${\tau_{21}}^{1}$.

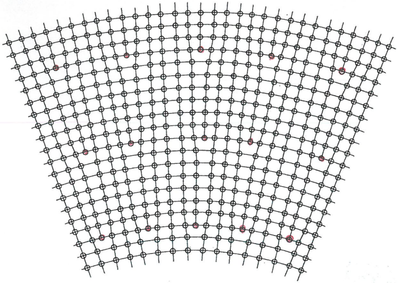


Homogeneous Cosserat rotation ω_{12} of the "particles" of a Cosserat continuum caused by the antisymmetric piece of the stress $\Sigma_{[12]}$.

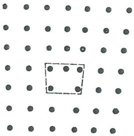


Conventional rotation $\partial_{[1}u_{2]}$ of the "particles" of a Cosserat continuum caused by an inhomogeneous strain.

- Macroscopic spin effects: Barnett, Einstein-de Haas
- $\rightarrow$ Elie Cartan 1922-24, inspired by the ideas of the Cosserats, developed an extension of GR to a theory with asymmetric energy-momentum tensors and spin: forerunner of EC-theory: curvature $\sim \kappa\cdot$ momentum, torsion $\sim \kappa\cdot$ spin
- Continuum theory of dislocations: Kazuo Kondo 1952, Bilby et al. 1956, Kröner & Seeger 1956..., see Kröner in Sommerfeld, Vol.2 (1964): (edge and screw) dislocation density $\alpha_{ab} \sim$ torsion, closure failure of infinitesimal parallelograms, 3d Riemann-Cartan space with torsion $T_{ab}^{c} = -T_{ba}{}^{c}$ and vanishing curvature $R_{abc}{}^{d} = 0$
- Kröner (1960) introduced moment stresses in the continuum theory of dislocations: $\tau_{ab}{}^{c} \sim T_{ab}{}^{c}$, similar to a 3d-version of the Einstein-Cartan theory. Plausibility considerations on the subsequent 2 pages:



NYE curvature (contortion) of a crystal lattice. No macroscopic elastic strain is involved.



a

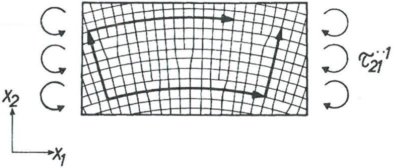
b

Couples as response to dislocations.

(E. Kröner)

Contortion and Moment Stress

Let us look at figure 3 in order to understand that a Riemannian space V_3 is too special to describe all types of deformations occuring



— Deformation of the crystal by edge dislocations of the $\alpha_{21}^{\cdot\cdot 1}$ type alters the relative orientation of the crystal structure. Thereby the vector in x_2-direction, parallelly displaced along the x_1-axis, will rotate : there occurs a closure failure of the infinitesimal parallelgram. The crystal's deformation will be maintained by the moment stress $\tau_{21}^{\cdot\cdot 1}$. The mean distances of the lattice points have *not* changed, hence no macroscopic strain and stress are produced.

2.2 Asymm. stress Σ_{ab} in superfluid Helium-3 in the A-phase

- Superfluid ^{3}He in the A-phase, is as spin fluid. Take the *angular momentum law*, see Vollhardt & Wölfle, *The Superfluid Phases of Helium 3*, Dover 2013: The antisym. piece of stress reads (p.427):

$$\underbrace{\epsilon_{ijk}\Pi_{jk}}_{\sim\epsilon_{abc}\mathfrak{T}^{bc}} = -\left(\frac{\partial}{\partial t} + \mathbf{v}_{\rm n}\cdot\boldsymbol{\nabla}\right)(t_0 l_i) + \nabla_j B_{ji} - \underbrace{\nabla_j}_{\sim\nabla^b}\{\underbrace{\frac{\hbar}{2m}g_{s,j}\, l_i}_{\sim\mathfrak{p}_b\epsilon_{acd}\mathfrak{s}^{cd}} + [\hat{\mathbf{l}}\times T\frac{\partial s}{\partial(\nabla_j\hat{\mathbf{l}})}]_i\}$$

$v_{\rm n}$ = velocity of normal fluid, t_0 = modulus of intrinsic angular momentum $\mathbf{t} = t_0\hat{\mathbf{l}}$, l_i = preferred direction of A-phase order parameter, s = entropy density, T = temperature, $g_{\rm s}$ = momentum density of superfluid component; this is an irrefutable proof that asymmetric stress tensors exist in nature

2.3 Spin angular momentum tensor $\mathfrak{S}_{ij}{}^k$ and the Dirac field

- Invariance under 3+3 infinitesimal Lorentz transformations: $x'^i = x^i + \omega^{ij}x_j$, with $\omega^{(ij)} = 0$, yields, via the Noether theorem and $\delta\mathcal{L}/\delta\Psi = 0$, angular-momentum conservation,

$$\partial_k\big(\underbrace{\mathfrak{S}_{ij}{}^k}_{\text{spin}} + \underbrace{x_{[i}\mathfrak{T}_{j]}{}^k}_{\text{orb. angular mom.}}\big) = 0\,, \qquad \underbrace{\mathfrak{S}_{ij}{}^k}_{6\times 4} := -\frac{\partial\mathcal{L}}{\partial\partial_k\Psi}\underbrace{f_{ij}}_{\text{Lor. gen.}}\Psi = -\mathfrak{S}_{ji}{}^k\,.$$

The canonical (Noether) spin $\mathfrak{S}_{ij}{}^k$, the spin current density, is a tensor of type $\binom{1}{2}$, plays a role in the interpretation of the Einstein-de Haas effect (1915). Components:

$$(\mathfrak{S}_{ij}{}^k) \sim \begin{pmatrix} \text{energy-dipole-moment d.} & \text{energy-dipole-moment flux d.} \\ \text{spin density} & \text{spin flux density} \end{pmatrix}$$

- We differentiate in the angular momentum law:

$$\boxed{\partial_k\mathfrak{S}_{ij}{}^k - \mathfrak{T}_{[ij]} = 0}\,.$$

Now it can be generalized to contorted and curved spacetimes.

- If $\mathfrak{S}_{ij}{}^{k} = 0$, then $\mathfrak{T}_{[ij]} = 0$, that is, the energy-momentum tensor is symmetric, but not necessarily vice versa.

- The irreducible decomposition, with the axial vector piece ${}^{\mathrm{AX}}\mathfrak{S}_{ijk} := \mathfrak{S}_{[ijk]}$ and the vector piece ${}^{\mathrm{VEC}}\mathfrak{S}_{ij}{}^{k} := \frac{2}{3}\mathfrak{S}_{[i|\ell}{}^{\ell}\delta^{k}_{|j]}$, reads:

$$
\begin{array}{rcl}
\mathfrak{S}_{ij}{}^{k} & = & {}^{\mathrm{TEN}}\mathfrak{S}_{ij}{}^{k} + {}^{\mathrm{VEC}}\mathfrak{S}_{ij}{}^{k} + {}^{\mathrm{AX}}\mathfrak{S}_{ij}{}^{k}\,, \\
24 & = & 16 \quad \oplus \quad 4 \quad \oplus \quad 4\,,
\end{array}
$$

- Point particle ansatz for a class. spin fluid of the convective type:

$$
\underbrace{\mathfrak{T}_{i}{}^{j}}_{\text{mom. curr. d.}} = \underbrace{\mathfrak{p}_{i}}_{\text{mom. d.}}\ \underbrace{u^{j}}_{\text{velocity}} \qquad \text{and} \qquad \underbrace{\mathfrak{S}_{ij}{}^{k}}_{\text{spin curr. d.}} = \underbrace{\mathfrak{s}_{ij}}_{\text{spin d.}}\ \underbrace{u^{k}}_{\text{velocity}} = -\mathfrak{S}_{ji}{}^{k}\,.
$$

The momentum density $\mathfrak{p}_i$ is no longer proportional to the velocity. Usually, the constraint $\mathfrak{s}_{ij}u^{j} = 0$ is assumed.

Dirac field in exterior calculus for illustration. Its Lagrangian reads,

$$L_{\rm D} = \frac{i}{2}(\overline{\Psi}{}^{\star}\gamma \wedge D\Psi + \overline{D\Psi} \wedge {}^{\star}\gamma\Psi) + {}^{\star}m\overline{\Psi}\Psi\,,$$

with $\gamma := \gamma_\alpha \vartheta^\alpha$ and $\gamma_{(\alpha}\gamma_{\beta)} = o_{\alpha\beta}\mathbf{1}_4$. The 3-forms of the canonical momentum and spin current densities are ($D_\alpha := e_\alpha \rfloor D$, here $\rfloor$ denotes the interior product sign):

$$\mathfrak{T}_\alpha = \frac{i}{2}(\overline{\Psi}\,{}^{\star}\gamma \wedge D_\alpha\Psi + \overline{D_\alpha\Psi} \wedge {}^{\star}\gamma\Psi)\,, \qquad \mathfrak{S}_{\alpha\beta} = \frac{1}{4}\vartheta_\alpha \wedge \vartheta_\beta \wedge \overline{\Psi}\gamma\gamma_5\Psi\,.$$

In Ricci calculus $\mathfrak{S}_{\alpha\beta\gamma} = \mathfrak{S}_{[\alpha\beta\gamma]} = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\overline{\Psi}\gamma_5\gamma^\delta\Psi$. Thus, the spin current is totally antisymmetric, $\overset{D}{\mathfrak{S}}_{ijk} = {}^{\rm AX}\overset{D}{\mathfrak{S}}_{ijk}$. Accordingly, we can introduce the spin flux *vector*

$$\mathcal{S}^i := \frac{1}{3!}\epsilon^{ijkl}\mathfrak{S}_{jkl} \quad \sim \quad \text{(spin flux density 1 comp., spin density 3 comps.)}\,.$$

Because of the equivalence principle, the inertial currents $\mathfrak{T}_\alpha$ and $\mathfrak{S}_{\alpha\beta}$ are, at the same time, the gravitational currents of the classical Dirac field. In general, fermionic fields generate asymmetric energy-momentum tensors.

2.4 Non-uniqueness of the momentum and spin currents

The Noether currents $\mathfrak{T}_\alpha{}^i$ and $\mathfrak{S}_{\alpha\beta}{}^i$ are only determined up to gradients. If we add gradients to $\mathfrak{T}_\alpha{}^i$ and $\mathfrak{S}_{\alpha\beta}{}^i$, we call it a *relocalization* of mom. and spin, see Hehl ROMP 1976 and Kirsch, Ryder, H. arXiv 2001:

Lemma 1: The canonical or Noether currents fulfill the conservation laws

$$D_i\,\mathfrak{T}_\alpha{}^i = 0\,, \qquad D_i\,\mathfrak{S}_{\alpha\beta}{}^i - 2\mathfrak{T}_{[\alpha\beta]} = 0$$

(here Ryder's conventions). Then the *relocalized* currents

$$\begin{aligned}
\hat{\mathfrak{T}}_\alpha{}^i(X) &= \mathfrak{T}_\alpha{}^i - D_j X_\alpha{}^{ij}\,, \\
\hat{\mathfrak{S}}_{\alpha\beta}{}^i(X,Y) &= \mathfrak{S}_{\alpha\beta}{}^i - 2X_{[\alpha\beta]}{}^i - D_j\,Y_{\alpha\beta}{}^{ij}\,,
\end{aligned}$$

satisfy the analog. cons. laws: $D_i\,\hat{\mathfrak{T}}_\alpha{}^i = 0$ and $D_i\,\hat{\mathfrak{S}}_{\alpha\beta}{}^i - 2\hat{\mathfrak{T}}_{[\alpha\beta]} = 0$.

The superpotentials $X_\alpha{}^{ij}(x) = -X_\alpha{}^{ji}$ and $Y_{\alpha\beta}{}^{ij}(x) = -Y_{\beta\alpha}{}^{ij} = -Y_{\alpha\beta}{}^{ji}$ represent $24 + 36$ *arbitrary* functions.

Lemma 2: The total energy-momentum and total angular momentum

$$P_\alpha :\overset{*}{=} \int_{H_t} \mathfrak{T}_\alpha{}^i \, dS_i \,, \quad J_{\alpha\beta} :\overset{*}{=} \int_{H_t} (\mathfrak{S}_{\alpha\beta}{}^i + 2x_{[\alpha} \mathfrak{T}_{\beta]}{}^i) \, dS_i \,,$$

are invariant under relocalization,

$$\hat{P}_\alpha \overset{*}{=} P_\alpha - \int_{\partial H_t} X_\alpha{}^{ij} \, da_{ij} \qquad \hat{J}_{\alpha\beta} \overset{*}{=} J_{\alpha\beta} - \int_{\partial H_t} \left(2x_{[\alpha} X_{\beta]}{}^{ij} + Y_{\alpha\beta}{}^{ij}\right) da_{ij} \,,$$

provided the superpotentials $X_\alpha{}^{ij}$ and $Y_{\alpha\beta}{}^{ij}$ approach zero at spacelike asymptotic infinity sufficiently fast. Here H_t denotes a spacelike hypersurface in Minkowski space with 3-volume element dS_i and ∂H_t its 2-dimensional boundary with area element $da_{ij} = -da_{ji}$. Orthonormal frames are used throughout.

Belinfante currents as example

The most straightforward approach to a relocalization is to put both, $\hat{\mathfrak{S}}_{\alpha\beta}{}^{i} = 0$ and $Y_{\alpha\beta}{}^{ij} = 0$. Then we find the Belinfante currents (1939)

$$t_\alpha{}^i := \overset{\mathrm{B}}{\mathfrak{T}}_\alpha{}^i(\overset{\mathrm{B}}{X}) \qquad \text{and} \qquad \overset{\mathrm{B}}{\mathfrak{S}}_{\alpha\beta}{}^i = 0 = \mathfrak{S}_{\alpha\beta}{}^i - 2\,\overset{\mathrm{B}}{X}_{[\alpha\beta]}{}^i$$

or

$$\overset{\mathrm{B}}{X}_\alpha{}^{ij} = \frac{1}{2}\left(\mathfrak{S}_\alpha{}^{ij} - \mathfrak{S}^{ij}{}_\alpha + \mathfrak{S}^j{}_\alpha{}^i\right) .$$

We collect our results with respect to the Belinfante relocalization in

$$D_i\, t_\alpha{}^i = 0\,, \quad t_{[\alpha\beta]} = 0\,, \quad t_\alpha{}^i = \mathfrak{T}_\alpha{}^i - \frac{1}{2} D_j\left(\mathfrak{S}_\alpha{}^{ij} - \mathfrak{S}^{ij}{}_\alpha + \mathfrak{S}^j{}_\alpha{}^i\right) .$$

This relocalization can be understood as one which kills the Belinfante spin current, i.e., the relocalized total angular momentum under this condition reduces to its orbital part alone, see also Rosenfeld 1940.

Since the canonical spin of the Dirac field is totally antisymmetric, it follows immediately that the Belinfante current $t_\alpha{}^i$ for a Dirac electron is the symmetric part of the canonical current $\mathfrak{T}_\alpha{}^i$,

$$t_{\alpha\beta} = \mathfrak{T}_{(\alpha\beta)}\,, \tag{2}$$

with $t_{\alpha\beta} = e_{i\beta}\, t_\alpha{}^i$ and $\mathfrak{T}_{\alpha\beta} = e_{i\beta}\, \mathfrak{T}_\alpha{}^i$.

Gordon decomposition of the currents of the Dirac field

- Gordon (1928) decomposed the Dirac current $\overline{\Psi}\gamma^i\Psi$ into a *convective* and a *polarization* part. An analogous procedure can be applied to the inertial currents. It yields, for $m \neq 0$, the gravitational moment densities of the Dirac field of the translation and of the Lorentz type, respectively:

$$M_\alpha{}^{ij} = \frac{i}{4m}\left[\overline{\Psi}\sigma^{ij}D_\alpha\Psi - D_\alpha\overline{\Psi}\sigma^{ij}\Psi\right], \quad M_{\alpha\beta}{}^{ij} = \frac{1}{8m}\overline{\Psi}(\sigma^{ij}\sigma_{\alpha\beta}+\sigma_{\alpha\beta}\sigma^{ij})\Psi\,.$$

  Universally valid procedure, has also been applied to the energy-momentum and spin currents of fields with spin 1, $\frac{3}{2}$, and 2.

- The gravitational moment densities, in the sense of the relocalization of Lemma 1, correspond to the choices

$$X_\alpha{}^{ij} = M_\alpha{}^{ij} + 2\delta_\alpha^{[i}\,M_k^{\cdot j]k}\,, \quad Y_{\alpha\beta}{}^{ij} = M_{\alpha\beta}{}^{ij}\,.$$

  Accordingly, for the relocalized currents we eventually find

$$\overset{\mathrm{G}}{\mathfrak{T}}_\alpha{}^i = \delta_\alpha^i\,\overset{\mathrm{G}}{\mathcal{L}} - \frac{1}{2m}\left(D^i\overline{\Psi}D_\alpha\Psi + D_\alpha\overline{\Psi}D^i\Psi\right)$$

$$\overset{\mathrm{G}}{\mathfrak{S}}_{\alpha\beta}{}^i = \frac{1}{4mi}\left[\left(D^i\overline{\Psi}\right)\sigma_{\alpha\beta}\Psi - \overline{\Psi}\sigma_{\alpha\beta}\left(D^i\Psi\right)\right]\,.$$

The Gordon Lagrangian reads

$$\overset{\mathrm{G}}{\mathcal{L}} := \frac{1}{2m}\left[(D_\alpha\overline{\Psi})D^\alpha\Psi - m^2\overline{\Psi}\Psi\right] . \tag{3}$$

In summary, for the Gordon type momentum and spin currents, we have

$$D_i \overset{\mathrm{G}}{\mathfrak{T}}_\alpha{}^i = 0\,, \qquad \overset{\mathrm{G}}{\mathfrak{T}}_{[\alpha\beta]} = 0\,, \qquad D_i \overset{\mathrm{G}}{\mathfrak{S}}_{\alpha\beta}{}^i = 0\,. \tag{4}$$

This seems to be the only way one can derive a relativistic *spin* density which is automatically *conserved* by itself. The Belinfante momentum $t_\alpha{}^i$ is also symmetric and the spin is conserved. However, in the Belinfante case the relocalized spin vanishes, i.e., the statement is trivial.

3. Currents of matter and gravitational theory

- Hilbert (1915); Weyl: Only the possibility of varying the metric in a Riemannian space leads to the true definition of the energy (in the context of GR): ${}^{\rm Hi}\mathfrak{t}_{ij} := 2\delta\mathfrak{L}_{\sf mat}/\delta g^{ij}$
- Sciama-Kibble (1961): Only the possibility of varying the Lorentz connection in a Riemann-Cartan space leads to the true definition of spin: ${}^{\rm SK}\mathfrak{S}_{\alpha\beta}{}^{i} = \delta\mathfrak{L}_{\sf mat}/\delta\Gamma_{i}{}^{\alpha\beta}$

3.1 Metric momentum current → general relativity (GR)

- How can we choose amongst the multitude of the relocalized energy-mom. tensors and spin tensors? Energy and spin distribution of matter (but not of gravity!) are observable quantities, at least in the classical domain. There must exist physically correct and unique energy-mom. and spin tensors in nature.
- Already in 1915, Hilbert defined the dynamic energy-momentum as

$$ {}^{\rm Hi}\mathfrak{t}_{ij} := 2\delta\mathfrak{L}_{\sf mat}(g, \Psi\,, \overset{\{\}}{\nabla}\Psi)/\delta g^{ij}\,; $$

g^{ij} (or its reciprocal g_{kl}) is the gravitational potential in GR. The matter Lagrangian is supposed to be *minimally coupled* to g^{ij}, in accordance with the equivalence principle.

The Hilbert definition is analogous to the relation from elasticity theory stress $\sim \delta$(elastic energy)/δ(strain). Recall that strain is defined as $\varepsilon^{ab} := \frac{1}{2}\left({}^{\rm def}g^{ab} - {}^{\rm undef}g^{ab}\right)$. Even the factor 2 is reflected in the Hilbert formula.

Rosenfeld (1940), via the Noether theorem, has shown that the Belinfante tensor ${}^{\mathsf{Bel}}\mathfrak{t}_{ij}$, derived within SR, coincides with the Hilbert tensor ${}^{\rm Hi}\mathfrak{t}_{ij}$ of GR. Thus, the Belinfante-Rosenfeld-Iskraut recipe yields

Lemma 3: *In the framework of GR, the Hilbert energy-momentum tensor*

$${}^{\rm Hi}\mathfrak{t}_{ij} = {}^{\mathsf{Bel}}\mathfrak{t}_{ij} = \mathfrak{T}_{ij} - \nabla_k\big(\mathfrak{S}_{ij}{}^k - \mathfrak{S}_j{}^k{}_i + \mathfrak{S}^k{}_{ij}\big) = {}^{\rm Hi}\mathfrak{t}_{ji}\,,$$

localizes the energy-momentum distribution correctly; here $(\mathfrak{T}_i{}^j, \mathfrak{S}_{ij}{}^k)$ are the canonical Noether currents. The spin tensor attached to ${}^{\rm Hi}\mathfrak{t}_{ij}$ vanishes.

3.2 Canonical momentum and spin currents → Einstein-Cartan (EC) and Poincaré gauge theory (PG)

- Gauging of the Poincaré group: gauge potentials orthonormal coframe $\vartheta^\alpha = e_i{}^\alpha dx^i$ (4×4) and the Lorentz connection $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$ (4×6). The spacetime arena of the emerging Poincaré gauge theory of gravity (PG) is a Riemann-Cartan space with Cartan's *torsion* (4×6) and with Riemann-Cartan *curvature* (6×6) as gauge field strength, respectively:

$$T_{ij}{}^\alpha := \nabla_{[i} e_{j]}{}^\alpha,\ R_{ij}{}^{\alpha\beta} := \text{“}\nabla\text{”}_{[i}\Gamma_{j]}{}^{\alpha\beta} \qquad (or\ T^\alpha = D\vartheta^\alpha,\ R^{\alpha\beta} = \text{“}D\text{”}\,\Gamma^{\alpha\beta}).$$

- The energy-momentum and angular momentum laws generalize to

$$\overset{*}{\nabla}_k\, \mathfrak{T}_i{}^k = \underbrace{T_{ik}{}^\ell}_{\text{torsion}}\, \mathfrak{T}_\ell{}^k + \underbrace{R_{ik}{}^{lm}}_{\text{curvature}}\, \mathfrak{S}_{lm}{}^k\,, \qquad \overset{*}{\nabla}_k\, \mathfrak{S}_{ij}{}^k - \mathfrak{T}_{[ij]} = 0\,;$$

here $\overset{*}{\nabla}_k := \nabla_k + T_{k\ell}{}^\ell$. GR is the subcase for $\mathfrak{S}_{ij}{}^k = 0$. The material currents are defined by variations with respect to the potentials:

$$^{\mathrm{SK}}\mathfrak{T}_\alpha{}^i = \frac{\delta\mathfrak{L}_{\mathrm{mat}}(e, \Gamma, \Psi\,, \overset{\Gamma}{D}\,\Psi)}{\delta e_i{}^\alpha}, \quad ^{\mathrm{SK}}\mathfrak{S}_{\alpha\beta}{}^i = \frac{\delta\mathfrak{L}_{\mathrm{mat}}(e, \Gamma, \Psi\,, \overset{\Gamma}{D}\,\Psi)}{\delta\Gamma_i{}^{\alpha\beta}}\,.$$

This Sciama-Kibble definition of the spin (1961) is only possible in the Riemann-Cartan spacetime of PG. It is analogous to the relation moment stress $\sim \delta$(elastic energy)$/\delta$(contortion) in a Cosserat type medium, the contortion being a "rotational strain".

The Lagrange-Noether machinery as applied to the minimally coupled action function yields,

$$ {}^{\rm SK}\mathfrak{T}_\alpha{}^i = \mathfrak{T}_\alpha{}^i\,, \qquad {}^{\rm SK}\mathfrak{S}_{\alpha\beta}{}^i = \mathfrak{S}_{\alpha\beta}{}^i\,. $$

The dynamically defined energy-momentum and spin currents à la Sciama-Kibble coincide with the canonical Noether currents.

Lemma 4: *Within PG, the energy-momentum and the spin of matter are distributed in accordance with the canonical Noether currents* $\mathfrak{T}_\alpha{}^i$ *and* $\mathfrak{S}_{\alpha\beta}{}^i$*, respectively.*

This is in marked contrast to the doctrine in the context of GR.

Express the canon. energy-mom. tensor in terms of the Hilbert one,

$$ {}^{\rm SK}\mathfrak{T}_\alpha{}^i = \mathfrak{T}_\alpha{}^i = {}^{\rm Hi}\mathfrak{t}_\alpha{}^i + \overset{*}{\nabla}_k \left(\mathfrak{S}_\alpha{}^{ik} - \mathfrak{S}^{ik}{}_\alpha + \mathfrak{S}^k{}_\alpha{}^i\right), \qquad {}^{\rm SK}\mathfrak{S}_{\alpha\beta}{}^i = \mathfrak{S}_{\alpha\beta}{}^i\,. $$

The new Rosenfeld formula reverses its original meaning in GR!

- These results on the correct distribution of material energy-momentum and spin in the framework of PG are are *independent* of a specific choice of the *gravitational* Lagrangian. However, if we choose the RC curvature scalar as a gravitational Lagrangian, we arrive at the Einstein-Cartan(-Sciama-Kibble) theory of gravitation (EC), which is a viable theory of gravity competing with GR.
- The way we treated the the spin current, one can also treat the dilation and the shear current. They are related to the $GL(4, R)$ and to possible violations of the Lorentz invariance...

Soli Deo Gloria.