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The Kummer tensor density in electrodynamics and in gravity

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ABSTRACT

Guided by results in the premetric electrodynamics of local and linear media, we introduce on 4-dimensional spacetime the new abstract notion of a Kummer tensor density of rank four, $\mathcal{K}^{ijkl}$. This tensor density is, by definition, a cubic algebraic functional of a tensor density of rank four $\mathcal{T}^{ijkl}$, which is antisymmetric in its first two and its last two indices: $\mathcal{T}^{ijkl} = -\mathcal{T}^{jikl} = -\mathcal{T}^{ijlk}$. Thus, $\mathcal{K} \sim \mathcal{T}^3$, see Eq. (46). (i) If $\mathcal{T}$ is identified with the electromagnetic response tensor of local and linear media, the Kummer tensor density encompasses the generalized *Fresnel wave surfaces* for propagating light. In the reversible case, the wave surfaces turn out to be *Kummer surfaces* as defined in algebraic geometry (Bateman 1910). (ii) If $\mathcal{T}$ is identified with the *curvature* tensor R^{ijkl} of a Riemann–Cartan spacetime, then $\mathcal{K} \sim R^3$ and, in the special case of general relativity, $\mathcal{K}$ reduces to the Kummer tensor of Zund (1969). This $\mathcal{K}$ is related to the *principal null directions* of the curvature. We discuss the properties of the general Kummer tensor density. In particular, we decompose $\mathcal{K}$ irreducibly under the 4-dimensional linear group $GL(4, R)$ and, subsequently, under the Lorentz group $SO(1, 3)$.

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1. Introduction

1.1. Fresnel surface

We consider electromagnetic waves propagating in a homogeneous, transparent, dispersionless, and nonconducting crystal. The response of the crystal to electric and magnetic perturbations is assumed to be *local* and *linear*. The permittivity tensor ε^{ab} $(a, b = 1, 2, 3)$ of the crystal[1] is anisotropic in general, and the same is true for its *im*permeability tensor μ_{ab}^{-1}. Such materials are called special bi-anisotropic (see [4–6]). If ε^{ab} and μ_{ab}^{-1} are assumed to be symmetric, these bi-anisotropic materials are characterized by 12 independent parameters. In many applications, however, μ_{ab}^{-1} can be considered approximately to be isotropic $\mu_{ab}^{-1} = \mu_0^{-1} g_{ab}$; here g_{ab} is the 3-dimensional Euclidean metric tensor. Such cases were already studied experimentally and theoretically in the early 19th century, before Maxwell recognized the electromagnetic nature of light in 1862 (see [7]). To carry out these investigations on geometric optics, one used the notions of light ray and of wave vector, and one was aware (Young, 1801)[2] that light was transverse and equipped with a polarization vector.

At each point inside a crystal, we have a ray vector and a wave covector (one-form). It is then possible to determine the *ray surface* and its dual, the *wave surface,* for visualizing how a pulse of light is propagating. The ray surface was first constructed by Fresnel (1822) and is conventionally called *Fresnel surface,* an expression also used for the wave surface, see the popular introduction by Knörrer [8]. Since the symmetric permittivity tensor can be diagonalized, the Fresnel surface is described by 3 principal values. In the case when all of them are equal, the Fresnel surface is an ordinary 2-dimensional sphere. For two unequal parameters, the surface is the union of two shells, a sphere and an ellipsoid. These two shells touch at two points. In Fig. 1, we display such a surface for a crystal with *three different principal* values of the permittivity: $(\varepsilon^{ab}) = \begin{pmatrix} \varepsilon^1 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix}$ and $(\mu_{ab}^{-1}) = \mu_0^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is a union of two shells that meet at 4 singular points. As already recognized by Hadamard [9], the wave surfaces are the characteristics of the corresponding partial differential equations describing the wave propagation.

The classical 3-parameter Fresnel surface is naturally generalized when the anisotropy of the impermeability μ_{ab}^{-1} is taken into account. Removing also the diagonalizability requirement, one deals with *generalized Fresnel surfaces* of less than 12 parameters. Such surfaces were derived in terms of positive definite symmetric dyadics by Lindell [11], see also [12].

Moreover, it seems to be rather natural to extend ε and μ to asymmetric tensors, which emerge if dissipative processes are involved. Recently, one of us [13] derived a tensorial expression of such a generalized 18-parameter Fresnel surface. The derivation does not require the corresponding matrices to be real, symmetric, positive definite, or even invertible.

We here will derive such tensorial expressions for generalized Fresnel surfaces by proceeding differently. We include first magnetoelectric effects and subsequently look for the corresponding 4-dimensional relativistic covariant generalizations of the 3-dimensional permittivity and impermeability tensors.

1.2. Magnetoelectricity

In the 1960s, substances were found that, if exposed to a magnetic field B^a, were electrically polarized $D^a = \alpha^a{}_b B^b$ and, reciprocally, if exposed to an electric field E_a, were magnetized, $H_a = \beta_a{}^b E_b$, see O'Dell [14]. These are small effects of the order $10^{-3}\sqrt{\varepsilon_0/\mu_0}$, or smaller.

Such materials are characterized by the constitutive moduli ε^{ab}, μ_{ab}^{-1}, $\alpha^a{}_b$, $\beta_b{}^a$, which can be accommodated in a 6 × 6 matrix. Originally, all these moduli were assumed to obey the *symmetry*

[1] The position of the indices are chosen always in accordance with the conventions of premetric electrodynamics, see [1–3].

[2] Thomas Young (1773–1829), English mathematician and physicist.

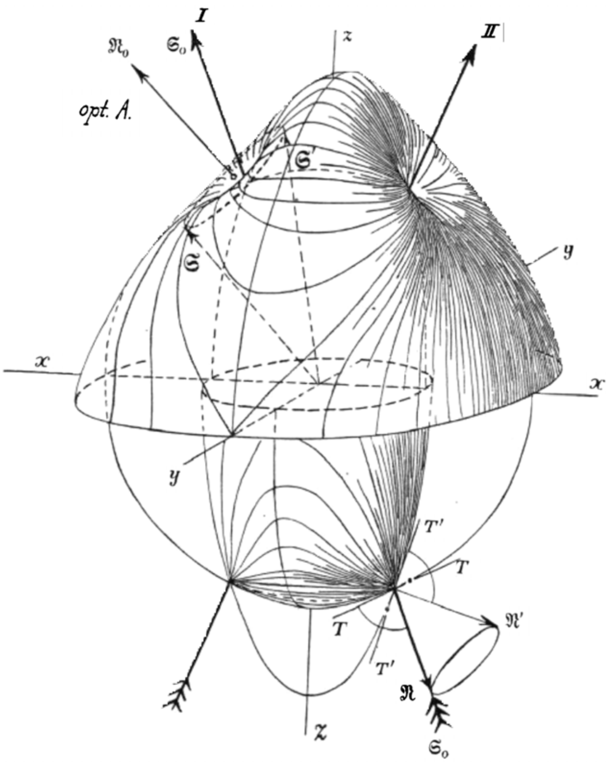

Fig. 1. Fresnel wave surface as the specific *quartic* surface $(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)(x^2+y^2+z^2)-[\alpha^2(\beta^2+\gamma^2)x^2+\beta^2(\gamma^2+\alpha^2)y^2+\gamma^2(\alpha^2+\beta^2)z^2]+\alpha^2\beta^2\gamma^2=0$, with the 3 parameters $\alpha := c/\sqrt{\varepsilon_1},\ \beta := c/\sqrt{\varepsilon_2},\ \gamma := c/\sqrt{\varepsilon_3}$ and $c=$ vacuum speed of light (from Schaefer [10], image by J. Jaumann). The upper half depicts the exterior shell with the funnel shaped singularities, the lower half the inner shell. The two optical axes are denoted by **I** and **II**. The two shells cross each other at four points forming cusps. The wave vectors are denoted by $\mathfrak{N}$ and $\mathfrak{N}'$. All essential facts of crystal optics are encoded in this drawing. Such quartic Fresnel surfaces and their non-trivial generalizations are at center stage of our article.

conditions

$$\varepsilon^{ab}=\varepsilon^{ba}, \qquad \mu^{-1}_{ab}=\mu^{-1}_{ba}, \quad \text{and} \quad \alpha^a{}_b=-\beta_b{}^a, \tag{1}$$

yielding altogether $2\times 6+9=21$ moduli characterizing a material.

Seemingly, Bateman in 1910 [15] was the first to investigate such materials. He studied electromagnetic wave propagation in Maxwell's theory. His constitutive relations, in his notation, were

$$\begin{aligned} -B_1 &= \kappa_{11}H_1+\kappa_{12}H_2+\kappa_{13}H_3+\kappa_{14}D_1+\kappa_{15}D_2+\kappa_{16}D_3,\\ &\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\ E_3 &= \kappa_{61}H_1+\kappa_{62}H_2+\kappa_{63}H_3+\kappa_{64}D_1+\kappa_{65}D_2+\kappa_{66}D_3. \end{aligned} \tag{2}$$

Bateman assumed the existence of 21 moduli $\kappa_{IJ}=\kappa_{JI}$, for $I,J=1,\ldots,6$, believing that *"These conditions* [the constitutive relations] *may not correspond to anything occurring in nature; nevertheless their investigation was thought to be of some mathematical interest on account of the connection which is established between line-geometry and the theory of partial differential equations."* Then, *"The general Kummer's surface appears to be the wave surface for a medium of a purely ideal character..."*.

Bateman's 21 independent moduli κ_{IJ} – in contrast to what he thought – are not only a mathematical abstraction. Rather, they do have applications in physics, since they coincide with the 21 moduli displayed in (1).

Of course, nowadays it is desirable to display the constitutive law (1) or (2), respectively, in a 4-dimensional covariant and premetric way, see [1,14,2]; Whittaker [16], vol. 2, pp. 192–196, provides a short history of the premetric program. We collect the fields D^a and H_a in the 4d electromagnetic excitation tensor density $\mathcal{H}^{ij}$ $(=-\mathcal{H}^{ji})$ and the fields E_a and B^a in the electromagnetic fields strength tensor F_{ij} $(=-F_{ji})$, with the coordinate indices $i, j, k, \ldots = 0, 1, 2, 3$. Then, Maxwell's equations, with the electric current density $\mathcal{J}^j$, read

$$\partial_j \mathcal{H}^{ij} = \mathcal{J}^j, \qquad \partial_{[i}F_{jk]} = 0. \tag{3}$$

We assume that the crystal considered above responds locally and linearly if exposed to electromagnetic fields. As a consequence, the constitutive law, with the *electromagnetic response tensor density* $\chi^{ijkl}(x)$, reads as follows:

$$\mathcal{H}^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}, \quad \text{where } \chi^{ijkl} = -\chi^{jikl},\ \chi^{ijkl} = -\chi^{ijlk}. \tag{4}$$

The response tensor density χ can be mapped to a 6×6 matrix with 36 independent components. This matrix can be decomposed in its trace-free symmetric part (20 independent components), its antisymmetric part (15 components), and its trace (1 component). On the level of χ, we have then the irreducible decomposition under $GL(4, R)$:

$$\begin{aligned} \chi^{ijkl} &= {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}, \\ 36 &= 20 \oplus 5 \oplus 1, \end{aligned} \tag{5}$$

see [2,5,3]. The third part, the *axion* part, is totally antisymmetric and as such proportional to the Levi-Civita symbol, ${}^{(3)}\chi^{ijkl} := \chi^{[ijkl]} = \alpha\, \epsilon^{ijkl}$ (see also [17–19]). The second part, the *skewon* part, is defined according to ${}^{(2)}\chi^{ijkl} := \frac{1}{2}(\chi^{ijkl} - \chi^{klij})$, see also [20,21]. If the constitutive equation can be derived from a Lagrangian, which is the case as long as only reversible processes are considered, then ${}^{(2)}\chi^{ijkl} = 0$. This corresponds to Bateman's symmetry postulate $\kappa_{[IJ]} = 0$. The *principal* part ${}^{(1)}\chi^{ijkl}$ fulfills the symmetries ${}^{(1)}\chi^{ijkl} = {}^{(1)}\chi^{klij}$ and ${}^{(1)}\chi^{[ijkl]} = 0$.

1.3. *Wave propagation: Tamm–Rubilar tensor density $\mathcal{G}^{ijkl}[\chi]$*

In Section 1.1, we studied the Fresnel wave surfaces of crystals with an anisotropic dielectric constant ε^{ab}. Let us generalize these considerations to the response tensor density χ^{ijkl} of (4), with its 36 independent components. The 4d wave covector is denoted by $q_i = (\omega, -\mathbf{q})$, with the frequency ω and the 3d wave vector $\mathbf{q}$. As known from the literature, see [2] and the references given there, one arrives at the generalized Fresnel equation, which is again *quartic* in the wave covector:

$$\mathcal{G}^{ijkl}[\chi]\, q_i q_j q_k q_l = 0. \tag{6}$$

The Tamm–Rubilar (TR) tensor density $\mathcal{G}$ is totally symmetric, $\mathcal{G}^{ijkl}[\chi] = \mathcal{G}^{(ijkl)}[\chi]$, and has 35 independent components.[3] The explicit definition reads, see [2], Eq. (D.2.22),

$$\mathcal{G}^{ijkl}[\chi] := \frac{1}{4!}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,\chi^{mnr(i}\,\chi^{j|ps|k}\,\chi^{l)qtu}, \tag{7}$$

that is, it is *cubic* in χ. The totally antisymmetric Levi-Civita tensor density is denoted by $\epsilon_{ijkl} = \pm 1, 0$, and the parentheses $^{(\,)}$ mark symmetrization over the enclosed indices; however, the indices standing between the two vertical strokes $^{|\ |}$, here p and s, are excluded from the symmetrization process, see [22]. It turns out that $\mathcal{G}[\chi]$ depends only on ${}^{(1)}\chi$ and ${}^{(2)}\chi$; the axion piece ${}^{(3)}\chi$ drops out.

Itin [23] developed a new and generally covariant method for deriving the generalized Fresnel equation ("dispersion relation"). His explicit expression for the TR-tensor density looks different from (7). However, Obukhov [24], by straightforward but cumbersome algebra, was able to show that the results are *equivalent*. In this context, Obukhov was able, by using the double dual, to put (7) into what

[3] Since by (6) the overall factor in the definition of $\mathcal{G}$ is conventional, $\mathcal{G}$ effectively has only 34 independent components.

Itin called "Probably the most symmetric form..." of the TR-tensor density, namely[4]

$$\mathcal{G}^{ijkl}[\chi] = \frac{1}{3!}\,\chi^{a(ij|b}\,{}^{\diamond}\chi^{\diamond}{}_{acbd}\,\chi^{c|kl)d}; \tag{8}$$

here we have the left-dual ${}^{\diamond}\chi_{ij}{}^{kl} := \frac{1}{2}\epsilon_{ijmn}\chi^{mnkl}$, the right-dual $\chi^{\diamond}{}^{ij}{}_{kl} := \frac{1}{2}\chi^{ijmn}\epsilon_{klmn}$, and, thus, the double dual ${}^{\diamond}\chi^{\diamond}{}_{ijkl} = \frac{1}{4}\epsilon_{ijmn}\chi^{mnrs}\epsilon_{klrs}$. This version was derived directly by Lindell [26] by using a dyadic calculus, see the discussion of Favaro [27].

The subject of our article is the Kummer tensor density $\mathcal{K}^{ijkl}[\chi]$ that we find from (8) by dropping the factor 1/3! and the symmetrization parentheses $^{(\cdot\cdot|\cdot\cdot|\cdot\cdot)}$. It should be well understood that the expressions of the TR-tensor density in Eqs. (7) and (8) are *equivalent* to each other. However, if we generalize the TR-tensor density in Section 2, it is decisive that we start exactly from Eq. (8).

1.4. Kummer surface and optics

We already saw that, as one examines increasingly structured materials, the Fresnel images of the wave surfaces become more complicated, and depend on a wider set of characteristic parameters. However, Eq. (6) specifies that, even in the general case, we are dealing with a *quartic* hypersurface in the 4d space of wave covectors $q_i = (\omega, -\mathbf{q})$. Plots such as Fig. 1 are derived by taking the section $\omega = 1$, and represent the inverse phase velocity of light.

In 1864, Kummer [28,29] discovered a family of quartic surfaces that would later play a major role in the development of algebraic geometry. Kummer surfaces belong to the 3d real projective space RP^3, but can be derived from hypersurfaces in the 4d space $R^4 - \{0\}$, by taking a section. This is usually regarded as a change of coordinates, from the homogeneous $(t, \mathbf{x})$ to the inhomogeneous $(1, \mathbf{x}/t)$, see also the explanations and visualizations in Lord [30].

As we establish below, for a dispersionless linear medium that is only required to have zero skewon part, the Fresnel surface coincides with the general Kummer surface. Is this equality a big surprise? Perhaps, not. Kummer's original motivation for investigating quartic surfaces came from optics. He wanted to improve Hamilton's geometric optics (1832) and was interested, for example, in how the atmosphere of the Earth modifies the image of the Sun or a planet [31]. In fact, he started to examine ray bundles of the second order, that is, ray bundles that have at one point two independent rays, like in a birefringent crystal. Given that rays in 4d space are projected in a 3d space, the tool for this analysis was projective geometry.

The surfaces considered in the articles of 1864 [28,29] were named after Kummer by Hudson, who wrote an authoritative book [32] on the subject. However, let us emphasize that already, in 1846, Cayley [33] found special Kummer surfaces – so-called tetrahedroids – whilst investigating light propagation, see also [32].

Let us turn to the visualization of Kummer surfaces. In Hudson [32] one can find, in suitably chosen coordinates, a parametrization of the general Kummer surface. However, we will take a parametrization of the Mathematica library [34]: the Kummer surfaces are a family of quartic surfaces. If we specify 4 coordinates (w, x, y, z), a representation is given by

$$(x^2 + y^2 + z^2 - \mu^2 w^2)^2 - \lambda pqrs = 0, \quad \text{where } \lambda := \frac{3\mu^2 - 1}{3 - \mu^2}, \tag{9}$$

and p, q, r, s are tetrahedral coordinates,

$$p := w - z - \sqrt{2}x, \tag{10}$$

$$q := w - z + \sqrt{2}x, \tag{11}$$

$$r := w + z + \sqrt{2}y, \tag{12}$$

$$s := w + z - \sqrt{2}y. \tag{13}$$

---

[4] By using the duality operators, Eq. (7) reads $\mathcal{G}^{ijkl}[\chi] = \frac{1}{3!}\,{}^{\diamond}\chi_{ab}{}^{c(i}\,\chi^{j|ad|k}\,\chi^{\diamond l)b}{}_{cd}$. Schuller et al. [25], in their work on the area metric, took also our original version of the TR-tensor, as displayed in (7).

In the projective space description, the frequency w can be put to 1. Decisive geometrical properties of a Kummer surface depend on the value of the parameter μ. This parameter turns out to be cubic in 20 of Bateman's 21 constitutive components κ_{IJ}, see Section 1.5. Originally, a Kummer surface is defined in a real 3-dimensional space, but extensions to 3 complex dimensions are straightforward.

A modern image of a Kummer surface, due to Rocchini [35], and to be compared with Fig. 1, looks as follows (Fig. 2):

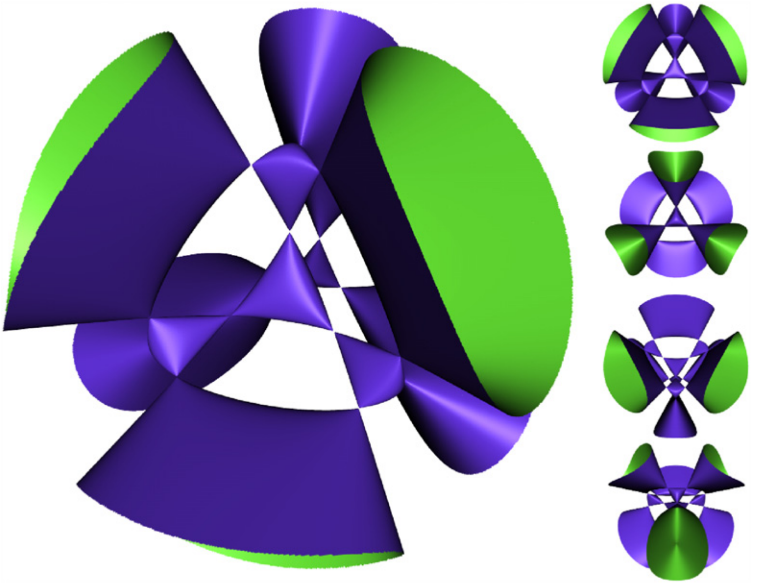

Fig. 2. A Kummer surface due to Rocchini [35]; for gypsum and other models of Kummer surfaces, see Rowe [36].

We now consider the equality, for media with zero skewon piece, of Fresnel and Kummer surfaces, which was first demonstrated by Bateman [15] in 1910. Electromagnetic waves have the property that the fields E_a and B^a are orthogonal. The same holds true for H_a and D^a. In addition, the electric and magnetic parts of the energy density are always equal. Even though the above statements invoke the notion of orthogonality, a premetric formulation is possible, see [2], p.126, as,

$$\epsilon^{ijkl}F_{ij}F_{kl} = 0 \quad \Leftrightarrow E_aB^a = 0, \tag{14}$$

$$\epsilon_{ijkl}\mathcal{H}^{ij}\mathcal{H}^{kl} = 0 \Leftrightarrow H_aD^a = 0, \tag{15}$$

$$\mathcal{H}^{kl}F_{kl} = 0 \quad \Leftrightarrow E_aD^a = H_aB^a. \tag{16}$$

The tensor density $\epsilon^{ijkl} = \pm 1, 0$ is the contravariant analog of ϵ_{ijkl}. Eqs. (14)–(16) define geometric optics itself. For the general local and linear electromagnetic response (4), they become:

$$\epsilon^{IJ}F_IF_J = 0, \tag{17}$$

$$\epsilon_{MN}\chi^{MI}\chi^{NJ}F_IF_J = 0, \tag{18}$$

$$\chi^{IJ}F_IF_J = 0, \tag{19}$$

where $I, J, M, N = 1, \ldots, 6$ denote pairs of antisymmetric indices. If the medium has a vanishing skewon part, (17)–(19) can be attributed a meaning in projective geometry. The first equation dictates that F_{ij} corresponds to a line in RP^3. As a matter of fact, one can recognize that (17) identifies, effectively, the Klein quadric. The antisymmetric (0, 2)-tensors that obey $\chi^{ijkl}F_{ij}F_{kl} = 0$ and $\epsilon^{ijkl}F_{ij}F_{kl} = 0$ correspond, in RP^3, to a quadratic line complex. As a matter of fact, the electromagnetic response tensor density χ^{ijkl} specifies a quadric, just like ϵ^{ijkl} specifies the Klein quadric. Finally, imposing (18) as well as (17) and (19) entails that F_{ij} corresponds, in RP^3, to a singular line of the quadratic line complex. These direct links between geometric optics and projective geometry are explained in Bateman's article [15]. Nonetheless, the reader may want to consult the papers [37–39] by Delphenich, for a modern treatment.

In RP^3, the singular lines of the quadratic line complex are tangent to the Kummer surface, as verified by Jessop [40]. Remarkably, the Fresnel surface is also a result of Eqs. (17)–(19). From this, it is possible to deduce that, for the local and linear media with ${}^{(2)}\chi^{ijkl} = 0$, the two surfaces coincide [15].

Alternatively, one can prove such equality as follows, cf. Ruse [41,42]. Eq. (17) is solved by $F_{ij} = 2q_{[i}a_{j]}$. The quantities q_i and a_i turn out to be the 4-dimensional wave and polarization covectors. A trivial substitution into (19) yields:

$$\mathcal{W}_q^{ij}[\chi]a_ia_j = 0, \tag{20}$$

$$\mathcal{W}_q^{ij}[\chi] := \chi^{ikjl}q_kq_l. \tag{21}$$

One recalls that q_i and a_i correspond to points in the real projective space. Hence, (20) states that $\mathcal{W}_q^{ij}$ determines a complex cone in RP^3 for a_i at q_i. To be precise, only the symmetric part of the tensor (21) contributes to (20). Accordingly, it is useful to pick a vector basis $\{e_{\bar{0}}, \ldots, e_{\bar{3}}\}$ such that $\mathcal{W}_q^{(\bar{i}\bar{j})} = \mathrm{diag}(\lambda_0, \ldots, \lambda_3)$, with real eigenvalues. This allows one to reformulate (20) as:

$$\lambda_0a_{\bar{0}}^2 + \lambda_1a_{\bar{1}}^2 + \lambda_2a_{\bar{2}}^2 + \lambda_3a_{\bar{3}}^2 = 0. \tag{22}$$

Notably, the Kummer surface is the locus of points q_i in RP^3 such that the complex cone (22) simplifies to two planes. By inspecting a table of the quadratic surfaces in 3-dimensions, one deduces that such factorization takes place exclusively if $\mathcal{W}_q^{(ij)}$ has two vanishing eigenvalues. For instance, when $\lambda_0 = \lambda_1 = 0, \lambda_2 < 0$ and $\lambda_3 > 0$, the complex cone is equal to two real intersecting planes. Moreover, if $\lambda_0 = \lambda_1 = 0, \lambda_2 > 0$ and $\lambda_3 > 0$, Eq. (22) specifies two imaginary intersecting planes. All other cases are discussed in Section 4.18 of the manual [43]. Demanding that $\mathcal{W}_q^{(ij)}$ has two zero eigenvalues implies that its rank must be 0, 1 or 2. If the electromagnetic response has zero skewon part, as we assumed, the tensor (21) is symmetric. So, when q_i corresponds, in RP^3, to a point on the Kummer surface, the rank of $\mathcal{W}_q^{ij}$ is 0, 1 or 2. Analogously, q_i belongs to the Fresnel surface if and only if $\mathcal{W}_q^{ij}$ has rank less than or equal to two [2,26,23]. In consequence, Kummer surfaces are also Fresnel surfaces, and vice-versa.

Before we collect our results, let us quote the precise formulation from Avritzer and Lange [44]: "*It is well-known that the singular surface of a generic quadratic complex is a Kummer surface, i.e. a quartic surface in* $\mathbb{P}^3$, *smooth apart from* 16 *ordinary double points. Moreover every Kummer surface appears as the singular surface of a generic quadratic complex*".

Summing up (see also [45]): the Kummer surface emerged during studies of the light propagation in local and linear media. Thus, light stood at the cradle of the Kummer surface. The Fresnel surface of an electromagnetic response tensor density χ^{ijkl}, whose skewon part vanishes, $\chi^{ijkl} - \chi^{klij} = 0$, is a Kummer surface. Moreover, every Kummer surface appears as the Fresnel surface of a generic electromagnetic response tensor density, whose skewon part vanishes.

1.5. Kummer surface and gravity

Yes, Kummer started from optical considerations, more exactly from the propagation of electromagnetic disturbances in the geometric optics limit. Why should then Kummer's considerations be relevant for gravity?

Well, both the electromagnetic and the gravitational fields are massless, that is, they propagate with the speed of light, and their waves are transversal with helicity 1 and helicity 2, respectively. However, electromagnetic waves solve the *linear* Maxwell equations, whereas gravitational waves obey the *nonlinear* Einstein equation. Of course, the Einstein equation is still linear in its highest (2nd) order derivatives.

Let us recall how the TR-tensor density emerges in *electrodynamics*. We substitute the local and linear constitutive law $\mathcal{H}^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}$ into the source-free inhomogeneous Maxwell equation $\partial_j\mathcal{H}^{ij} = 0$.

Moreover, we express the field strength in terms of the potential, $F_{kl} = 2\partial_{[k}A_{l]}$, and achieve

$$\partial_j\big(\chi^{ijkl}\partial_k A_l\big) = 0 \quad \Rightarrow \quad q_j\big(\chi^{ijkl}q_k a_l\big) = 0 \tag{23}$$

with $q_i \neq 0$ being the wave-covector. Here, the geometric optics limit is shown immediately—see [2] for details. By introducing the characteristic matrix (21), one can rewrite (23) as a homogeneous system of linear equations,[5]

$$\mathcal{W}_q^{ij}[\chi]a_j = 0. \tag{24}$$

The field strength as determined in the geometric optics limit, $F_{ij} = 2q_{[i}a_{j]}$, is non-zero iff (24) has two linearly independent solutions, or more. We are thus prompted to identify the wave-covectors such that the rank of $\mathcal{W}_q^{ij}$ is lower than 3. The search for all $q_i \neq 0$ that have this property leads to the Fresnel equation (6). Next, (7) specifies the Tamm–Rubilar and eventually, though not uniquely, the Kummer tensor density (Section 1.3, last paragraph).

Before we turn to the gravitational case in more detail, let us first fix our corresponding notation. Our conventions are taken from [46], unless stated otherwise. We work in the 4-dimensional *Riemann–Cartan* (RC) *spacetime* of the Poincaré gauge theory of gravitation (see [47]), a generalization of GR. Such a spacetime is endowed with torsion $T_{ij}{}^k$ and curvature $R_{ijk}{}^l$. If the torsion vanishes, we recover the Riemannian spacetime, and the Riemannian quantities are denoted by a tilde $\widetilde{\ }$, see also Obukhov [48].

In *general relativity* (GR), the field strength, representing the tidal forces, is the Riemann curvature tensor,[6]

$$\widetilde{R}_{ijk}{}^l := 2(\partial_{[i}\widetilde{\Gamma}_{j]k}{}^l + \widetilde{\Gamma}_{[i|m}{}^l\,\widetilde{\Gamma}_{|j]k}{}^m), \tag{25}$$

$$\text{with } \widetilde{\Gamma}_{ij}{}^k := \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{26}$$

Here, the Riemann (or Levi-Civita) connection $\widetilde{\Gamma}_{ij}{}^k$ is defined in terms of the gravitational potential, the metric g_{ij}.

Owing to the symmetries $\widetilde{R}^{(ij)kl} = \widetilde{R}^{ij(kl)} = 0$, $\widetilde{R}^{ijkl} = \widetilde{R}^{klij}$ and $\widetilde{R}^{[ijkl]} = 0$, the Riemann curvature tensor has 20 independent components. We separate these into 10 independent entries for the Ricci tensor $\widetilde{\mathrm{Ric}}_{ij} := \widetilde{R}_{kij}{}^k$, and 10 independent entries for the Weyl tensor $\widetilde{C}^{ijkl}$. The latter, also known as the conformal Weyl curvature, is the trace-free part. Contracting the Ricci tensor with the inverse metric leads to the curvature scalar, $\widetilde{R} := g^{ij}\widetilde{\mathrm{Ric}}_{ij}$. We remove this single independent component from the Ricci tensor, and obtain the trace-free part "Ricsymf". In conclusion, see [46],

$$\begin{aligned} \text{Riem} &= \text{Weyl} \oplus \text{Ricsymf} \oplus \text{Scalar}. \\ 20 &= \quad 10 \ \oplus \quad 9 \quad \oplus 1. \end{aligned} \tag{27}$$

This was the kinematics of the gravitational field strength $\widetilde{R}^{ijkl}$. Let us now turn to the dynamics: the Einstein field equation without cosmological constant and sources,

$$\widetilde{\mathrm{Ric}}_{ij} - \frac{1}{2}g_{ij}\widetilde{R} = 0 \quad \text{or} \quad \widetilde{\mathrm{Ric}}_{ij} = 0, \tag{28}$$

determines that, in vacuum, only the Weyl tensor $\widetilde{C}^{ijkl}$ is left over for describing the gravitational field strength. The *Petrov classification* (1954) of the gravitational vacuum field, see Penrose et al. [49] and Stephani et al. [50], is based on the algebraic properties of the Weyl tensor alone. One way is to study, for the bivector $X^{ij} = -X^{ji}$, the eigenvalue equation

$$\frac{1}{2}\widetilde{C}^{ij}{}_{kl}X^{kl} = \lambda X^{ij}. \tag{29}$$

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[5] The name *characteristic matrix*, for the tensor (21), was used in the paper [23].

[6] Often in GR, the $\widetilde{\Gamma}_{ij}{}^k$'s are named as "gravitational field strength". However, since they are non-tensorial, it seems more appropriate to us to call instead the tidal forces by this name.

Another, equivalent way is to determine the *principal null directions* of the Weyl tensor according to [50]

$$\xi^{[i}\big(\xi_m\widetilde{C}^{j]mn[k}\xi_n\big)\xi^{l]}=0,\quad \text{with } g^{ij}\xi_i\xi_j=0. \tag{30}$$

So, the covector $\xi_i \neq 0$ is explicitly required to be null (light-like). The 2nd rank tensor occurring inside the parentheses of (30),

$$W^{ij}_{\xi}[\widetilde{C}]:=\widetilde{C}^{ikjl}\xi_k\xi_l, \tag{31}$$

is the analog of $\mathcal{W}^{ij}_q[\chi]$, see (21). Thus, the Weyl tensor $\widetilde{C}^{ikjl}$ mimics a corresponding electromagnetic response tensor χ^{ikjl}, and there is small wonder that demanding $W^{ij}_{\xi}[\widetilde{C}]$ to have rank 0, 1 or 2 yields a Kummer tensor $K^{ijkl}[\widetilde{C}]$.

In this analogy, we compare electromagnetic wave propagation in local and linear media, that is, Maxwell's linear field theory *in matter,* with the nonlinear gravitational wave propagation *in vacuum*, that is, with the nonlinear Einstein equation in free space.

But this analogy teaches us one more thing: The electromagnetic response tensor χ^{ijkl}, only in the simplest nontrivial case, depends on 20 independent components, see (5) for ${}^{(2)}\chi^{ijkl}={}^{(3)}\chi^{ijkl}=0$. Can we generalize the 20 components' Riemann curvature $\widetilde{R}_{ijk}{}^{l}$ to an object with 36 independent components? Yes, we can! In the gauge theory of the Poincaré group, see [47], Part B, the field strength, besides Cartan's torsion, is the curvature tensor of a Riemann–Cartan space,

$$R_{ijk}{}^{l}:=2(\partial_{[i}\Gamma_{j]k}{}^{l}+\Gamma_{[i|m}{}^{l}\,\Gamma_{|j]k}{}^{m}), \tag{32}$$

with 36 independent components, in strict analogy to χ^{ijkl}. The Riemann–Cartan connection $\Gamma_{ij}{}^{k}$, entering the definition (32), is metric-compatible and carries the torsion tensor $T_{ij}{}^{k}:=2\Gamma_{[ij]}{}^{k}$. The Weyl tensor of a Riemann–Cartan space C^{ijkl} has, like its Riemannian counterpart, 10 independent components and is traceless.

In GR, starting with the curvature $\widetilde{R}_{ijk}{}^{l}$, we can construct the Kummer tensor $K^{ijkl}[\widetilde{R}]$. It is possible to show, see Ruse [42], that the surface generated by $K^{ijkl}[\widetilde{R}]\,\xi_i\xi_j\xi_k\xi_l=0$ is a Kummer surface.[7] Let us now generalize to a Riemann–Cartan spacetime, having curvature tensor $R_{ijk}{}^{l}$. It is an open problem whether

$$K^{ijkl}[R]\,\xi_i\xi_j\xi_k\xi_l=0 \tag{33}$$

always leads to a Kummer surface, since it may occur that $R^{ijkl}\neq R^{klij}$. As a matter of fact, this is algebraically the same as having a medium with a finite skewon piece (drawing the analogy involves raising three indices of the curvature tensor with the metric g^{ij}). Nevertheless, Eq. (33) does result in a surface that is equivalent to a Fresnel surface, and which encodes information about the principal null directions of $R_{ijk}{}^{l}$.

We define the principal null directions of a curvature tensor in Riemann–Cartan spacetime as the covectors such that

$$\xi^{[i}\big(\xi_m R^{j]mn[k}\xi_n\big)\xi^{l]}=0, \tag{34}$$

$$g^{ij}\xi_i\xi_j=0. \tag{35}$$

In particular, (34) is obtained by generalizing (30) from Riemannian to Riemann–Cartan spacetimes. As an aside, we were able to prove that, if $R^{ikjl}\xi_k\xi_l\neq 0$, (34) implies Eq. (35). One could then hope that all covectors that satisfy (34) are null, whence the requirement (35) could be dropped. This hope is, in fact, untenable. Let us consider the case where $R^{ijkl}=\alpha\epsilon^{ijkl}$, so that $R^{ikjl}\xi_k\xi_l=0$. Then, the requirement (34) is fulfilled by an arbitrary covector, which may not be null. One then concludes that (35) must be kept as an explicit requirement.

The following observation motivated, to a good extent, our interest in Kummer surfaces: the principal null directions of $R_{ijk}{}^{l}$ belong to the intersection of the hypersurface generated by (33) and the

7 Incidentally, the nodes of the Kummer surface can be expressed by $K^{ijkl}[\widetilde{R}]\,\xi_i\xi_j\xi_k=0$, see Ruse [41].

light-cone. The key is to demonstrate that, because of (34), the covector ξ_i fulfills (33). In addition, (35) states that ξ_i must also belong to the light-cone.

Let us look at this proof more in detail. The first step is to define an equivalent of the tensor (31), namely $W_\xi^{ij}[R] := R^{ikjl}\xi_k\xi_l$. One can then rewrite (34) as

$$\xi^{[i}W_\xi{}^{j][k}\xi^{l]} = 0. \tag{36}$$

Decomposing (36) in space + time yields a set of four simultaneous equations

$$\begin{cases} \xi^{[0}W_\xi{}^{a][0}\xi^{b]} = 0, \\ \xi^{[0}W_\xi{}^{a][b}\xi^{c]} = 0, \\ \xi^{[a}W_\xi{}^{b][0}\xi^{c]} = 0, \\ \xi^{[a}W_\xi{}^{b][c}\xi^{d]} = 0. \end{cases} \tag{37}$$

The indices a, b, c, d range from 1 to 3. Next, one considers a vector basis $\{e_{\bar{0}}, e_{\bar{a}}\}$ such that $\xi^{\bar{0}} = 1$ and $\xi^{\bar{a}} = 0$. In this vector basis, the second, third and fourth equations of (37) are trivially fulfilled. The condition (36) is then equivalent to the first equation of the set, which takes the form

$$W_\xi^{\bar{a}\bar{b}} = 0. \tag{38}$$

By contrast, $W_\xi^{\bar{0}\bar{0}}$, $W_\xi^{\bar{0}\bar{a}}$ and $W_\xi^{\bar{a}\bar{0}}$ remain arbitrary. It follows that, using the tensor product $\otimes$,

$$\begin{aligned} W_\xi &= W_\xi^{\bar{0}\bar{0}} e_{\bar{0}} \otimes e_{\bar{0}} + W_\xi^{\bar{0}\bar{a}} e_{\bar{0}} \otimes e_{\bar{a}} + W_\xi^{\bar{b}\bar{0}} e_{\bar{b}} \otimes e_{\bar{0}} \\ &= e_{\bar{0}} \otimes (W_\xi^{\bar{0}\bar{0}} e_{\bar{0}} + W_\xi^{\bar{0}\bar{a}} e_{\bar{a}}) + W_\xi^{\bar{b}\bar{0}} e_{\bar{b}} \otimes e_{\bar{0}}. \end{aligned} \tag{39}$$

When one defines the vectors $S_\xi := W_\xi^{\bar{0}\bar{0}} e_{\bar{0}} + W_\xi^{\bar{0}\bar{a}} e_{\bar{a}}$ and $T_\xi := W_\xi^{\bar{b}\bar{0}} e_{\bar{b}}$, (39) translates into $W_\xi = e_{\bar{0}} \otimes S_\xi + T_\xi \otimes e_{\bar{0}}$. In an arbitrary 4-dimensional vector basis $\{e_i\}$, this reads

$$W_\xi^{ij} = e_{\bar{0}}{}^i S_\xi^j + T_\xi^i e_{\bar{0}}{}^j, \tag{40}$$

where $e_{\bar{0}} = e_{\bar{0}}{}^i e_i$. One recalls that $e_{\bar{0}}{}^i = \xi^i$, and finds that (34) is equivalent to

$$W_\xi^{ij} = \xi^i S_\xi^j + T_\xi^i \xi^j, \tag{41}$$

for some S_ξ^i and T_ξ^i. Hence, the condition (34) implies that W_ξ^{ij} has rank 0, 1 or 2. Let us momentarily leave this result to one side, and proceed on a different front. As we mentioned before, the surface generated by (33) is algebraically the same as a Fresnel surface. Consequently, one can exploit the results of [2,26,23] in geometric optics, and deduce that ξ_i belongs to the hypersurface generated by (33) if and only if the rank of W_ξ^{ij} is 0, 1 or 2. Bringing it all together, when ξ_i fulfills (34), it also satisfies (33). At the same time, ξ_i is required to be null (35). Therefore, the principal null directions of $R_{ijk}{}^l$ belong to the intersection of the hypersurface generated by (33) and the light-cone. If $R^{ijkl} = R^{klij}$, one can further conclude that the hypersurface describes, in RP^3, a Kummer surface.

Let us observe that one may classify the principal null directions of $R_{ijk}{}^l$ according to the rank of $W_\xi^{ij}[R]$.

1.6. Outline, conventions

In Section 3, we *decompose* the premetric Kummer tensor density $\mathcal{K}^{ijkl}[\mathcal{T}]$, which has 136 independent components, into smaller pieces. We determine its irreducible pieces under the linear group $GL(4, R)$, see Eq. (104). Especially, we also spell out the irreducible decomposition of $\mathcal{K}^{ijkl}[\mathcal{T}]$, if the latter is expressed only in terms of one single irreducible piece of $\mathcal{T}^{ijkl}$, see Fig. 5.

In Section 4, we assume the presence of a spacetime *metric* that allows to contract the Kummer tensor once or twice. Accordingly, we extract the completely trace-free part of the Kummer tensor and, moreover, we decompose $\mathcal{K}$ under the Lorentz group $SO(1, 3)$. As a special case, this yields the $SO(1, 3)$ decomposition of the Tamm–Rubilar tensor into 3 pieces according to $35 = 25+9+1$, which is the new result.

Our conventions of physics are basically taken from Landau–Lifshitz [51]: (holonomic) coordinates are denoted by Latin indices $i, j, k, \ldots = 0, 1, 2, 3$, spatial coordinate indices by $a, b, \ldots = 1, 2, 3$, (anholonomic) frames (tetrads) by Greek indices $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$. Our metric has signature $(-+++)$. Our conventions in differential geometry are taken from Schouten [22], see also Penrose and Rindler [49]. Only our torsion $T_{ij}^{k} = 2S_{ij}^{k}|_{\text{Schouten}}$. Symmetrizing of indices is denoted by parentheses, $(ij) := \{ij + ji\}/2!$, antisymmetrization by brackets $[ij] := \{ij + ji\}/2!$, with corresponding generalizations $(ijk) := \{ijk + jik + kij + \cdots\}/3!$, etc.; indices standing between two vertical strokes $|\,|$ are excluded from the (anti)symmetrization process.

Our conventions in exterior calculus are displayed in [46,2]. The Hodge star is denoted by *, the dual with respect to the Lie-algebra indices by $^{(*)}$. In tensor analysis, it is not necessary to distinguish between the Hodge dual and the Lie dual: we simply use *. The totally antisymmetric Levi-Civita symbol (a tensor density) is $\epsilon_{ijkl} = \pm 1, 0$. The "diamond" dual $^\diamond$, see [2] (sometimes called Poincaré dual, see [37]), is built with the premetric ϵ_{ijkl}.

2. Kummer tensor density $\mathcal{K}^{ijkl}[\mathcal{T}]$ in space–time newly defined

2.1. Its definition in terms of a doubly antisymmetric 4th rank tensor density $\mathcal{T}^{ijkl}$

Consider an arbitrary fourth rank tensor density of type $\binom{4}{0}$ that is antisymmetric in its first and its last pair of indices:

$$\mathcal{T}^{ijkl} \quad \text{with } \mathcal{T}^{(ij)kl} = 0 \quad \text{and} \quad \mathcal{T}^{ij(kl)} = 0. \tag{42}$$

Then, $\mathcal{T}$ can be thought of as a 6×6 matrix, which can be decomposed into a symmetric traceless, an antisymmetric, and a trace part. This corresponds, see the analogous case of χ in (5), to an irreducible decomposition of $\mathcal{T}^{ijkl}$ under the action of the group $GL(4, R)$[8]:

$$\begin{aligned} \mathcal{T}^{ijkl} &= {}^{(1)}\mathcal{T}^{ijkl} + {}^{(2)}\mathcal{T}^{ijkl} + {}^{(3)}\mathcal{T}^{ijkl}. \\ 36 &= \;\; 20 \;\; \oplus \;\; 15 \;\; \oplus \;\; 1. \end{aligned} \tag{43}$$

Motivated by the existence, see (7), of the Tamm–Rubilar tensor density $\mathcal{G}^{ijkl}[\chi]$ of classical electrodynamics, and starting from (8), we want to define, as a prototype, the Kummer tensor density $\mathcal{K}^{ijkl}[\mathcal{T}]$, which is *cubic* in the fourth rank tensor density $\mathcal{T}$ of Eq. (42).

As long as one has no metric available – we call this the *premetric* situation – one can only use the totally antisymmetric Levi-Civita symbol $\epsilon_{ijkl} = \pm 1, 0$ for lowering indices. We define, see above, the "diamond" (single) dual by

$$\mathcal{T}^{\diamond\, ij}{}_{kl} = \frac{1}{2}\mathcal{T}^{ijcd}\epsilon_{cdkl} \tag{44}$$

and the double dual by

$${}^\diamond\mathcal{T}^\diamond_{ijkl} = \frac{1}{2}\epsilon_{ijab}\mathcal{T}^{\diamond\, ab}{}_{kl} = \frac{1}{4}\epsilon_{ijab}\mathcal{T}^{abcd}\epsilon_{cdkl}. \tag{45}$$

Clearly, we have once again the symmetries ${}^\diamond\mathcal{T}^\diamond{}_{(ij)kl} = {}^\diamond\mathcal{T}^\diamond{}_{ij(kl)} = 0$, and this double dual can also be decomposed according to the scheme (43).

Keeping in mind the definition (8) and desymmetrizing it, let us now introduce the prototype of a Kummer tensor density that corresponds to the fourth rank tensor density $\mathcal{T}$ of Eq. (42):

$$\boxed{\mathcal{K}^{ijkl}[\mathcal{T}] := \mathcal{T}^{aibj}\, {}^\diamond\mathcal{T}^\diamond_{acbd}\, \mathcal{T}^{ckdl}.} \tag{46}$$

[8] The *principal part* of $\mathcal{T}^{ijkl}$ can be put into the explicit form

$${}^{(1)}\mathcal{T}^{ijkl} = \frac{1}{6}\Big[2\left(\mathcal{T}^{ijkl} + \mathcal{T}^{klij}\right) - \left(\mathcal{T}^{iklj} + \mathcal{T}^{ljik}\right) - \left(\mathcal{T}^{iljk} + \mathcal{T}^{jkil}\right)\Big].$$

If we switch the index pairs according to $\mathcal{K}^{klij}[\mathcal{T}] := \mathcal{T}^{akbl}\,{}^{\diamond}\mathcal{T}^{\diamond}{}_{acbd}\mathcal{T}^{cidj} = \mathcal{T}^{cidj}\,{}^{\diamond}\mathcal{T}^{\diamond}{}_{acbd}\mathcal{T}^{akbl}$, then, by renaming the indices c, d and a, b, we immediately recognize the symmetry

$$\boxed{\mathcal{K}^{ijkl}[\mathcal{T}] = \mathcal{K}^{klij}[\mathcal{T}].} \tag{47}$$

No other algebraic symmetries of the Kummer tensor density are known, provided $\mathcal{T}$ carries no additional symmetries beyond (42). Thus, we can think of $\mathcal{K}^{ij|kl}[\mathcal{T}]$ as a 16×16 matrix. Because of (47), it is a symmetric 16×16 matrix *with* 136 *independent components.*

A Kummer tensor belonging to the Riemann curvature tensor $\widetilde{R}^{ijkl}$ of general relativity was defined by Zund [52] explicitly in his Eq. (16) and earlier by Ruse [42] implicitly in his Eqs. (5.15) and (5.7), see also [53,41]. Our convention for the position of the indices of the Kummer tensor density turned out to be in harmony with that of Ruse and Zund. Note, however, that they worked in a Riemannian space with a Riemannian metric, whereas our definition is *premetric* and, as such, appreciably more general. In premetric electrodynamics, this more general definition (46) is required, which we propose here *for the first time.*

2.2. In premetric electrodynamics

In exterior calculus, the premetric form of Maxwell's equations reads,

$$dH = J, \qquad dF = 0, \tag{48}$$

with the excitation 2-form $H = (-H_a, D^b)$, the electric current 3-form $J = (-j^a, \rho)$, and the field strength 2-form $F = (E_a, B^b)$, see [2]. Maxwell's equations, as displayed in (48), are manifestly invariant under coordinate and frame transformations and are visibly independent of the metric, since the d's denote exterior (*not* exterior *covariant*) differentiation. With the formulas

$$H = \frac{1}{2}\epsilon_{ijkl}\mathcal{H}^{kl}dx^i \wedge dx^j, \qquad J = \frac{1}{6}\epsilon_{ijkl}\mathcal{J}^l dx^i \wedge dx^j \wedge dx^k, \qquad F = \frac{1}{2}F_{ij}dx^i \wedge dx^j, \tag{49}$$

we recover the coordinate component version (3) of Maxwell's equations.

With the local and linear constitutive law (4), $\mathcal{H}^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}$, one finds the Kummer tensor density of premetric electrodynamics as

$$\mathcal{K}^{ijkl}[\chi] = \chi^{aibj}\,{}^{\diamond}\chi^{\diamond}_{acbd}\chi^{ckdl}. \tag{50}$$

Thus, recalling how we arrived at $\mathcal{K}^{ijkl}[\chi]$ in the context of (8), we immediately retrieve the TR-tensor density

$$\mathcal{G}^{ijkl}[\chi] = \frac{1}{6}\,\mathcal{K}^{(ijkl)}[\chi]. \tag{51}$$

Accordingly, the totally symmetric piece $\mathcal{K}^{(ijkl)}[\chi]$, with its 35 independent components, can, up to a factor, be observed in crystal optics. In Figs. 3 and 4, we display generalized Fresnel wave surfaces for media that carry a skewon piece, in order to convey an idea of the numerous possible physical situations.

The TR-tensor density has an operational interpretation. This is not yet the case for the other, less symmetric pieces of the Kummer tensor density $\mathcal{K}^{ijkl}[\chi]$. It is one of the goals of our paper to come to a better understanding of these new pieces, which are beyond the TR-tensor density.

If we restrict ourselves to local and linear *reversible* electrodynamics, that is, if the skewon piece vanishes, ${}^{(2)}\chi^{ijkl} = 0$, then we recover the case which was considered by Bateman [15], see our Eq. (2) with $\kappa_{IJ} = \kappa_{JI}$. Then, $\chi = {}^{(1)}\chi + {}^{(3)}\chi$ has only $20 + 1$ independent components. Exactly for this reversible case, Bateman, by using tools of projective geometry, see Hudson [32] and Jessop [40], has demonstrated that the generalized Fresnel wave surfaces are *Kummer surfaces*. This fact justifies to call $\mathcal{K}^{ijkl}$ the Kummer tensor—at least for ${}^{(2)}\chi = 0$. However, we do not know whether the wave surfaces in Figs. 3 and 4 represent Kummer surfaces or not, since they incorporate a non-vanishing ${}^{(2)}\chi$.

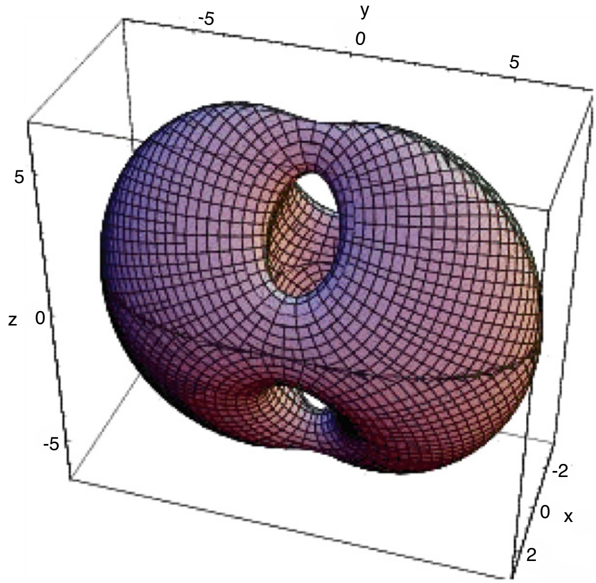


Fig. 3. Generalized Fresnel wave surface for anisotropic permittivity, trivial impermeability, and with a spatially isotropic skewon piece. We use the dimensionless variables $x := cq_1/q_0,\ y := cq_2/q_0,\ z := cq_3/q_0$, see [54].

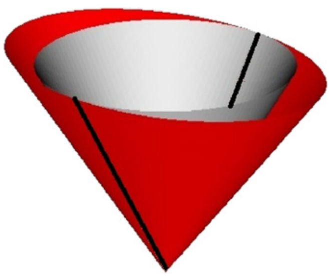


Fig. 4. The two cones of the generalized Fresnel wave surface for trivial permittivity and impermeability, with a one-component skewon piece χ^{1213} (or $\not{S}_{01}$). The axis of rotation is directed along the time coordinate. The two optical axes are lying in the plane $q_2 = 0$, the fourth coordinate q_3 is suppressed. In contrast, double cone media *for vanishing skewon piece* have been investigated by Dahl and Favaro [55,56].

2.3. In general relativity (Riemann spacetime)

Let us turn to the metric case: if a Riemannian metric with Lorentz signature is prescribed, we can define the scalar density of weight $+1$ as $\sqrt{-g}$, with $g := \det g_{rs}$. Then the densities can be transformed to tensors (see [2], p. 218),

$$T^{ijkl} = (\sqrt{-g})^{-1}\mathcal{T}^{ijkl}, \qquad K^{ijkl} = (\sqrt{-g})^{-1}\mathcal{K}^{ijkl}, \qquad \eta_{ijkl} := \sqrt{-g}\,\epsilon_{ijkl}, \tag{52}$$

and the dual $^* := \sqrt{-g}\ ^\diamond$ is built with this unit tensor.

In general relativity (GR), T^{ijkl} is identified with the Riemann curvature tensor $\widetilde{R}^{ijkl}$, that is, $T^{ijkl} = \widetilde{R}^{ijkl}$. We recall the algebraic symmetries of the Riemann curvature tensor:

$$\widetilde{R}^{(ij)kl} = 0, \qquad \widetilde{R}^{ij(kl)} = 0; \qquad \widetilde{R}^{ijkl} = \widetilde{R}^{klij}; \qquad \widetilde{R}^{[ijkl]} = 0. \tag{53}$$

The first two symmetries qualify $\widetilde{R}$ to be identified with T, that is, we have 36 independent components left. The pair commutator symmetry corresponds, algebraically, to the vanishing of the skewon piece in electrodynamics. Thus, 21 independent components are left over, in the "reversible" case. Eventually, the vanishing of the totally antisymmetric piece (1 component) leaves us with 20 independent components for the Riemann curvature $\widetilde{R}^{ijkl}$. Electrodynamically speaking, only the principal part of the curvature survives. In the parlance of, for example, Gilkey [57], ${}^{(1)}\chi$ is an *algebraic* Riemann curvature tensor.

With the identification $T = \widetilde{R}$, the Kummer tensor density becomes the plain gravitational Kummer tensor of Zund [52],

$$K^{ijkl}[\widetilde{R}] = \widetilde{R}^{aibj}\,{}^{*}\widetilde{R}^{*}_{acbd}\,\widetilde{R}^{ckdl}. \tag{54}$$

Since $\widetilde{R}^{ijkl}$ exhibits only on 20 independent components, its Kummer tensor possesses, besides the conventional pairwise symmetry

$$K^{ijkl}[\widetilde{R}] = K^{klij}[\widetilde{R}], \tag{55}$$

also an additional algebraic symmetry, namely

$$K^{ijkl}[\widetilde{R}] = K^{jilk}[\widetilde{R}]. \tag{56}$$

This can be shown directly, but later, in Eq. (118), the proof of (56) will be much more illuminating.

Incidentally, to our knowledge, the Kummer tensor $K^{ijkl}[\widetilde{R}]$ of Zund [52] was forgotten altogether. We could not find any new reference to it.

The curvature tensor of GR, $\widetilde{R}^{ijkl}$ (20 independent components), can be irreducibly decomposed, under the local Lorentz group, into the Weyl tensor $\widetilde{C}^{ijkl}$ (10 components), the trace-free Ricci tensor $\widetilde{\mathrm{Ric}}_{ij} - \frac{1}{4}\widetilde{R}\,g_{ij}$ (9 components, with $\widetilde{\mathrm{Ric}}_{ij} := \widetilde{R}^{k}_{kij}$), and the curvature scalar $\widetilde{R} := \widetilde{\mathrm{Ric}}_{k}{}^{k}$ (1 component).

Accordingly, we can define three different Kummer tensors for each of these pieces. In practical applications, however, only the Kummer tensor attached to the Weyl tensor, $K^{ijkl}[\widetilde{C}]$, will be of importance. For a matter-free region of spacetime (vacuum), we have, according to Einstein's field equation without cosmological constant, $\widetilde{\mathrm{Ric}}_{ij} = 0$. In other words, a *vacuum gravitational field* is described by a non-vanishing *Weyl tensor*, $\widetilde{C}^{ijkl} \neq 0$. In turn, its Kummer tensor, $K^{ijkl}[\widetilde{C}]$, will be relevant in this context. We will discuss this in Section 2.5.

2.4. In Poincaré gauge theory of gravity (Riemann–Cartan spacetime)

In the RC-space, the curvature tensor $R_{ijk}{}^{l}$ has only the following algebraic symmetries,

$$R^{(ij)kl} = 0, \qquad R^{ij(kl)} = 0. \tag{57}$$

Thus, it has 36 independent components, just like the electromagnetic response tensor χ^{ijkl}, which, accordingly, is called an *algebraic* RC-curvature tensor. The Kummer tensor of the RC-curvature reads

$$K^{ijkl}[R] = R^{aibj}\,{}^{*}R^{*}_{acbd}R^{ckdl} = K^{klij}[R]. \tag{58}$$

It has the same algebraic symmetries as $K[\chi]$. Hence, it carries 136 independent components.

For *vanishing torsion*, the RC-curvature $R_{ijk}{}^{l}$ becomes the Riemann curvature $\widetilde{R}_{ijk}{}^{l}$, with the corresponding Kummer tensor $K^{ijkl}[\widetilde{R}]$ that, for vanishing Ricci tensor, $\widetilde{\mathrm{Ric}}_{ij} = 0$, eventually specializes to Zund's Kummer tensor $K^{ijkl}[\widetilde{C}]$ in [58]. In each step, the tensor fed into the "Kummer machine" loses some independent components. As a consequence, the corresponding Kummer tensor picks up additional algebraic symmetries.

2.5. Kummer–Weyl tensor $K^{ijkl}[C]$ in Riemann–Cartan and in Riemann spacetime

In a RC-space, the curvature can be decomposed under the $SO(1,3)$ into 6 independent pieces, see [46] for details,

$$R_{ijk}{}^{l} = \sum_{I=1}^{6} {}^{(I)}R_{ijk}{}^{l}, \qquad 36 = 10 \oplus 9 \oplus 1 \oplus 9 \oplus 6 \oplus 1. \tag{59}$$

The first piece ${}^{(1)}R_{ijk}{}^{l} =: C_{ijk}{}^{l}$ is the Weyl tensor of the RC-space with 10 independent components. It so happens that the corresponding Weyl tensor of the Riemann space $\widetilde{C}_{ijk}{}^{l}$ has 10 independent components as well.

This can be understood in the following way: the Weyl tensor C^{ijkl}, as curvature tensor of a RC-space, obeys the symmetries (57); moreover, the irreducible pieces ${}^{(I)}C^{ijkl}$, for $I = 2, 3, 4, 5, 6$, have to vanish.[9] This yields the algebraic symmetries of the Weyl tensor,

$$C^{(ij)kl} = 0, \qquad C^{ij(kl)} = 0, \qquad C^{ijkl} = C^{klij}, \qquad C^{[ijkl]} = 0, \qquad g_{kl}C^{kijl} = 0, \tag{60}$$

leaving us with 10 independent components for the Weyl tensor C^{ijkl}.

In a Riemann space, the vanishing of the torsion induces the following symmetries for the Riemann curvature $\widetilde{R}^{ijkl} = \widetilde{R}^{klij}$, $\widetilde{R}^{[ijkl]} = 0$. As a consequence, since also $\widetilde{\mathrm{Ric}}_{[ij]} = 0$, the Weyl tensor of a Riemann space obeys the same symmetries as the Weyl tensor of a RC-space:

$$\widetilde{C}^{(ij)kl} = 0, \qquad \widetilde{C}^{ij(kl)} = 0, \qquad \widetilde{C}^{ijkl} = \widetilde{C}^{klij}, \qquad \widetilde{C}^{[ijkl]} = 0, \qquad g_{kl}\widetilde{C}^{kijl} = 0. \tag{61}$$

Algebraically, C^{ijkl} and $\widetilde{C}^{ijkl}$ share the same symmetries. Hence, C^{ijkl} and $\widetilde{C}^{ijkl}$, both have 10 independent components. The anti-self double duality of $\widetilde{C}^{ijkl}$, known from GR, translates into the corresponding symmetry of C^{ijkl}:

$$C^{ijkl} = -{}^{*}C^{*\,ijkl}, \qquad \widetilde{C}^{ijkl} = -{}^{*}\widetilde{C}^{*\,ijkl}. \tag{62}$$

These relations can be proven by substituting the definitions of the dualities and by using some of the symmetries in (60) or (61), respectively.

Let us turn to the Kummer–Weyl tensor, see the definition (58):

$$K^{ijkl}[C] = C^{aibj\,*}C^{*}_{acbd}\,C^{ckdl}. \tag{63}$$

The analogous relation is valid for vanishing torsion, that is, for $\widetilde{C}^{\,ijkl}$. Such statements are valid for all the Kummer formulas relating to $K[C]$. We will not mention this further. Because of (62), the definition simplifies appreciably:

$$K^{ijkl}[C] = -C^{aibj}C_{acbd}C^{ckdl}. \tag{64}$$

We have again the symmetries (55) and (56):

$$K^{ijkl}[C] = K^{klij}[C], \qquad K^{ijkl}[C] = K^{jilk}[C]. \tag{65}$$

Since $K^{ijkl}[\widetilde{R}]$ should have more independent components than $K^{ijkl}[C]$, we have to expect additional symmetries for $K^{ijkl}[C]$. Indeed, one finds that

$$K^{[ij](kl)}[C] = 0, \qquad K^{(kl)[ij]}[C] = 0. \tag{66}$$

For the proof we expand the parentheses and the brackets:

$$\begin{aligned} K^{[ij](kl)}[C] &= \frac{1}{2}\left(K^{[ij]kl} + K^{[ij]lk}\right) = \frac{1}{4}\left(K^{ijkl} - K^{jikl} + K^{ijlk} - K^{jilk}\right) \\ &= \frac{1}{4}\left[\left(K^{ijkl} - K^{jilk}\right) - \left(K^{jikl} - K^{ijlk}\right)\right] = 0. \end{aligned} \tag{67}$$

Here we made use of $(65)_2$. Accordingly, the symmetries (66) are not independent from $(65)_2$. We collect our results:

Proposition. *The Kummer–Weyl tensors $K^{ijkl}[C]$ and $K^{ijkl}[\widetilde{C}]$ fulfill the algebraic symmetries*

$$K^{ijkl} = K^{klij}, \qquad K^{ijkl} = K^{jilk}, \quad K^{[ij](kl)} = 0, \qquad K^{(kl)[ij]} = 0. \tag{68}$$

9 These vanishing pieces add up to $9 + 1 + 9 + 6 + 1 = 26$ independent pieces. Thus, for the Weyl tensor, $36 - 26 = 10$ independent pieces are left over.

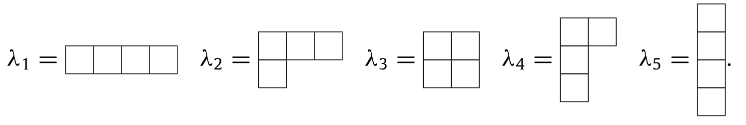
3. Canonical decomposition of $\mathcal{K}^{ijkl}$ under $GL(4, R)$

We are looking for a decomposition of the premetric Kummer tensor density $\mathcal{K}^{ijkl}$ that is irreducible under the action of the general linear group $GL(4, R)$. For this purpose it is convenient to turn first to a …

3.1. General fourth rank tensor density $\mathcal{S}^{ijkl}$

Due to the Schur–Weyl duality theorem, see [59], the decomposition of $\mathcal{S}^{ijkl}$ is related to the irreducible finite-dimensional representations of the symmetric group S_4. An interplay between the groups S_4 and $GL(4, R)$ determines the decomposition of $\mathcal{S}^{ijkl}$.

First we have to consider the Young[10] diagrams of $\mathcal{S}^{ijkl}$, see Boerner [60] or Hamermesh [61]. These diagrams are a graphical representations of the S_4:


(69)

The Schur–Weyl duality theorem can be stated as follows: Let be given a tensor (or a tensor density) of a rank p over the space R^n. For $p \geq 2$, the tensor space $\mathcal{T}_n^p$ is decomposed into a direct sum of its subspaces,

$$\mathcal{T}_n^p = \oplus_\lambda \left(S_p^\lambda \otimes G_n^\lambda \right). \tag{70}$$

Here S_p^λ is a representation of a direct sub-group of S_p and G_n^λ that of $GL(n, R)$. The tensor products of pairs of the irreducible moduli for these two groups are subspaces of $\mathcal{T}_n^p$ that are summed over all Young diagrams λ_i.

The dimension of S_p^λ is calculated by the *hook-length formula*[11]

$$\dim S_p^\lambda = \frac{p!}{\prod\limits_{x \in \lambda} h(x)}; \tag{71}$$

here the *hook-length* $h(x)$ of a box x is the number of boxes that are in the same row to the right of it plus the number of boxes in the same column below it plus 1. Using (71), we find for the different Young diagrams λ_i,

$$\dim S_4^{\lambda_1} = \dim S_4^{\lambda_5} = 1, \qquad \dim S_4^{\lambda_2} = \dim S_4^{\lambda_4} = 3, \qquad \dim S_4^{\lambda_3} = 2. \tag{72}$$

The dimension of the second factor G_n^λ is calculated by the *hook-content formula*, with the content $c(x)$ of the box x—the number of its rows minus the number of its columns:

$$\dim G_n^\lambda = \prod_{x \in \lambda} \frac{n + c(x)}{h(x)}. \tag{73}$$

We obtain for the different diagrams λ_i of (69),

$$\dim G_4^{\lambda_1} = 35, \qquad \dim G_4^{\lambda_2} = 45, \qquad \dim G_4^{\lambda_3} = 20,$$
$$\dim G_4^{\lambda_4} = 15, \qquad \dim G_4^{\lambda_5} = 1. \tag{74}$$

Consequently, for an arbitrary fourth rank tensor density $\mathcal{S}^{ijkl}$ the set of its 256 independent components is decomposed into the subsets

$$\mathbf{256} = \mathbf{1} \times \mathbf{35} + \mathbf{3} \times \mathbf{45} + \mathbf{2} \times \mathbf{20} + \mathbf{3} \times \mathbf{15} + \mathbf{1} \times \mathbf{1}. \tag{75}$$

10 Alfred Young (1873–1940), English mathematician and priest.

11 See also Wikipedia http://en.wikipedia.org/wiki/Young_tableau.

In terms of Young diagrams, the decomposition can be represented as

$$\yng(1)\otimes\yng(1)\otimes\yng(1)\otimes\yng(1)$$
$$= (\mathbf{1})\yng(4)\oplus(\mathbf{3})\yng(3,1)\oplus(\mathbf{2})\yng(2,2)\oplus(\mathbf{3})\yng(2,1,1)\oplus(\mathbf{1})\yng(1,1,1,1). \tag{76}$$

The left-hand-side describes a general fourth rank tensor (density). On the right-hand side, the *first* diagram represents the completely symmetric tensor, the *three middle* diagrams describe tensors that are partially symmetric and partially antisymmetric, and the *last* diagram represents the completely antisymmetric tensor. The numbers in brackets denote the dimension of the irreducible representation, that is, the number of Young tableaux associated with a given diagram.

The interpretation is as follows: The 256-dimensional tensor space is *uniquely* ("canonically") decomposed into the direct sum of 5 tensor spaces with the dimensions indicated. Their explicit form can be found, for instance, in Wade [62]:

$$^{[1]}\mathcal{S}^{ijkl} := \frac{1}{4!}\,(A+B+C+D+E), \tag{77}$$

$$^{[2]}\mathcal{S}^{ijkl} := \frac{3}{4!}\,(3A+B-D-E), \tag{78}$$

$$^{[3]}\mathcal{S}^{ijkl} := \frac{2}{4!}\,(2A-C+2E), \tag{79}$$

$$^{[4]}\mathcal{S}^{ijkl} := \frac{3}{4!}\,(3A-B+D-E), \tag{80}$$

$$^{[5]}\mathcal{S}^{ijkl} := \frac{1}{4!}\,(A-B+C-D+E). \tag{81}$$

Here, for conciseness, we suppressed the indices i, j, k, l on the right-hand sides. The numerators in the coefficients are the dimensions given in (72). These five so-called cycles are defined according to

$$A^{ijkl}[\mathcal{S}] := \mathcal{S}^{ijkl}, \tag{82}$$

$$B^{ijkl}[\mathcal{S}] := \mathcal{S}^{jikl} + \mathcal{S}^{kjil} + \mathcal{S}^{ljki} + \mathcal{S}^{ikjl} + \mathcal{S}^{ilkj} + \mathcal{S}^{ijlk}, \tag{83}$$

$$C^{ijkl}[\mathcal{S}] := \mathcal{S}^{jkil} + \mathcal{S}^{kijl} + \mathcal{S}^{jlki} + \mathcal{S}^{likj} + \mathcal{S}^{kjli} + \mathcal{S}^{ljik} + \mathcal{S}^{iklj} + \mathcal{S}^{iljk}, \tag{84}$$

$$D^{ijkl}[\mathcal{S}] := \mathcal{S}^{jkli} + \mathcal{S}^{klji} + \mathcal{S}^{jlik} + \mathcal{S}^{lijk} + \mathcal{S}^{kilj} + \mathcal{S}^{lkij}, \tag{85}$$

$$E^{ijkl}[\mathcal{S}] := \mathcal{S}^{jilk} + \mathcal{S}^{klij} + \mathcal{S}^{lkji}. \tag{86}$$

The term A corresponds to the identity operator, B is the sum of 2-cycles (permutations of two indices), C is used for 3-cycles, and D for 4-cycles. The term E presents the products of disjoint 2-cycles.

However, this decomposition is reducible. The tensor spaces corresponding to the diagrams with the dimensions higher than one, that is, λ_2, λ_3, and λ_4, see (76), are successively decomposed into the direct sum of 3, 2, and 3 isomorphic subspaces, respectively. Accordingly, the 135-dimensional tensor space, depicted in the second diagram of (76), is decomposed into a direct sum of three isomorphic subspaces of dimension 45. Similarly, the 40-dimensional space corresponding to the third diagram of (76) is decomposed into a direct sum of two isomorphic 20-dimensional subspaces. Finally the 45-dimensional space is decomposed into a direct sum of three isomorphic 15-dimensional subspaces.

In this way, we arrive at an irreducible decomposition. Accordingly, there are no minimal subspaces of dimensions different from those specified above; or, in terms of tensors: there are no tensors, the number of components of which is different from those given in (75). Note, however, that the irreducible decomposition in (75) is not necessarily unique. There is an infinite number of ways to decompose the tensor spaces corresponding to λ_2, λ_3, and λ_4 into their isomorphic subspaces.

3.2. Specializing to the Kummer tensor density $\mathcal{K}^{ijkl}$

The Kummer tensor density $\mathcal{K}^{ijkl}$ satisfies the symmetry property $\mathcal{K}^{ijkl} = \mathcal{K}^{klij}$ and can be represented as a symmetric 16×16 matrix, the dimension of which is 136. The question is now: Into which subspaces can this 136-dimensional space be decomposed irreducibly under the $GL(4, R)$?

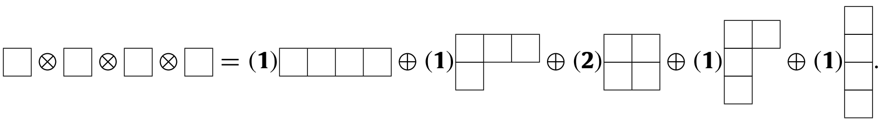
We turn to a dimensional analysis. We must have a partition of 136 into a sum of the numbers representing the dimensions of the invariant subspaces (72). This yields the Diophantine equation

$$\mathbf{136} = \alpha \times \mathbf{35} \oplus \beta \times \mathbf{45} \oplus \gamma \times \mathbf{20} \oplus \delta \times \mathbf{15} \oplus \epsilon \times \mathbf{1}. \tag{87}$$

The coefficients have to be less or equal to the corresponding dimensions specified in (72). Then, $0 \leq \alpha, \epsilon \leq 1$, $0 \leq \beta, \delta \leq 3$, and $0 \leq \gamma \leq 2$. Since the coefficient before the total symmetric part $\alpha = 1$, the unique solution is

$$\alpha = \beta = \delta = \epsilon = 1, \qquad \gamma = 2. \tag{88}$$

Thus, we have a unique decomposition of the Kummer tensor into 5 pieces:



(89)

Using the formulas (77)–(81) and the symmetry of the Kummer tensor $\mathcal{K}^{ijkl} = \mathcal{K}^{klij}$, we find for the five canonical pieces,

$$^{[1]}\mathcal{K}^{ijkl} = \mathcal{K}^{(ijkl)}, \tag{90}$$

$$^{[2]}\mathcal{K}^{ijkl} = \frac{1}{2}\left(\mathcal{K}^{(i|j|k)l} - \mathcal{K}^{j(i|l|k)}\right), \tag{91}$$

$$^{[3]}\mathcal{K}^{ijkl} = \frac{1}{6}\left(2\mathcal{K}^{ijkl} - \mathcal{K}^{jkil} - \mathcal{K}^{kijl} - \mathcal{K}^{likj} - \mathcal{K}^{ljik} + 2\mathcal{K}^{jilk}\right), \tag{92}$$

$$^{[4]}\mathcal{K}^{ijkl} = \frac{1}{2}\left(\mathcal{K}^{[i|j|k]l} - \mathcal{K}^{j[i|l|k]}\right), \tag{93}$$

$$^{[5]}\mathcal{K}^{ijkl} = \mathcal{K}^{[ijkl]}. \tag{94}$$

Each piece has then inherited the symmetry of the Kummer tensor:

$$^{[I]}\mathcal{K}^{ijkl} = ^{[I]}\mathcal{K}^{klij}, \qquad \text{for } I = 1, 2, 3, 4, 5. \tag{95}$$

Let us look for the projection properties. If $^{[I]}\mathcal{Y}$ denotes the five tensor operators corresponding to the decomposition in (90)–(94), then we find $(I, J = 1, \ldots, 5)$:

$$^{[I]}\mathcal{Y} \circ ^{[J]}\mathcal{Y} = ^{[J]}\mathcal{Y}\delta^I_J \qquad \text{(no summation over } J). \tag{96}$$

The orthogonality of two operators corresponding to different diagrams and their projection property is a generic fact [63].

We can collect our results in the following proposition.

• **Proposition:** *With the Young tableau technique, the Kummer tensor density $\mathcal{K}^{ijkl}$ can be decomposed uniquely into the five canonical pieces $^{[I]}\mathcal{K}^{ijkl}$: of Eqs. (90)–(94) $(I = 1, \ldots, 5)$. Let the five tensorial projection operators $^{[I]}\mathcal{Y}$ be given that create the canonical pieces $^{[I]}\mathcal{K}^{ijkl}$. Then these operators satisfy the projection equations (96). The dimensions of the canonical pieces turn out to be as follows:*

$$\begin{aligned} \mathcal{K}^{ijkl} &= {}^{[1]}\mathcal{K}^{ijkl} + {}^{[2]}\mathcal{K}^{ijkl} + {}^{[3]}\mathcal{K}^{ijkl} + {}^{[4]}\mathcal{K}^{ijkl} + {}^{[5]}\mathcal{K}^{ijkl}, \\ 136 &= \quad 35 \quad \oplus \quad 45 \quad \oplus \quad 40 \quad \oplus \quad 15 \quad \oplus \quad 1. \end{aligned} \tag{97}$$

Only the piece $^{[3]}\mathcal{K}^{ijkl}$ can be decomposed further into a sum of two independent terms. However, this decomposition is not uniquely defined.

The splitting of the piece ${}^{[3]}\mathcal{K}^{ijkl}$ is the last problem to be solved in this context. We compute two tableaux embraced by the term ${}^{[3]}\mathcal{K}^{ijkl}$:

$$ {}^{[3a]}\mathcal{K}^{ijkl} := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\ \mathcal{K}^{ijkl}, \qquad {}^{[3b]}\mathcal{K}^{ijkl} := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\ \mathcal{K}^{ijkl}. \tag{98} $$

By successive permutations of the indices, we obtain

$$ \begin{aligned} {}^{[3a]}\mathcal{K}^{ijkl} &= \frac{1}{12}\big(I+(1,2)\big)\big(I+(3,4)\big)\big(I-(1,3)\big)\big(I-(2,4)\big)\mathcal{K}^{ijkl} \\ &= \frac{1}{12}\big(I+(1,2)\big)\big(I+(3,4)\big)\big(I-(1,3)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ilkj}\big) \\ &= \frac{1}{6}\big(I+(1,2)\big)\big(I+(3,4)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ilkj}\big) \\ &= \frac{1}{6}\big(I+(1,2)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ilkj}+\mathcal{K}^{ijlk}-\mathcal{K}^{iklj}\big) \\ &= \frac{1}{3}\big(\mathcal{K}^{ij(kl)}-\mathcal{K}^{i(lk)j}+\mathcal{K}^{ji(kl)}-\mathcal{K}^{j(lk)i}\big), \end{aligned} \tag{99} $$

$$ \begin{aligned} {}^{[3b]}\mathcal{K}^{ijkl} &= \frac{1}{12}\big(I+(1,3)\big)\big(I+(2,4)\big)\big(I-(1,2)\big)\big(I-(3,4)\big)\mathcal{K}^{ijkl} \\ &= \frac{1}{12}\big(I+(1,3)\big)\big(I+(2,4)\big)\big(I-(1,2)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ijlk}\big) \\ &= \frac{1}{12}\big(I+(1,3)\big)\big(I+(2,4)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ijlk}-\mathcal{K}^{jikl}+\mathcal{K}^{jilk}\big) \\ &= \frac{1}{12}\big(I+(1,3)\big)\big(\mathcal{K}^{ijkl}-\mathcal{K}^{ijlk}-\mathcal{K}^{jikl}+\mathcal{K}^{jilk}\mathcal{K}^{ilkj}-\mathcal{K}^{iljk}-\mathcal{K}^{likj}+\mathcal{K}^{lijk}\big) \\ &= \frac{1}{3}\big(\mathcal{K}^{ij[kl]}+\mathcal{K}^{kj[il]}+\mathcal{K}^{ji[lk]}+\mathcal{K}^{jk[li]}\big). \end{aligned} \tag{100} $$

Observe that the sum of these terms indeed equals to ${}^{[3]}\mathcal{K}^{ijkl}$:

$$ {}^{[3a]}\mathcal{K}^{ijkl} + {}^{[3b]}\mathcal{K}^{ijkl} = {}^{[3]}\mathcal{K}^{ijkl}. \tag{101} $$

Moreover, both terms satisfy the symmetry of a Kummer tensor density:

$$ {}^{[3a]}\mathcal{K}^{ijkl} = {}^{[3a]}\mathcal{K}^{klij}, \qquad {}^{[3b]}\mathcal{K}^{ijkl} = {}^{[3b]}\mathcal{K}^{klij}. \tag{102} $$

Accordingly, these pieces can serve as Kummer tensor densities themselves.

It should be remarked, however, that two operators associated with the same Young diagram are not necessary orthogonal, see the examples in [63]. Still, the projective property (96) holds true also for the terms ${}^{[3a]}\mathcal{K}^{ijkl}$ and ${}^{[3b]}\mathcal{K}^{ijkl}$. Indeed, the corresponding operators are orthogonal to ${}^{[I]}\mathcal{Y}$, for $I = 1, 2, 4, 5$, since their images are lying in ${}^{[3]}\mathcal{Y}$.

To demonstrate the orthogonality between ${}^{[3a]}\mathcal{K}^{ijkl}$ and ${}^{[3b]}\mathcal{K}^{ijkl}$, it is enough to consider the products of the operators specified by (99) and (100). We have

$$ {}^{[3a]}\mathcal{Y} \circ {}^{[3b]}\mathcal{Y} = \cdots \big(I-(2,4)\big)\big(I+(2,4)\big)\cdots = 0. \tag{103} $$

A similar relation holds for the product ${}^{[3b]}\mathcal{Y} \circ {}^{[3a]}\mathcal{Y}$. The projective property $\mathcal{Y} \circ \mathcal{Y} = \mathcal{Y}$ is a generic fact for every Young diagram with normalized operators. Note that in [63] a non-normalized version of the operators is used. Thus, finally, we arrive at the...

• **Proposition on the irreducible decomposition of the Kummer tensor density:** *The Kummer tensor density $\mathcal{K}^{ijkl}[T]$ decomposes into six pieces according to*

$$ \begin{aligned} \mathcal{K}^{ijkl} &= {}^{[1]}\mathcal{K}^{ijkl} + {}^{[2]}\mathcal{K}^{ijkl} + {}^{[3a]}\mathcal{K}^{ijkl} + {}^{[3b]}\mathcal{K}^{ijkl} + {}^{[4]}\mathcal{K}^{ijkl} + {}^{[5]}\mathcal{K}^{ijkl}, \\ 136 &= 35 \oplus 45 \oplus 20 \oplus 20 \oplus 15 \oplus 1, \\ \mathcal{K}^{ijkl} &= {}^{(1)}\mathcal{K}^{ijkl} + {}^{(2)}\mathcal{K}^{ijkl} + {}^{(3)}\mathcal{K}^{ijkl} + {}^{(4)}\mathcal{K}^{ijkl} + {}^{(5)}\mathcal{K}^{ijkl} + {}^{(6)}\mathcal{K}^{ijkl}. \end{aligned} \tag{104} $$

The pieces ${}^{(3)}\mathcal{K}^{ijkl}$ and ${}^{(4)}\mathcal{K}^{ijkl}$ are not uniquely determined.

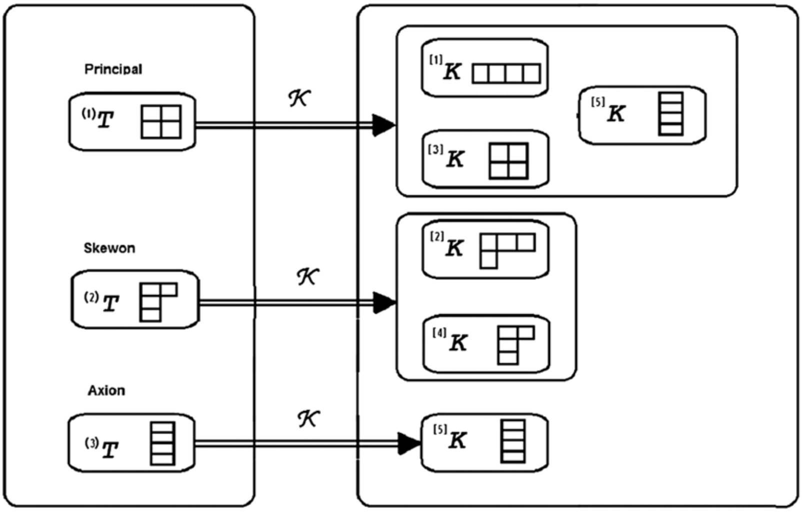


Fig. 5. If a single irreducible piece $^{(I)}\mathcal{T}^{ijkl}$, for $I = 1, 2$, or 3, of the doubly antisymmetric tensor density $\mathcal{T}^{ijkl}$ is given, we depict how it maps into the irreducible pieces of the Kummer tensor density $^{[J]}\mathcal{K}^{ijkl}$, for $J = 1, \ldots, 5$.

The *new notation* in the last line with parentheses, that is, $^{(I)}K^{ijkl}$, for $I = 1, 2, 3, 4, 5, 6$, has to be carefully distinguished from $^{[J]}K^{ijkl}$, for $J = 1, 2, 3, 4, 5$. We introduced this notation because it is often more convenient to work with successive Arabic numbers.

3.3. Kummer tensor density $\mathcal{K}^{ijkl}[\mathcal{T}]$ associated with irreducible pieces of $\mathcal{T}$

Axion piece

The simplest example is the axion piece $^{(3)}\mathcal{T}^{ijkl} = \alpha\,\epsilon^{ijkl}$. Its "diamond" double dual (45) can be calculated straightforwardly:

$$^{\diamond\,(3)}\mathcal{T}^{\diamond}_{ijkl} = \frac{1}{4}\epsilon_{ijab}\,\alpha\epsilon^{abcd}\,\epsilon_{klcd} = \frac{1}{4}\alpha\epsilon_{ijab}2(\delta^a_k\delta^b_l - \delta^a_l\delta^b_k) = \alpha\,\epsilon_{ijkl}. \tag{105}$$

Thus, the corresponding Kummer tensor density takes the form

$$\mathcal{K}^{ijkl}[^{(3)}\mathcal{T}] = \alpha^3\epsilon^{aibj}\epsilon_{acbd}\epsilon^{ckdl} = 2\alpha^3\epsilon^{ijkl} = {}^{(6)}\mathcal{K}^{ijkl}[^{(3)}\mathcal{T}], \tag{106}$$

that is, it is totally antisymmetric.

Skewon piece

The Kummer tensor density of the skewon piece is a bit more involved. First we observe that the skewon part, by definition, satisfies the pair antisymmetry

$$^{(2)}\mathcal{T}^{ijkl} = -{}^{(2)}\mathcal{T}^{klij}. \tag{107}$$

The same property holds for its diamond double dual:

$$^{\diamond\,(2)}\mathcal{T}^{\diamond}_{ijkl} = -{}^{\diamond\,(2)}\mathcal{T}^{\diamond}_{klij}. \tag{108}$$

Indeed,

$$^{\diamond\,(2)}\mathcal{T}^{\diamond}_{ijkl} = \frac{1}{4}\epsilon_{ijab}\,{}^{(2)}\mathcal{T}^{abcd}\epsilon_{klcd} = -\frac{1}{4}\epsilon_{klcd}\,{}^{(2)}\mathcal{T}^{cdab}\epsilon_{ijab} = -{}^{\diamond\,(2)}\mathcal{T}^{\diamond}_{klij}. \tag{109}$$

Eqs. (107) and (108) yield an additional antisymmetry of the Kummer tensor density associated with the skewon piece:

$$\mathcal{K}^{ijkl}\left[^{(2)}\mathcal{T}\right] = -\mathcal{K}^{jilk}\left[^{(2)}\mathcal{T}\right]. \tag{110}$$

This relation is proved by applying the antisymmetries (107) and (108) in the definition of the Kummer tensor density and subsequently its symmetry (47):

$$\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right] = {}^{(2)}\mathcal{T}^{aibj}\,{}^{\diamond(2)}\mathcal{T}^{\diamond}_{acbd}\,{}^{(2)}\mathcal{T}^{ckdl}$$
$$= -{}^{(2)}\mathcal{T}^{bjai}\,{}^{\diamond(2)}\mathcal{T}^{\diamond}_{bdac}\,{}^{(2)}\mathcal{T}^{dlck} = -\mathcal{K}^{jilk}\left[{}^{(2)}\mathcal{T}\right]. \tag{111}$$

Consequently, in the irreducible decomposition of $\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right]$ only those pieces can appear that obey (110). This is the case with the pieces ${}^{(2)}\mathcal{K}^{ijkl}$ and ${}^{(5)}\mathcal{K}^{ijkl}$. Thus,

$$\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right] = {}^{(2)}\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right] + {}^{(5)}\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right]. \tag{112}$$

In terms of Young tableaux, this decomposition is represented by

$$\mathcal{K}^{ijkl}\left[{}^{(2)}\mathcal{T}\right] = (\mathbf{1})\,\ydiagram{3,1} \oplus (\mathbf{1})\,\ydiagram{2,1,1}\,. \tag{113}$$

We can also check explicitly that our conclusion (112) is correct. Using the expressions (90)–(94), we calculate

$$
\begin{aligned}
{}^{[1]}\mathcal{K}^{ijkl} &= \mathcal{K}^{(ijkl)} = 0,\\
{}^{[2]}\mathcal{K}^{ijkl} &= \frac{1}{4}(\mathcal{K}^{ijkl} + \mathcal{K}^{kjil} - \mathcal{K}^{jilk} - \mathcal{K}^{jkli}) = \mathcal{K}^{(i|j|k)l} \neq 0,\\
{}^{[3]}\mathcal{K}^{ijkl} &= \frac{1}{6}(2\mathcal{K}^{ijkl} - \mathcal{K}^{jkil} - \mathcal{K}^{kijl} - \mathcal{K}^{likj} - \mathcal{K}^{ljik} + 2\mathcal{K}^{jilk})\\
&= \frac{1}{6}(2(\mathcal{K}^{ijkl} + \mathcal{K}^{jilk}) - (\mathcal{K}^{jkil} + \mathcal{K}^{likj}) - (\mathcal{K}^{kijl} + \mathcal{K}^{ljik})) = 0,\\
{}^{[4]}\mathcal{K}^{ijkl} &= \frac{1}{4}(\mathcal{K}^{ijkl} - \mathcal{K}^{kjil} - \mathcal{K}^{jilk} + \mathcal{K}^{jkli}) = \mathcal{K}^{[i|j|k]l} \neq 0,\\
{}^{[5]}\mathcal{K}^{ijkl} &= \mathcal{K}^{[ijkl]} = 0.
\end{aligned}
$$

Principal piece

It is characterized by two symmetry relations

$${}^{(1)}\mathcal{T}^{ijkl} = {}^{(1)}\mathcal{T}^{klij}, \tag{114}$$

and

$${}^{(1)}\mathcal{T}^{[ijkl]} = 0. \tag{115}$$

Again, its diamond double dual reflects the symmetry (114),

$${}^{\diamond(1)}\mathcal{T}^{\diamond}_{ijkl} = {}^{\diamond(1)}\mathcal{T}^{\diamond}_{klij}, \tag{116}$$

as can be seen by the definition of the double dual and by using (114):

$${}^{\diamond(1)}\mathcal{T}^{\diamond}_{ijkl} = \frac{1}{4}\epsilon_{ijab}\,{}^{(1)}\mathcal{T}^{abcd}\epsilon_{klcd} = \frac{1}{4}\epsilon_{klcd}\,{}^{(1)}\mathcal{T}^{cdab}\epsilon_{ijab} = {}^{\diamond(1)}\mathcal{T}^{\diamond}_{klij}. \tag{117}$$

Similar as in the skewon case, the symmetries (114) and (116) yield the relation

$$\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] = \mathcal{K}^{jilk}\left[{}^{(1)}\mathcal{T}\right]. \tag{118}$$

Only the irreducible pieces ${}^{(1)}\mathcal{K}^{ijkl}$, ${}^{(3)}\mathcal{K}^{ijkl}$, ${}^{(4)}\mathcal{K}^{ijkl}$, and ${}^{(6)}\mathcal{K}^{ijkl}$ carry this symmetry. Consequently, $\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right]$ can include only these pieces. Accordingly, we are left with the decomposition

$$\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] = {}^{(1)}\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] + {}^{(3)}\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] + {}^{(4)}\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] + {}^{(6)}\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right]. \tag{119}$$

In the notation of Young tableaux, this decomposition is represented by

$$\mathcal{K}^{ijkl}\left[{}^{(1)}\mathcal{T}\right] = (\mathbf{1})\,\yng(4) \oplus (\mathbf{2})\,\yng(2,2) \oplus (\mathbf{1})\,\yng(1,1,1,1). \tag{120}$$

4. Presence of a metric and decomposition of $K^{\alpha\beta\gamma\delta}$ under $SO(1,3)$

As soon as a metric is available, we can, according to (52), pass over from the Kummer tensor density $\mathcal{K}^{ijkl}$ to the Kummer tensor K^{ijkl}. Its six irreducible $GL(4,R)$ pieces ${}^{(I)}K^{ijkl}$ can now be decomposed still finer yielding pieces that are invariant under the Lorentz group $SO(1,3)$.

Having now locally the $SO(1,3)$ available, it is useful for later applications to introduce a *local orthonormal frame* $\mathbf{e}_\alpha = e^i{}_\alpha\,\partial_i$, with $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} \stackrel{*}{=} \mathrm{diag}(-1,1,1,1)$. Besides the Latin (holonomic) coordinate indices $i,j,k,\ldots$, we have now Greek (anholonomic) frame indices $\alpha,\beta,\gamma,\ldots$. The metric transforms according to $g_{\alpha\beta} = e^i{}_\alpha e^j{}_\beta\, g_{ij}$. The dual coframe $\boldsymbol{\vartheta}^\alpha = e_i{}^\alpha dx^i$ is then also orthonormal, with $e_k{}^\alpha\, e^k{}_\beta = \delta^\alpha_\beta$ and $e_i{}^\gamma\, e^j{}_\gamma = \delta^j_i$.

We will refer the Kummer tensor to the orthonormal frame,

$$K^{\alpha\beta\gamma\delta} = e_i{}^\alpha e_j{}^\beta e_k{}^\gamma e_l{}^\delta K^{ijkl}. \tag{121}$$

In the subsequent sections, $T^{\alpha\beta\gamma\delta}$ in $K^{\alpha\beta\gamma\delta}[T]$ will become the Riemann–Cartan curvature tensor $R^{\alpha\beta\gamma\delta}$, which is originally defined as a 2-*form* according to $R_\gamma{}^\delta = \frac{1}{2}R_{\alpha\beta\gamma}{}^\delta\,\vartheta^\alpha \wedge \vartheta^\beta$. Then one can apply the Hodge dual * to it, ${}^\star R_{\gamma\delta}$, and the Lie dual $^{(*)}$ to the Lie-algebra indices γ,δ, namely $R^{(*)\alpha\beta} := \frac{1}{2}R_{\gamma\delta}\,\eta^{\alpha\beta\gamma\delta}$, for details see [46]. In the present section, just read * (the ordinary dual of tensor calculus) for both stars, for the Hodge star * as well as for the Lie star $^{(*)}$.

4.1. First and second contractions of the Kummer tensor $K^{\alpha\beta\gamma\delta}[T]$

As a 4th rank tensor, the Kummer tensor $K^{\alpha\beta\gamma\delta}[T]$ has *six* possible contractions:

$$\begin{aligned} &{}_{(1)}\Phi^{\gamma\delta} := g_{\alpha\beta}K^{\alpha\beta\gamma\delta}, \qquad {}_{(2)}\Phi^{\alpha\beta} := g_{\gamma\delta}K^{\alpha\beta\gamma\delta}, \qquad {}_{(3)}\Phi^{\beta\delta} := g_{\alpha\gamma}K^{\alpha\beta\gamma\delta},\\ &{}_{(4)}\Phi^{\alpha\delta} := g_{\beta\gamma}K^{\alpha\beta\gamma\delta}, \qquad {}_{(5)}\Phi^{\beta\gamma} := g_{\alpha\delta}K^{\alpha\beta\gamma\delta}, \qquad {}_{(6)}\Phi^{\alpha\gamma} := g_{\beta\delta}K^{\alpha\beta\gamma\delta}. \end{aligned} \tag{122}$$

In other words, we have ${}_{(A)}\Phi^{\alpha\beta}$, for $A = 1,\ldots,6$; here we used for the contractions the mapping $\{1,2\}\to 1$, $\{3,4\}\to 2$, $\{1,3\}\to 3$, $\{2,3\}\to 4$, $\{1,4\}\to 5$, and $\{2,4\}\to 6$, which can be read off directly from (122). The symmetry (47) is also valid in anholonomic indices,

$$K^{\alpha\beta\gamma\delta} = K^{\gamma\delta\beta\alpha}, \tag{123}$$

as can be seen from (121). Performing the contractions (122) on both sides of (123), we find

$${}_{(1)}\Phi^{\gamma\delta} = {}_{(2)}\Phi^{\gamma\delta}, \qquad {}_{(3)}\Phi^{\beta\delta} = {}_{(3)}\Phi^{\delta\beta}, \qquad {}_{(4)}\Phi^{\alpha\delta} = {}_{(5)}\Phi^{\delta\alpha}, \qquad {}_{(6)}\Phi^{\alpha\gamma} = {}_{(6)}\Phi^{\gamma\alpha}. \tag{124}$$

Consequently there are six independent first contractions of the full Kummer tensor, namely

$${}_{(1)}\Phi^{(\alpha\beta)},\ {}_{(1)}\Phi^{[\alpha\beta]},\ {}_{(3)}\Phi^{(\alpha\beta)},\ {}_{(4)}\Phi^{(\alpha\beta)},\ {}_{(4)}\Phi^{[\alpha\beta]},\ {}_{(6)}\Phi^{(\alpha\beta)}, \tag{125}$$

which could be used to define a basis for the first contractions of the Kummer tensor.

Because of the pair symmetry (123), the second contractions do not turn out to be independent:

$$\begin{aligned} {}_{(1)}\Phi^{\alpha\beta} &= K^\mu{}_\mu{}^{\alpha\beta} &&\Longrightarrow\quad {}_{(1)}\Phi^\rho{}_\rho = K^\mu{}_\mu{}^\lambda{}_\lambda =: M,\\ {}_{(2)}\Phi^{\alpha\beta} &= K^{\alpha\beta\mu}{}_\mu &&\Longrightarrow\quad {}_{(2)}\Phi^\rho{}_\rho = K^\mu{}_\mu{}^\lambda{}_\lambda = M,\\ {}_{(3)}\Phi^{\alpha\beta} &= K^{\mu\alpha}{}_\mu{}^\beta &&\Longrightarrow\quad {}_{(3)}\Phi^\rho{}_\rho = K^{\mu\lambda}{}_{\mu\lambda} =: L,\\ {}_{(4)}\Phi^{\alpha\beta} &= K^{\alpha\mu}{}_\mu{}^\beta &&\Longrightarrow\quad {}_{(4)}\Phi^\rho{}_\rho = K^{\lambda\mu}{}_{\mu\lambda} =: K,\\ {}_{(5)}\Phi^{\alpha\beta} &= K^{\mu\alpha\beta}{}_\mu &&\Longrightarrow\quad {}_{(5)}\Phi^\rho{}_\rho = K^{\mu\lambda}{}_{\lambda\mu} = K,\\ {}_{(6)}\Phi^{\alpha\beta} &= K^{\alpha\mu\beta}{}_\mu &&\Longrightarrow\quad {}_{(6)}\Phi^\rho{}_\rho = K^{\mu\lambda}{}_{\mu\lambda} = L. \end{aligned} \tag{126}$$

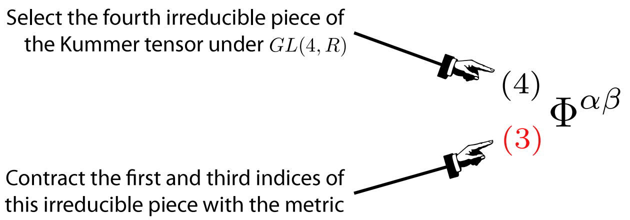


Fig. 6. Our use of the different types of indices: The 4th irreducible piece of the Kummer tensor density can be found in Eq. (104) together with Eq. (100). The number 3 denotes, according to Eq. (122), a contraction over its first and third index.

Table 1
First and second contractions of *Kummer*.

	$K^{\alpha\beta\gamma\delta}[T] = T^{\mu\alpha\lambda\beta}\,{}^{\star}T^{*}_{\mu\rho\lambda\sigma}T^{\rho\gamma\sigma\delta}$
${}^{(1)}K^{\alpha\beta\gamma\delta}$	${}^{(1)}_{(1)}\Phi^{\alpha\beta} = {}^{(1)}_{(2)}\Phi^{\alpha\beta} = {}^{(1)}_{(3)}\Phi^{\alpha\beta} = {}^{(1)}_{(4)}\Phi^{\alpha\beta} = {}^{(1)}_{(5)}\Phi^{\alpha\beta} = {}^{(1)}_{(6)}\Phi^{\alpha\beta} \neq 0$
	$\phi_{(1)} := {}^{(1)}_{(A)}\Phi^{\mu}{}_{\mu}\ (A = 1, \dots, 6)$
${}^{(2)}K^{\alpha\beta\gamma\delta}$	${}^{(2)}_{(2)}\Phi^{\alpha\beta} = {}^{(2)}_{(1)}\Phi^{\alpha\beta}, {}^{(2)}_{(3)}\Phi^{\alpha\beta} = -{}^{(2)}_{(6)}\Phi^{\alpha\beta}, {}^{(2)}_{(4)}\Phi^{\alpha\beta} = {}^{(2)}_{(1)}\Phi^{\alpha\beta},$
	${}^{(2)}_{(5)}\Phi^{\alpha\beta} = -{}^{(2)}_{(1)}\Phi^{\alpha\beta}$
	$\phi_{(2)} := {}^{(2)}_{(A)}\Phi^{\mu}{}_{\mu} = 0\ (A = 1, \dots, 6)$
${}^{(3)}K^{\alpha\beta\gamma\delta}$	${}^{(3)}_{(2)}\Phi^{\alpha\beta} = {}^{(3)}_{(4)}\Phi^{\alpha\beta} = {}^{(3)}_{(5)}\Phi^{\alpha\beta} = {}^{(3)}_{(1)}\Phi^{\alpha\beta},$
	${}^{(3)}_{(3)}\Phi^{\alpha\beta} = {}^{(3)}_{(6)}\Phi^{\alpha\beta} = -2{}^{(3)}_{(1)}\Phi^{\alpha\beta}$
	$\phi_{(3)} := {}^{(3)}_{(1)}\Phi^{\mu}{}_{\mu}$
${}^{(4)}K^{\alpha\beta\gamma\delta}$	${}^{(4)}_{(1)}\Phi^{\alpha\beta} = {}^{(4)}_{(2)}\Phi^{\alpha\beta}, {}^{(4)}_{(4)}\Phi^{\alpha\beta} = {}^{(4)}_{(5)}\Phi^{\alpha\beta} = -{}^{(4)}_{(1)}\Phi^{\alpha\beta},$
	${}^{(4)}_{(3)}\Phi^{\alpha\beta} = {}^{(4)}_{(6)}\Phi^{\alpha\beta} = 0$
	$\phi_{(4)} := {}^{(4)}_{(1)}\Phi^{\mu}{}_{\mu}$
${}^{(5)}K^{\alpha\beta\gamma\delta}$	${}^{(5)}_{(2)}\Phi^{\alpha\beta} = {}^{(5)}_{(1)}\Phi^{\alpha\beta}, {}^{(5)}_{(5)}\Phi^{\alpha\beta} = {}^{(5)}_{(1)}\Phi^{\alpha\beta}, {}^{(5)}_{(4)}\Phi^{\alpha\beta} = -{}^{(5)}_{(1)}\Phi^{\alpha\beta},$
	${}^{(5)}_{(3)}\Phi^{\alpha\beta} = 0,\ {}^{(5)}_{(6)}\Phi^{\alpha\beta} = 0$
	$\phi_{(5)} := {}^{(5)}_{(A)}\Phi^{\mu}{}_{\mu} = 0\ (A = 1, \dots, 6)$

Thus, the Kummer tensor entails the three independent scalars $\{K, L, M\}$. If the "Kummer machinery" is fed with a more symmetric tensor, the Weyl tensor $C_{\alpha\beta\gamma}{}^{\delta}$, for instance, we find further algebraic relations between these scalars. In the case of $T = C$, only one independent scalar, say K, is left over.

The analogous contractions can be made for each irreducible piece ${}^{(I)}K^{\alpha\beta\gamma\delta}[T]$ of the Kummer tensor. We find (see Fig. 6),

$$ {}^{(I)}_{(A)}\Phi^{\alpha\beta}, \quad I = 1, \dots, 6, A = 1, \dots, 6. \tag{127} $$

Since ${}^{(6)}K^{\alpha\beta\gamma\delta}$ is totally antisymmetric, its traces vanish and we have

$$ {}^{(6)}_{(A)}\Phi^{\alpha\beta} \equiv 0, \quad \text{for } A = 1, \dots, 6. \tag{128} $$

Thus, in the future, we need to number the index I only up to 5. Since each irreducible piece of K has six traces and the traces of ${}^{(6)}K$ are identically zero, the first contractions yield altogether $5 \times 6 = 30$ potentially nonvanishing traces. Because of the existing symmetries of the ${}^{(I)}K^{\alpha\beta\gamma\delta}$, not all of these contractions will be independent. With the help of the computer algebra package Reduce-Excalc, see [64–67], we were led to the results collected in Table 1.

In Table 1, the first contractions of ${}^{(2)}K^{\alpha\beta\gamma\delta}$ can be written slightly more compactly as

$$ {}^{(2)}_{(1)}\Phi^{\alpha\beta} = {}^{(2)}_{(2)}\Phi^{\alpha\beta} = {}^{(2)}_{(4)}\Phi^{\alpha\beta} = -{}^{(2)}_{(5)}\Phi^{\alpha\beta}, {}^{(2)}_{(3)}\Phi^{\alpha\beta} = -{}^{(2)}_{(6)}\Phi^{\alpha\beta}. \tag{129} $$

Table 2
Antisymmetric parts of first contractions of *Kummer*.

	$K^{\alpha\beta\gamma\delta}[T] = T^{\mu\alpha\lambda\beta}\,{}^{\star}T^{*}_{\mu\rho\lambda\sigma}\,T^{\rho\gamma\sigma\delta}$
${}^{(1)}K^{\alpha\beta\gamma\delta}$	$\overset{(1)}{\underset{(A)}{}}\Phi^{[\alpha\beta]} = 0\ (A = 1,\ldots,6)$
${}^{(2)}K^{\alpha\beta\gamma\delta}$	$\overset{(2)}{\underset{(1)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(2)}{\underset{(2)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(2)}{\underset{(4)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(2)}{\underset{(5)}{}}\Phi^{[\alpha\beta]} \neq 0,$
	$\overset{(2)}{\underset{(1)}{}}\Phi^{[\alpha\beta]} = \overset{(2)}{\underset{(2)}{}}\Phi^{[\alpha\beta]} = \overset{(2)}{\underset{(4)}{}}\Phi^{[\alpha\beta]} = -\overset{(2)}{\underset{(5)}{}}\Phi^{[\alpha\beta]},$
	$\overset{(2)}{\underset{(3)}{}}\Phi^{[\alpha\beta]} = \overset{(2)}{\underset{(6)}{}}\Phi^{[\alpha\beta]} = 0$
${}^{(3)}K^{\alpha\beta\gamma\delta}$	$\overset{(3)}{\underset{(A)}{}}\Phi^{[\alpha\beta]} = 0\ (A = 1,\ldots,6)$
${}^{(4)}K^{\alpha\beta\gamma\delta}$	$\overset{(4)}{\underset{(A)}{}}\Phi^{[\alpha\beta]} = 0\ (A = 1,\ldots,6)$
${}^{(5)}K^{\alpha\beta\gamma\delta}$	$\overset{(5)}{\underset{(1)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(5)}{\underset{(2)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(5)}{\underset{(4)}{}}\Phi^{[\alpha\beta]} \neq 0,\ \overset{(5)}{\underset{(5)}{}}\Phi^{[\alpha\beta]} \neq 0,$
	$\overset{(5)}{\underset{(1)}{}}\Phi^{[\alpha\beta]} = \overset{(5)}{\underset{(2)}{}}\Phi^{[\alpha\beta]} = -\overset{(5)}{\underset{(4)}{}}\Phi^{[\alpha\beta]} = \overset{(5)}{\underset{(5)}{}}\Phi^{[\alpha\beta]},$
	$\overset{(5)}{\underset{(3)}{}}\Phi^{[\alpha\beta]} = \overset{(5)}{\underset{(6)}{}}\Phi^{[\alpha\beta]} = 0$

Inspection of Table 1 shows that there are six independent first contractions of the Kummer tensor, namely one from each of ${}^{(1)}K$, ${}^{(3)}K$, ${}^{(4)}K$, and ${}^{(5)}K$, two from ${}^{(2)}K$, and none from ${}^{(6)}K$. As a basis for the first contractions $\overset{(I)}{\underset{(A)}{}}\Phi^{\alpha\beta}$, we use the set,

$$\left\{\overset{(1)}{\underset{(1)}{}}\Phi^{\alpha\beta},\ \overset{(2)}{\underset{(1)}{}}\Phi^{\alpha\beta},\ \overset{(2)}{\underset{(3)}{}}\Phi^{\alpha\beta},\ \overset{(3)}{\underset{(1)}{}}\Phi^{\alpha\beta},\ \overset{(4)}{\underset{(1)}{}}\Phi^{\alpha\beta},\ \overset{(5)}{\underset{(1)}{}}\Phi^{\alpha\beta}\right\}. \tag{130}$$

This set corresponds to one realization. Then, with Reduce-Excalc, we find, the following.

• **Proposition on first contractions of *Kummer***: *Define a tensor $\Lambda^{\alpha\beta}$ as a superposition of the elements of the set (130) according to*

$$\Lambda^{\alpha\beta} := \xi_1\,\overset{(1)}{\underset{(1)}{}}\Phi^{\alpha\beta} + \xi_2\,\overset{(2)}{\underset{(1)}{}}\Phi^{\alpha\beta} + \xi_3\,\overset{(2)}{\underset{(3)}{}}\Phi^{\alpha\beta} + \xi_4\,\overset{(3)}{\underset{(1)}{}}\Phi^{\alpha\beta} + \xi_5\,\overset{(4)}{\underset{(1)}{}}\Phi^{\alpha\beta} + \xi_6\,\overset{(5)}{\underset{(1)}{}}\Phi^{\alpha\beta}, \tag{131}$$

with arbitrary constants ξ_I, for $I = 1,\ldots,6$. Then $\Lambda^{\alpha\beta} = 0$ is equivalent to

$$\xi_I = 0, \quad \text{for } I = 1,\ldots,6. \tag{132}$$

Thus, the elements of set (130) are algebraically independent. Reversely, every 2nd rank tensor, constructed by contraction of the Kummer tensor, can be represented as the linear combination (131). Thus, the elements of set (130) span the vector space of the 2nd rank tensors.

Let us now turn in Table 1 to the second contractions of Kummer, the traces. Because the traces $\overset{(2)}{\underset{(A)}{}}\Phi^{\mu}{}_{\mu}$ and $\overset{(5)}{\underset{(A)}{}}\Phi^{\mu}{}_{\mu}$ vanish identically for all $A = 1,\ldots,6$, we will have only three independent scalars. As the corresponding set we choose

$$\{\phi_{(1)}, \phi_{(3)}, \phi_{(4)}\}. \tag{133}$$

These 3 scalars are independent linear combinations of the scalars K, L, M defined in (126).

• **Proposition on second contractions of** *Kummer: The Kummer tensor $K^{\alpha\beta\gamma\delta}[T]$ has three algebraically independent second contraction, such as $\{\phi_{(1)}, \phi_{(3)}, \phi_{(4)}\}$.*

For later use, we collect the antisymmetric pieces of the first contractions of *Kummer* in Table 2.

*4.2. Completely tracefree parts of **Kummer** $K^{\alpha\beta\gamma\delta}[T]$*

In order to form $SO(1,3)$-invariant tensors from the $GL(4,R)$-invariant tensors ${}^{(I)}K^{\alpha\beta\gamma\delta}[T]$, we have to subtract out all traces. Thus, we need the maximal number of the independent tracefree symmetric contractions $\overset{(I)}{\underset{(A)}{}}\widehat{\Phi}^{(\alpha\beta)}$, their antisymmetric counterparts $\overset{(I)}{\underset{(A)}{}}\Phi^{[\alpha\beta]}$, and the traces $\phi_{(I)}$. Here, tracefree objects will be denoted by a hat,

$$\overset{(I)}{\underset{(A)}{}}\widehat{\Phi}^{\alpha\beta} := \overset{(I)}{\underset{(A)}{}}\Phi^{\alpha\beta} - \frac{1}{4}\,\overset{(I)}{\underset{(A)}{}}\Phi^{\mu}{}_{\mu}\,g^{\alpha\beta} \quad (\text{for } I \text{ and } A = 1,\ldots,5). \tag{134}$$

Furthermore, we have to take into consideration all symmetrized or antisymmetrized third rank tensors that can be built from ${}^{(I)}\widehat{K}^{\alpha\beta\gamma\delta}$.

For the completely tracefree parts of ${}^{(I)}K^{\alpha\beta\gamma\delta}$, we find

$$ {}^{(1)}\widehat{K}^{\alpha\beta\gamma\delta} = {}^{(1)}K^{\alpha\beta\gamma\delta} - \frac{3}{4}\,{}^{(1)}_{(1)}\widehat{\Phi}^{(\alpha\beta}g^{\gamma\delta)} - \frac{1}{8}\,\phi_{(1)}\,g^{(\alpha\beta}g^{\gamma\delta)}, \tag{135} $$

$$ \begin{aligned} {}^{(2)}\widehat{K}^{\alpha\beta\gamma\delta} &= {}^{(2)}K^{\alpha\beta\gamma\delta} + \frac{1}{4}\left({}^{(2)}_{(3)}\widehat{\Phi}^{\alpha\gamma}g^{\beta\delta} - {}^{(2)}_{(3)}\widehat{\Phi}^{\beta\delta}g^{\alpha\gamma}\right) \\ &\quad - \frac{1}{3}\left({}^{(2)}_{(1)}\widehat{\Phi}^{\alpha(\beta|}g^{\gamma|\delta)} + {}^{(2)}_{(1)}\widehat{\Phi}^{\gamma(\delta|}g^{\alpha|\beta)}\right), \end{aligned} \tag{136} $$

$$ \begin{aligned} {}^{(3)}\widehat{K}^{\alpha\beta\gamma\delta} &= {}^{(3)}K^{\alpha\beta\gamma\delta} + {}^{(3)}_{(1)}\widehat{\Phi}^{\alpha\gamma}g^{\beta\delta} + {}^{(3)}_{(1)}\widehat{\Phi}^{\beta\delta}g^{\alpha\gamma} \\ &\quad - {}^{(3)}_{(1)}\widehat{\Phi}^{\alpha(\beta|}g^{\gamma|\delta)} - {}^{(3)}_{(1)}\widehat{\Phi}^{\gamma(\beta|}g^{\alpha|\delta)} + \frac{1}{6}\phi_{(3)}g^{\alpha[\beta|}g^{\gamma|\delta]}, \end{aligned} \tag{137} $$

$$ {}^{(4)}\widehat{K}^{\alpha\beta\gamma\delta} = {}^{(4)}K^{\alpha\beta\gamma\delta} - {}^{(4)}_{(1)}\widehat{\Phi}^{\alpha[\beta|}g^{\gamma|\delta]} - {}^{(4)}_{(1)}\widehat{\Phi}^{\gamma[\delta|}g^{\alpha|\beta]} - \frac{1}{6}\phi_{(4)}g^{\alpha[\beta|}g^{\gamma|\delta]}, \tag{138} $$

$$ {}^{(5)}\widehat{K}^{\alpha\beta\gamma\delta} = {}^{(5)}K^{\alpha\beta\gamma\delta} - {}^{(5)}_{(1)}\widehat{\Phi}^{\alpha[\beta|}g^{\gamma|\delta]} - {}^{(5)}_{(1)}\widehat{\Phi}^{\gamma[\delta|}g^{\alpha|\beta]}, \tag{139} $$

$$ {}^{(6)}\widehat{K}^{\alpha\beta\gamma\delta} = {}^{(6)}K^{\alpha\beta\gamma\delta} = K^{[\alpha\beta\gamma\delta]}. \tag{140} $$

The first piece in (135), which, in the electromagnetic case, corresponds to the Tamm–Rubilar tensor, is already irreducibly decomposed under the $SO(1,3)$:

• **Proposition on the irreducible decomposition of the totally symmetric part of *Kummer***: *$K^{(\alpha\beta\gamma\delta)}[T] = {}^{(1)}\widehat{K}^{\alpha\beta\gamma\delta}[T]$ – and with it in the electromagnetic case the Tamm–Rubilar tensor density $\mathcal{G}^{\alpha\beta\gamma\delta}[\chi]$ – decompose irreducibly as follows:*

$$ \begin{aligned} {}^{(1)}K^{\alpha\beta\gamma\delta} &= {}^{(1)}\widehat{K}^{\alpha\beta\gamma\delta} + \frac{3}{4}\,{}^{(1)}_{(1)}\widehat{\Phi}^{(\alpha\beta}g^{\gamma\delta)} + \frac{1}{8}\,\phi_{(1)}\,g^{(\alpha\beta}g^{\gamma\delta)}, \\ 35 &= \quad 25 \quad \oplus \quad 9 \quad \oplus \quad 1. \end{aligned} \tag{141} $$

Note, however, that the tensors (136)–(139) are still reducible. In a next step for a finer decomposition, one has to determine all tensors with symmetric or antisymmetric pairs or triplets of indices. Furthermore, one has to consider tensors like $\widehat{K}^{\alpha[\beta\gamma\delta]}$ and their permutations, together with their corresponding symmetric counterparts. For example we find

$$ \begin{aligned} &{}^{(1)}\widehat{K}^{\alpha[\beta\gamma\delta]} = {}^{(2)}\widehat{K}^{\alpha[\beta\gamma\delta]} = {}^{(3)}\widehat{K}^{\alpha[\beta\gamma\delta]} = {}^{(4)}\widehat{K}^{\alpha[\beta\gamma\delta]} = 0, \\ &\text{and} \quad {}^{(5)}\widehat{K}^{\alpha[\beta\gamma\delta]} \neq 0, \qquad {}^{(6)}K^{\alpha[\beta\gamma\delta]} \neq 0. \end{aligned} $$

In an accompanying paper we will work out a complete list of those tensors.

4.3. Decomposition of $K^{\alpha\beta\gamma\delta}[C]$ in Riemann–Cartan space

The Kummer tensor density $\mathcal{K}^{\alpha\beta\gamma\delta}[\mathcal{T}]$ has 136, see (104), and $\mathcal{K}^{\alpha\beta\gamma\delta}[{}^{(1)}\mathcal{T}]$ only 76 independent components, see (119). Since the principal part ${}^{(1)}\mathcal{T}^{\alpha\beta\gamma\delta}$ has 20, but the Weyl tensor $C^{\alpha\beta\gamma\delta}$ merely 10 independent components, see the decomposition (59), $K^{\alpha\beta\gamma\delta}[C]$ must then have appreciably fewer independent components than 76. If we find its irreducible decomposition, we should be able to determine them.

In analogy to Table 1, we list in Table 3 the contractions of the Kummer–Weyl tensor $K^{\alpha\beta\gamma\delta}[C]$.

We find only one independent first contraction $K^{\alpha\beta}$ of the Kummer–Weyl tensor, and, in turn, only one independent scalar K as second contraction. We choose $\{K^{\alpha\beta}, K\}$ as independent objects for both contractions.

Moreover, we can additionally relate irreducible pieces to each other:

$$ -2\,{}^{(3)}_{(1)}\Phi^{\alpha\beta} = {}^{(1)}_{(1)}\Phi^{\alpha\beta}, \qquad -2\,{}^{(4)}_{(1)}\Phi^{\alpha\beta} = {}^{(1)}_{(1)}\Phi^{\alpha\beta}. \tag{142} $$

Table 3
Contractions of *Kummer–Weyl*.

	$K^{\alpha\beta\gamma\delta}[C] = C^{\mu\alpha\lambda\beta}\,{}^{\star}C^{*}_{\mu\rho\lambda\sigma}C^{\rho\gamma\sigma\delta}$
${}^{(1)}K^{\alpha\beta\gamma\delta}$	${}^{(1)}K_{\mu}^{\mu\alpha\beta} = {}^{(1)}_{(1)}\Phi^{\alpha\beta} = \cdots = {}^{(1)}_{(6)}\Phi^{\alpha\beta} =: K^{\alpha\beta}$
	$K := {}^{(1)}_{(1)}\Phi^{\mu}{}_{\mu} = {}^{(1)}K^{\mu}{}_{\mu\lambda}{}^{\lambda}$
${}^{(2)}K^{\alpha\beta\gamma\delta}$	${}^{(2)}K^{\alpha\beta\gamma\delta} = 0$
${}^{(3)}K^{\alpha\beta\gamma\delta}$	${}^{(3)}_{(1)}\Phi^{\alpha\beta} = {}^{(3)}_{(2)}\Phi^{\alpha\beta} = {}^{(3)}_{(4)}\Phi^{\alpha\beta} = {}^{(3)}_{(5)}\Phi^{\alpha\beta}$,
	${}^{(3)}_{(3)}\Phi^{\alpha\beta} = {}^{(3)}_{(6)}\Phi^{\alpha\beta} = -{}^{(3)}_{(1)}\Phi^{\alpha\beta}/2$
${}^{(4)}K^{\alpha\beta\gamma\delta}$	${}^{(4)}_{(2)}\Phi^{\alpha\beta} = {}^{(4)}_{(1)}\Phi^{\alpha\beta}$,
	${}^{(4)}_{(4)}\Phi^{\alpha\beta} = {}^{(4)}_{(5)}\Phi^{\alpha\beta} = -{}^{(4)}_{(1)}\Phi^{\alpha\beta}$,
	${}^{(4)}_{(3)}\Phi^{\alpha\beta} = {}^{(4)}_{(6)}\Phi^{\alpha\beta} = 0$
${}^{(5)}K^{\alpha\beta\gamma\delta}$	${}^{(5)}K^{\alpha\beta\gamma\delta} = 0$
${}^{(6)}K^{\alpha\beta\gamma\delta}$	$K^{[\alpha\beta\gamma\delta]} \neq 0$

Note that all contractions are symmetric, that is, we do not have antisymmetric pieces of $\Phi^{\alpha\beta}$,

$$ {}^{(I)}_{(A)}\Phi^{[\alpha\beta]} = 0 \quad (I = 1, \ldots, 5\; A = 1, \ldots, 6). \tag{143} $$

Table 3 allows also to prove interrelations between contractions of the various irreducible pieces as, for example,

$$ {}_{(1)}\widehat{K}^{\alpha\beta}[C] := K^{\mu}{}_{\mu}{}^{\alpha\beta} - \frac{1}{4}K^{\mu}{}_{\mu}{}^{\lambda}{}_{\lambda}g^{\alpha\beta} = {}^{(1)}_{(1)}\Phi^{\alpha\beta} + {}^{(1)}_{(3)}\Phi^{\alpha\beta} + {}^{(1)}_{(4)}\Phi^{\alpha\beta} - \frac{1}{4}\left[{}^{(1)}_{(1)}\Phi^{\mu}{}_{\mu} + {}^{(1)}_{(3)}\Phi^{\mu}{}_{\mu} + {}^{(1)}_{(4)}\Phi^{\mu}{}_{\mu}\right]g^{\alpha\beta} = 0. \tag{144} $$

The five remaining contractions, defined analogously, fulfill the trace-free condition, that is, for the full Kummer–Weyl tensor we have

$$ {}_{(A)}\widehat{K}^{\alpha\beta}[C] = 0 \quad (A = 1, \ldots, 6). \tag{145} $$

5. Outlook

In order to win more insight into the properties of the Kummer tensor, we will apply it to exact vacuum solutions of general relativity (GR) and of the Poincaré gauge theory of gravitation (PG). In particular, we will investigate the Kerr solution of GR and the Kerr metric *with torsion* [68,69] within PG. It could be that the considerations of Burinskii [70], who finds Kummer surfaces inside a Kerr black hole, have some relation to our investigations.

In current string theory, the Kummer surface plays a role as a special case of a so-called K3 (Kummer–Kähler–Kodaira) surface, see [71]. In turn, a K3 surface is a special case of a 3-dimensional complex Calabi–Yau manifold, see [72]. These manifolds are used for the compactification of higher dimensions. However, in this application, light propagation does not seem to play a role.

Quite generally, we wonder whether one can also attach a Kummer-like tensor of rank four to the K3 surfaces and the Calabi–Yau manifolds in differential geometry and in string theory, respectively.

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