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GEOMETRY OF THE RENORMALIZATION GROUP WITH AN APPLICATION IN TWO DIMENSIONS

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The renormalization group is viewed as a theory of the geometry of action space. A general covariant relation between coupling constant and field renormalization is derived. As an application, the crossover between the two-dimensional minimal modes M_m and M_{m-1} is calculated to two-loop order in a minimal subtraction scheme.

1. Introduction

Recent years' work has revealed an amazing richness of critical phenomena in two dimensions (for a review see ref. [1]). At a critical point and at distances much larger than any microscopic scale, a system is described by a massless euclidean quantum field theory which characterizes the universality class of the fixed point. In many physically interesting cases, it can be solved exactly by analyzing its infinite-dimensional conformal symmetry [2]. Thus one knows all scaling dimensions and correlation functions of the critical theory directly and need not construct them perturbatively from the gaussian theory.

What is then the rôle of the renormalization group (RG)? It describes the embedding scenario of the conformally invariant theories, which are its fixed points: (a) locally, the field theories in the neighborhood of a fixed point represent its scaling region and (b) globally, the attraction domains of the fixed points characterize the topology of the system's phase diagram.

In two dimensions, one can prove under mild assumptions a striking property of the RG flow, namely Zamolodchikov's $\mathscr{C}$-theorem [3]: there exists a function $\mathscr{C}$ on the space of two-dimensional reflection-positive field theories which is monotonically decreasing along RG trajectories and is stationary only at fixed points, where it equals the central charge c of the corresponding conformal field theory. Thus the central charge, which characterizes the symmetry of the system at a critical point, is related to an entropy-like quantity away from criticality, where that symmetry is broken. Moreover, as shown by Cardy [4], the difference in central charge between two fixed points linked by a RG trajectory can be measured in terms of correlation functions of any noncritical theory on that trajectory.

Perhaps equally important, Zamolodchikov's analysis also sheds a new light on the RG scenario in a general dimension of space: the space of actions, the stage on which the RG acts [5], is endowed with a metric that is nonsingular at the RG fixed points. This suggests to reformulate the RG which, at present, exists in a multitude of different schemes that correspond to different coordinate systems on action space. Under coordinate transformations, physical quantities are covariant, but this is not obvious. New insight may be gained by conceiving the RG as the theory that describes the geometry of action space in which the covariance of physical quantities is manifest.

The present paper is a step in this direction. In sect. 2, we outline the geometric formulation and show that universal quantities are covariant. A simple but important consequence is derived: there is always a geometric link between coupling constant and field renormalization.

In sect. 3, we apply these ideas to a crossover between two-dimensional conformally invariant fixed points that can be treated in perturbation theory [3]. An asymptotic expansion can be performed with a small parameter ε whose geometric significance is the distance between the (infrared-)unstable fixed point and the stable fixed point*. This is quite remarkable since in general even the unstable fixed point theory has, unlike the gaussian theory, a complicated structure of multipoint correlation functions. Here we present a systematic minimal subtraction calculation of the minimal model $M_m(m \gg 1)$, perturbed by its weakest relevant scaling field (i.e. the one with the smallest positive RG eigenvalue). Consistently to the order of two loops, we find that this perturbation induces a crossover to the fixed point M_{m-1}, which confirms Zamolodchikov's result. This crossover is distinguished geometrically by the fact that the RG trajectory joining the two fixed points is a geodesic.

2. Geometry of the renormalization group

Critical phenomena occur at length scales R much larger than any microscopic scale, where the system can be described by an euclidean quantum field theory. The fundamental objects of a field theory are its (connected) correlation functions [3]

$$\left\langle \phi_{a_1}(r_1) \dots \phi_{a_k}(r_k) \right\rangle. \tag{2.1}$$

The spatial coordinates $r \in \mathbb{R}^d$ and the local fields $\phi^a(r) \in \mathbb{R}$ are conveniently taken to be dimensionless quantities scaled by the length unit R that characterizes the

* The parameter in the usual ε-expansion about the upper critical dimension can be interpreted in the same way.

observer rather than the system. The fields $\phi^a(r)$ span the infinite-dimensional operator-algebra $\mathscr{A}$. The k-point correlation functions are maps $\otimes_k \mathscr{A} \to \mathbb{R}$, i.e. tensors of rank $(0, k)$. (In the sequel, Latin letters $a, b, c, \ldots$ are used as geometric indices, Greek letters $\alpha, \beta, \gamma, \ldots$ as coordinate indices; see ref. [6].) Since all measurable quantities can be expressed in terms of the correlation functions, we may define the field theory, independently of any functional integral description, as the collection of all its correlation functions (2.1).

The measurements depend on a number of (scalar) intensive thermodynamic parameters $u^\alpha(\alpha = 1, \ldots, n)$, which for simplicity are assumed to be in equilibrium, i.e. spatially homogeneous. The set of field theories related to each other by a continuous change of these experimental parameters is called the thermodynamic state manifold $\mathscr{M}$ of the system. The u^α are regarded as coordinates on $\mathscr{M}$. Their choice is a gauge freedom; equivalent coordinate systems are related by diffeomorphisms.

Associated with these coordinates, there are local sources $u^\alpha(r)$ and their conjugate rotation-scalar fields $\phi_\alpha(r) \in \mathscr{A}$, by means of which the correlation functions can be generated from a scalar functional*

$$\left\langle \phi_{\alpha_1}(r_1) \ldots \phi_{\alpha_k}(r_k) \right\rangle_u = \left. \frac{\delta^n}{\delta u^{\alpha_1}(r_1) \ldots \delta u^{\alpha_k}(r_k)} \right|_{u(r)=u} F\{u(r)\} . \tag{2.2}$$

It can be defined perturbatively in special "bare" coordinates u_0 on a neighborhood of some point on $\mathscr{M}$ (taken to be the coordinate origin) by the exponential mapping

$$F\{u_0(r)\} = \ln \left\langle \exp \int u_0{}^\alpha(r) \phi^0{}_\alpha(r) \, \mathrm{d}^d r \right\rangle_{u_0 = 0} . \tag{2.3}$$

All fields may appear in the exponential; hence this formula defines a neighborhood of the point $u_0{}^\alpha = 0$ in an infinite-dimensional manifold $\mathscr{S}$ (parametrized by the coordinates $u_0{}^\alpha$) in which the finite-dimensional manifold $\mathscr{M}$ is embedded. The space of zero-momentum fields $\phi_a \equiv \int \phi_a(r) \, \mathrm{d}^d r$ can be identified with the tangent space $T\mathscr{S}$; that is, these fields act as derivative operators on the tensor fields (the correlation functions). In particular, the bare coordinate derivatives $\partial^0{}_\gamma \equiv \partial / \partial u_0{}^\gamma$ read

$$\partial^0{}_\gamma \left\langle \phi^0{}_{\alpha_1}(r_1) \ldots \phi^0{}_{\alpha_k}(r_k) \right\rangle_{u_0} = \left\langle \phi^0{}_\gamma \phi^0{}_{\alpha_1}(r_1) \ldots \phi^0{}_{\alpha_k}(r_k) \right\rangle_{u_0} . \tag{2.4}$$

In general, this expression contains divergences that make the bare perturbation series (2.3) meaningless. New renormalized coordinates have to be defined. To this

* The function $F(u) \equiv F\{u(r) = u\}$ on $\mathscr{M}$ is the grand canonical entropy of the system, the Legendre transform of the entropy $S(\langle \phi_a \rangle)$ with respect to the extensive thermodynamic parameters $\langle \phi_\alpha \rangle = \partial F / \partial u^\alpha$.

end, it is convenient to express F as a functional integral

$$\exp F\{u(r)\} = \int \mathscr{D}\Phi \exp S\{\Phi(r), u(r)\}. \tag{2.5}$$

Here $\Phi(r)$ represents a subset of $\mathscr{A}$ chosen as "elementary" fields, in terms of which the remaining fields are written as "composite" fields [3]. $\mathscr{S}$ is now interpreted as the space of actions S, parameterized by the coupling constants u^α. (The choice of bare coordinates $u_0^{\ \alpha}$ corresponds to the decomposition $S = S_0 + S_i$ into an "unperturbed" action S_0 at the point $u_0^{\ \alpha} = 0$ and the perturbation $S_i = \int u_0^{\ \alpha}(r)\phi^0_{\ \alpha}(r)\,\mathrm{d}^d r$.) The gauge freedom in the choice of coordinates is realized in the functional integral as follows. Any transformation of variables $\Phi(r) \mapsto \Phi'(r)$ that leaves the partition function (2.5) invariant induces a change in the action

$$S'\{\Phi'\} = S\{\Phi\{\Phi'\}\} - \int \ln \frac{\partial \Phi'(r)}{\partial \Phi(r)}, \tag{2.6}$$

that is, it acts as a diffeomorphism on $\mathscr{S}$ under which F transforms as a scalar and the correlation functions (2.1) transform as tensors of rank $(0, k)$.

Consider now a one-parameter group of diffeomorphisms with group parameter s and generating vector field $\sigma = \sigma^\alpha \phi_\alpha$. The differential change of a tensor field $\Xi^{b_1 \dots b_l}_{a_1 \dots a_k}$ of rank (l, k) under this group of transformations is given by its Lie-derivative* with respect to the vector field σ,

$$\mathscr{L}_\sigma \Xi^{b_1 \dots b_j \dots b_l}_{a_1 \dots a_i \dots a_k} = \sigma^c \cdot \nabla_c \Xi^{b_1 \dots b_j \dots b_l}_{a_1 \dots a_i \dots a_k} + \sum_{i=1}^{k} \left(\nabla_{a_i} \cdot \sigma^c\right) \Xi^{b_1 \dots b_j \dots b_l}_{a_1 \dots c \dots a_k} - \sum_{j=1}^{l} \left(\nabla_c \cdot \sigma^{b_j}\right) \Xi^{b_1 \dots c \dots b_l}_{a_1 \dots a_i \dots a_k}. \tag{2.7}$$

In particular, for the correlation functions (2.1) one obtains

$$\begin{aligned} &\left(-\frac{\mathrm{d}}{\mathrm{d}s} + \mathscr{L}_\sigma\right)\left\langle \phi_{a_1}(r_1) \dots \phi_{a_k}(r_k)\right\rangle_u \\ &\quad = \left(-\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^c \cdot \nabla_c + \sum_{i=1}^{k} \gamma_{(i)}\right)\left\langle \phi_{a_1}(r_1) \dots \phi_{a_k}(r_k)\right\rangle_u = 0. \end{aligned} \tag{2.8}$$

Here γ is the $(1,1)$-tensor with components $\gamma_a^{\ c} = \nabla_a \sigma^c$; the index i indicates that it multiplies the field $\phi_{a_i}(r_i)$:

$$(\gamma\phi)_{a_i}(r_i) = \gamma_{a_i}^{\ \ c} \phi_c(r_i). \tag{2.9}$$

* A concise introduction can e.g. be found in ref. [6].

Notice that the Lie-derivative does not depend on the connection ∇_c (provided it is torsion-free). Hence, at least formally, it may be evaluated using the coordinate derivative operator (2.4),

$$\sigma_0{}^{\gamma}\partial^0{}_{\gamma}\left\langle \phi^0{}_{\alpha_1}(r_1)\dots\phi^0{}_{\alpha_k}(r_k)\right\rangle = \left\langle \sigma\,\phi^0{}_{\alpha_1}(r_1)\dots\phi^0{}_{\alpha_k}(r_k)\right\rangle. \tag{2.10}$$

The functional integral (2.5) is meaningful only if it is regularized at distances smaller than some scale $a \ll R$. A RG transformation is a change of the observer's scale R at a fixed cutoff a (or equivalently [7], a change of a at fixed R) that leaves F invariant, i.e. that acts a gauge transformation on the functional integral. We shall assume that the cutoff in (2.5) is implicit in the form of the interaction (decaying rapidly at momentum scales $q \gtrsim a^{-1}$) rather than given by a restriction on the functional measure. As shown by Wegner [8], a RG transformation can then be regarded as a transformation of the elementary fields with a judicious choice of $\Phi'\{\Phi\}$. It follows that *the renormalization group is the one-parameter group of action space diffeomorphisms $\mathscr{R}_t$ with group parameter* $t = \ln(a/R)$. Its generating vector field $\theta = \beta^{\alpha}\phi_{\alpha}$ is the trace of the stress–energy tensor; the components $\beta^{\alpha} = (\mathrm{d}/\mathrm{d}t)u^{\alpha}$ are called beta-functions. A gauge transformation of the functional integral that leaves t fixed is redundant [8] in the sense of the RG.

The finitely many thermodynamic parameters determine the long-distance behavior of the system in a universal way. Therefore, the underlying field theory should be renormalizable, i.e. in $\mathscr{S}$ there exists a finite-dimensional submanifold $\mathscr{M}$ which is an attractor under the RG flow (and hence left invariant by it: $\theta(u) \in T\mathscr{M}$ for all $u \in \mathscr{M}$). This submanifold is to be identified with the thermodynamic manifold $\mathscr{M}$.

The correlation functions at scale R obey the Ward identity

$$\left(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathscr{L}_{\theta}\right)\left\langle \phi_{a_1}(r_1)\dots\phi_{a_k}(r_k)\right\rangle_u$$
$$= \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \beta^c\cdot\nabla_c + \sum_{i=1}^{k}\gamma_{(i)}\right)\left\langle \phi_{a_1}(r_1)\dots\phi_{a_k}(r_k)\right\rangle_u = 0, \tag{2.11}$$

which may again be evaluated with the coordinate derivative (2.4),

$$\beta_0{}^{\gamma}\partial^0{}_{\gamma}\left\langle \phi^0{}_{\alpha_1}(r_1)\dots\phi^0{}_{\alpha_k}(r_k)\right\rangle = \left\langle \theta\,\phi^0{}_{\alpha_1}(r_1)\dots\phi^0{}_{\alpha_k}(r_k)\right\rangle. \tag{2.12}$$

The $(1,1)$-tensor

$$\gamma_c{}^a(u) = \nabla_c\beta^a(u) \tag{2.13}$$

is the representation of the RG on the dual vector space $\mathscr{A}^*$. Since the RG is abelian, $\gamma_c{}^a$ decomposes into one-dimensional irreducible representations, i.e. it is diagonalizable. Its eigenvalues are invariant under diffeomorphisms and are called (anomalous) scaling dimensions; the corresponding eigenvectors are the scaling fields.

At a fixed point u^* of the RG, $\theta(u^*)$ vanishes and (2.11) expresses the scale invariance of the correlation functions. The scaling dimensions at the fixed point,

$$\gamma^{*}{}_{c}{}^{a} = \nabla_c \beta^a(u^*) \tag{2.14}$$

are independent of the connection ∇_c on action space. Renormalizability implies that only finitely many of them are negative. One can define a metric on $\mathscr{S}$ which is regular at all RG fixed points by [3]

$$g_{ab}(u) \equiv \langle \phi_a(0) \phi_b(1) \rangle_u , \tag{2.15}$$

and let ∇_c denote its Christoffel connection. Then it is easy to show* that the structure constants $C_{bc}{}^{a}$ of the operator algebra $\mathscr{A}$ at the fixed point u^* are given by**

$$C_{bc}{}^{a} = -\frac{1}{2\pi} \nabla_b \nabla_c \beta^a(u^*) . \tag{2.16}$$

Hence *the covariant RG flow in the neighborhood of a fixed point contains all information about the fixed point theory.*

In the special case of a conformally invariant fixed point in $d = 2$, the metric and the beta-functions determine the RG flow of Zamolodchikov's $\mathscr{C}$-function***

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{C}(u) = 6\pi^2 \beta^a(u) g_{ab}(u) \beta^b(u) . \tag{2.17}$$

The scaling dimensions (2.14) and the structure constants (2.16) can then be expressed covariantly in terms of the $\mathscr{C}$-function,

$$\gamma^{*}{}_{c}{}^{a} = \frac{1}{6\pi^2} g^{ad} \nabla_c \nabla_d \mathscr{C}(u^*) , \qquad C_{bc}{}^{a} = \frac{1}{12\pi^3} g^{ad} \nabla_b \nabla_c \nabla_d \mathscr{C}(u^*) . \quad (2.18),(2.19)$$

Consider now the crossover from an unstable fixed point u^* to other fixed points located at a finite distance from u^*, defined by the metric (2.15). The bare coordinates u_0 in a neighborhood of u^* are given in terms of generic coordinates u

* This follows from the one-loop beta-functions; a particularly straightforward derivation of them can be found in ref. [9], sect. 6.4.

** Since $[\nabla_b, \nabla_c]\beta^a(u^*) = R_{bcd}{}^{a}(u^*)\beta^d(u^*) = 0$ at the fixed point u^*, there is no factor ordering ambiguity in eqs. (2.16) and (2.19).

*** The numerical constant in this equation is determined by the normalization of the stress–energy tensor; see ref. [1].

by the nonlinear transformation

$$u_0 = \lim_{t_0 \to \infty} \mathrm{e}^{-t_0 \gamma^*}\left(\mathscr{R}_{t_0}(u) - u^*\right). \tag{2.20}$$

They are exactly Wegner's scaling coordinates (he calls them scaling fields) that linearize the beta-functions,

$$\beta_0{}^{\alpha} = u_0{}^{\nu} \gamma^*{}_{\nu}{}^{\alpha}. \tag{2.21}$$

Hence all the other fixed points are at infinity in these coordinates. This ties in with the singularity of the coordinate derivative (2.4). But all covariant quantities remain regular; these singularities are coordinate singularities*. They can be absorbed, order by order in perturbation theory, into a coordinate transformation

$$u^{\alpha} = Z^{\alpha}{}_{\nu}(u) u_0{}^{\nu}, \qquad \phi_{\alpha}(r) = \phi^0{}_{\nu}(r) \tilde{Z}^{\nu}{}_{\alpha}(u) \qquad (2.22),(2.23)$$

to renormalized coordinates u^{α} and fields $\phi_{\alpha}(r)$. In the renormalized coordinates, the other fixed points are at a finite coordinate distance. This transformation is a matter of calculational convenience only since the interesting physical quantities (2.14) and (2.16) are covariant. Different sets of renormalized coordinates are related by diffeomorphisms that leave the other fixed points at a finite coordinate distance. This is a residual gauge freedom that corresponds to the choice of the RG scheme.

The Z-factors Z and $\tilde{Z}$ determine the beta-functions β^a and the anomalous dimensions $\gamma_c{}^a$, respectively. Hence, the important interpretation of eq. (2.13) is that *the fact that the coupling constants are coordinates and the fields are tangent vectors on a manifold geometrically unifies their renormalization.*

This fact can be put to use whenever the renormalization is carried out at the level of the operator algebra, since then the field renormalization $\tilde{Z}$ is relatively straightforward to compute, while the coupling constant renormalization Z is not directly accessible. An example is given in the next section.

3. The crossover between minimal models

The two-dimensional minimal models M_m are well-studied conformally invariant [2] and reflection-positive [10] euclidean quantum field theories. The minimal models for $m = 3, 4, 5$ have been argued to describe critical melting transitions of atomic monolayers on crystal surfaces [12].

* Coordinate singularities are familiar in general relativity; the most famous example is Schwarzschild coordinates at the event horizon of a Schwarzschild black hole [6].

The diagonal series $m=3,4,5,\ldots$ represents the universality class of the Landau–Ginzburg model

$$\int \mathscr{D}\Phi \exp\left\{-\int\left[(\nabla\Phi)^2+\sum_{j=1}^{2m-2} u^{(j)}:\Phi^j:\right]\mathrm{d}^2 r\right\} \tag{3.1}$$

at its multicritical point $u^{(1)}=u^{(2)}=\cdots=u^{(2m-4)}=0$ [11]. The operator algebra has a basis of spinless scaling fields. It consists of

(i) $m(m-1)/2$ primary fields $\phi_{(p,q)}$ ($1 \leqslant p \leqslant m-1$, $1 \leqslant q \leqslant m$, with the identification $\phi_{(p,q)}=\phi_{(m-p,m+1-q)}$), having scaling dimensions

$$x_{(p,q)} = 2 \times \frac{[(m+1)p-mq]^2-1}{4m(m+1)}\,. \tag{3.2}$$

Of them, $2m-4$ fields have scaling dimension <2; i.e. they are relevant in the sense of the RG. They can be identified with the composite Landau–Ginzburg fields $:\Phi^j:$ ($1 \leqslant j \leqslant 2m-4$). The field $:\Phi^{2m-3}:$ is redundant in the sense of the RG and does not appear in the operator algebra of the conformal field theory.

(ii) infinitely many secondary fields generated from each primary field $\phi_{(p,q)}$ by the Virasoro generators L_{-j} and $\bar{L}_{-\bar{j}}$ ($j, \bar{j}=1,2,3,\ldots$), with scaling dimensions $x_{(p,q)}+N$ ($N=1,2,3,\ldots$). All rotation-scalar secondary fields are irrelevant in the sense of the RG.

The two-point function of primary fields is (with a suitable normalization)

$$\left\langle \phi_\alpha(r_1)\phi_\beta(r_2)\right\rangle = \frac{1}{r_{12}^{x_\alpha+x_\beta}}\delta_{\alpha\beta}\,, \tag{3.3}$$

where $r_{12} \equiv |r_1-r_2|$. Hence, at the fixed point corresponding to M_m, the scaling basis diagonalizes the metric (2.15) in the subspace of primary fields.

The existence of an operator algebra implies that all multipoint correlation functions can be expressed in terms of two-point functions and operator product coefficients; these are the basic constituents of the theory. For primary fields, one obtains the three-point function

$$\left\langle \phi_\alpha(r_1)\phi_\beta(r_2)\phi_\gamma(r_3)\right\rangle = \frac{1}{r_{12}^{x_\alpha+x_\beta-x_\gamma}r_{23}^{x_\beta+x_\gamma-x_\alpha}r_{31}^{x_\gamma+x_\alpha-x_\beta}}C_{\alpha\beta\gamma} \tag{3.4}$$

and the full four-point function (in boldface notation to distinguish it from its connected part)

$$\langle \phi_\alpha(r_1)\phi_\beta(r_2)\phi_\gamma(r_3)\phi_\delta(r_4)\rangle = \frac{r_{13}^{x_\beta+x_\delta} r_{24}^{x_\alpha+x_\gamma}}{r_{12}^{x_\alpha+x_\beta} r_{23}^{x_\beta+x_\gamma} r_{34}^{x_\gamma+x_\delta} r_{41}^{x_\delta+x_\alpha}} F_{\alpha\beta\gamma\delta}(\rho), \quad (3.5)$$

with $\rho \equiv (r_{13}r_{24})/(r_{12}r_{34})$. The scaling function $F_{\alpha\beta\gamma\delta}(\rho)$ is a sum

$$F_{\alpha\beta\gamma\delta}(\rho) = \sum_\nu C_{\alpha\beta}{}^\nu C_{\nu\gamma\delta} F^{(\nu)}(\rho) \quad (3.6)$$

in which each term $F^{(\nu)}$ is in turn an infinite sum of two-point functions of the primary field ϕ_ν and its secondaries [2, 13].

We wish to calculate the RG flow of the model M_m under the perturbation

$$S_i = \int u_0 \phi^0(r)\, \mathrm{d}^2 r \quad (3.7)$$

with the relevant energy-like scaling field $\phi^0 \equiv \phi_{(1,3)} = :\Phi^{2m-4}:$. Via (2.3), this perturbation defines a one-dimensional submanifold $\mathscr{M}_1$ of $\mathscr{M}$. Two properties of $\phi_{(1,3)}$ are important:

(i) it has a finite coupling $C_{(1,3)(1,3)(1,1)} = 1$ to the identity $\phi_{(1,1)}$ (corresponding to the normalization (3.3) of the two-point function) and a finite self-coupling $C_{(1,3)(1,3)(1,3)} \equiv C$, but decouples from all other relevant scaling fields ϕ_ν (i.e. $C_{(1,3)(1,3)\nu} = 0$). This implies that $\mathscr{M}_1$ is a geodesic RG trajectory and can therefore be left invariant by the transformation (2.22), (2.23) to renormalized coordinates and fields. Within this subspace, the transformation then reads

$$u = Z(u) u_0, \qquad \phi = \phi^0 \tilde{Z}(u). \quad (3.8), (3.9)$$

The Z-factors determine β and γ [7],

$$\beta(u) = \gamma^* u \Big/ \left(1 - u\frac{\mathrm{d}}{\mathrm{d}u}\ln Z\right), \qquad \gamma(u) = \gamma^* - \beta\frac{\mathrm{d}}{\mathrm{d}u}\ln \tilde{Z}. \quad (3.10), (3.11)$$

In this case, the relation (2.13), $\gamma = (\mathrm{d}/\mathrm{d}u)\beta$, can be integrated and yields a relation between Z and $\tilde{Z}$:

$$\tilde{Z} = \left[1 - u\frac{\mathrm{d}}{\mathrm{d}u}\ln Z\right] \Big/ Z. \quad (3.12)$$

(ii) for $m \gg 1$, it becomes nearly marginal:

$$x_{(1,3)} = 2 - 4/(m+1) \equiv 2 - \varepsilon. \quad (3.13)$$

Hence, at a distance of order ε from the unstable fixed point $u^* = 0$, the trajectory $\mathscr{M}_1$ contains a nontrivial fixed point u^{**} (the point $u_0 = \infty$).

We make an ε-expansion by analytic continuation in the central charge

$$c = 1 - 6/m(m+1) \tag{3.14}$$

to $c = 1$, the accumulation point of the sequence of minimal models. This replaces the usual ε-expansion about the upper critical spatial dimension.

From (2.5) one obtains the bare perturbation series for the two-point function to order u_0^2,

$$\begin{aligned}\left\langle \phi^0(0)\phi^0(1)\right\rangle_{u_0} = 1 &+ u_0 \int \left\langle \phi^0(0)\phi^0(1)\phi^0(r_1)\right\rangle \mathrm{d}^2 r_1 \\ &+ \tfrac{1}{2}u_0^2 \int \left\langle \phi^0(0)\phi^0(1)\phi^0(r_1)\phi^0(r_2)\right\rangle \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2 + \mathrm{O}\left(u_0^3\right).\end{aligned} \tag{3.15}$$

Here the three-point function is of the form (3.4); it can be integrated exactly in terms of gamma-functions. The four-point function is more difficult. Only in the limit $c \to 1$, it has a simple form and it can be constructed from its crossing symmetry [2] and meromorphy properties (see appendix A). In general, each term $F^{(\nu)}$ of the scaling function (3.6) is the square of a generalized hypergeometric function (the solution of a third order differential equation) for which no closed expression exists. Perhaps somewhat surprisingly, the singular part of the integral in (3.15) may still be expressed in terms of gamma-functions. The resulting Laurent series for the bare two-point function is

$$\left\langle \phi^0(0)\phi^0(1)\right\rangle = 1 + \left[\frac{4\pi C}{\varepsilon} + \mathrm{O}(\varepsilon^2)\right]u_0 + \left[\frac{10\pi^2 C^2}{\varepsilon^2} - \frac{3\pi^2 C^2}{2\varepsilon} + \mathrm{O}(\varepsilon^0)\right]u_0^2 + \mathrm{O}\left(u_0^3\right). \tag{3.16}$$

This formula is derived in appendix B. Now the transformation (3.8), (3.9) has to be constructed order by order in such a way that the renormalized two-point function

$$\left\langle \phi(0)\phi(1)\right\rangle_u = \tilde{Z}^2(u)\left\langle \phi^0(0)\phi^0(1)\right\rangle_{u_0 = Z^{-1}(u)u} \tag{3.17}$$

is regular. This determines the renormalized quantities only up to finite coordinate reparametrizations. We fix this residual gauge freedom by choosing a minimal subtraction scheme, i.e. by requiring that Z and $\tilde{Z}$ have no regular part. Upon inserting (3.16) and (3.12) in (3.17), we then obtain to order u^2

$$\tilde{Z}(u) = 1 - \frac{2\pi C}{\varepsilon} u + \left[\frac{3\pi^2 C^2}{\varepsilon^2} + \frac{3\pi^2 C^2}{4\varepsilon} \right] u^2 + \mathrm{O}(u^3), \tag{3.18}$$

$$Z(u) = 1 + \frac{\pi C}{\varepsilon} u - \frac{\pi^2 C^2}{4\varepsilon} u^2 + \mathrm{O}(u^3) \tag{3.19}$$

and

$$\langle \phi(0) \phi(1) \rangle_u = 1 + \mathrm{O}(\varepsilon^2 u, u^2). \tag{3.20}$$

Hence u is a normal coordinate on $\mathscr{M}_1$ about the fixed point $u^* = 0$ to order ε^2. From (3.10), (3.11), (3.18), (3.19), (3.20) and (2.17), we have

$$\gamma(u) = -\varepsilon - 2\pi C u + \tfrac{3}{2}\pi^2 C^2 u^2 + \mathrm{O}(u^3), \tag{3.21}$$

$$\beta(u) = -\varepsilon u - \pi C u^2 + \tfrac{1}{2}\pi^2 C^2 u^3 + \mathrm{O}(u^4), \tag{3.22}$$

$$\mathscr{C}(u) - \mathscr{C}(u^* = 0) = 6\pi^2 \left[-\tfrac{1}{2}\varepsilon u^2 - \tfrac{1}{3}\pi C u^3 + \tfrac{1}{8}\pi^2 C^2 u^4 + \mathrm{O}(u^5) \right]. \tag{3.23}$$

The beta-function has the infrared-stable fixed point

$$u^{**} = \frac{1}{\pi C} \left(-\varepsilon + \tfrac{1}{2}\varepsilon^2 + \mathrm{O}(\varepsilon^3) \right), \tag{3.24}$$

where

$$\gamma^{**} = \varepsilon + \tfrac{1}{2}\varepsilon^2 + \mathrm{O}(\varepsilon^3), \tag{3.25}$$

$$\mathscr{C}(u^{**}) - \mathscr{C}(u^* = 0) = -\frac{1}{C^2}\varepsilon^3 + \frac{3}{4C^2}\varepsilon^4 + \mathrm{O}(\varepsilon^5) = -\tfrac{3}{16}\varepsilon^3 - \tfrac{9}{64}\varepsilon^4 + \mathrm{O}(\varepsilon^5). \tag{3.26}$$

Comparison with (3.14) shows that $\mathscr{C}(u^{**})$ equals to the calculated order the central charge of the fixed point M_{m-1}. At this infrared-stable fixed point, the perturbing field has to be irrelevant. Eqs. (3.25) and (3.2) say that it becomes the leading irrelevant scaling field $\phi_{(3,1)}$. This is precisely what one would expect from the Landau–Ginzburg interpretation.

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Appendix A

In this appendix, the exact scaling function (3.6) of the perturbing field $\phi^0 \equiv \phi_{(1,3)}$ is constructed in the limit $c = 1$, where $\phi_{(1,3)}$ becomes marginal ($\varepsilon = 0$).

The infinite conformal symmetry implies that, after reparametrizing $r = (x, y)$ in terms of complex coordinates ($z = x + iy$, $\bar{z} = x - iy$), each term $F^{(\nu)}$ of the scaling function (3.6) factorizes into an analytic function of the holomorphic cross-ratio $\zeta \equiv (z_1(z_2 - 1))/((z_1 - 1)z_2)$ and its complex conjugate function,

$$F^{(\nu)}(\rho) \equiv F^{(\nu)}(\zeta, \bar{\zeta}) = \mathscr{F}^{(\nu)}(\zeta)\mathscr{F}^{(\nu)}(\bar{\zeta}). \tag{A.1}$$

No closed expressions are known in general for these "conformal blocks", but they may be computed as power series about any of their singular points 0, 1, and ∞ [2]. In the case of the minimal models, the $\mathscr{F}^{(\nu)}$ are solutions of differential equations, and there is a divergent contribution to the coefficients of the power series from the null fields. After this has been projected out, the coefficients are well behaved, even in the limit $c = 1$.

Specifically for the field $\phi_{(1,3)}$, there are three conformal blocks corresponding to the three intermediate channels $\nu = (1,1)$, $\nu = (1,3)$, and $\nu = (1,5)$ in (3.6). A lengthy calculation yields in the limit $\varepsilon = 0$ the power series about $\zeta = 0$,

$$\begin{aligned}
\mathscr{F}_{\varepsilon=0}^{(1,1)}(\zeta) &= \zeta^{-2} + 2 + 2\zeta + \tfrac{7}{3}\zeta^2 + \mathrm{O}(\zeta^3), \\
\mathscr{F}_{\varepsilon=0}^{(1,3)}(\zeta) &= \zeta^{-1} + \tfrac{1}{2} + \zeta + \tfrac{5}{4}\zeta^2 + \mathrm{O}(\zeta^3), \\
\mathscr{F}_{\varepsilon=0}^{(1,5)}(\zeta) &= \zeta^2 + \mathrm{O}(\zeta^3).
\end{aligned} \tag{A.2}$$

The important simplification for $\varepsilon = 0$ is that the conformal blocks become meromorphic functions, which are completely determined by their poles. These poles are at $\zeta = 0$, 1, and ∞, and of first and second order; from (A.2), it is a mere algebraic exercise to compute their coefficients,

$$\begin{aligned}
\mathscr{F}_{\varepsilon=0}^{(1,1)} &= \zeta^{-2} + \tfrac{1}{3}(1-\zeta)^{-2} + \tfrac{4}{3}(1-\zeta)^{-1} + \tfrac{1}{3}, \\
\mathscr{F}_{\varepsilon=0}^{(1,3)} &= \zeta^{-1} + \tfrac{1}{4}(1-\zeta)^{-2} + \tfrac{1}{2}(1-\zeta)^{-1} + \tfrac{1}{4}, \\
\mathscr{F}_{\varepsilon=0}^{(1,5)} &= (1-\zeta)^{-2} - 2(1-\zeta)^{-1} + 1.
\end{aligned} \tag{A.3}$$

The scaling function is then

$$\begin{aligned}
F_{\varepsilon=0}(\zeta, \bar{\zeta}) = {}& \mathscr{F}_{\varepsilon=0}^{(1,1)}(\zeta)\mathscr{F}_{\varepsilon=0}^{(1,1)}(\bar{\zeta}) + C_{\varepsilon=0}^2 \mathscr{F}_{\varepsilon=0}^{(1,3)}(\zeta)\mathscr{F}_{\varepsilon=0}^{(1,3)}(\bar{\zeta}) \\
& + \check{C}_{\varepsilon=0}^2 \mathscr{F}_{\varepsilon=0}^{(1,5)}(\zeta)\mathscr{F}_{\varepsilon=0}^{(1,5)}(\bar{\zeta}).
\end{aligned} \tag{A.4}$$

The as yet unknown couplings $C \equiv C_{(1,3)(1,3)(1,3)}$ and $\check{C} \equiv C_{(1,3)(1,3)(1,5)}$ can be readily determined by imposing crossing symmetry,

$$F_{\epsilon=0}(\zeta, \bar{\zeta}) = F_{\epsilon=0}(1-\zeta, 1-\bar{\zeta}) = \zeta^{-2}\bar{\zeta}^{-2} F_{\epsilon=0}(\zeta^{-1}, \bar{\zeta}^{-1}). \tag{A.5}$$

One obtains the correct limit value of the expression known from Coulomb gas methods [13], namely

$$C^2_{\epsilon=0} = \tfrac{16}{3}, \qquad \check{C}^2_{\epsilon=0} = \tfrac{5}{9}. \tag{A.6}$$

The scaling function can now be written in a manifestly crossing-symmetric way as a sum of its disconnected part and its connected part:

$$F_{\epsilon=0}(\zeta, \bar{\zeta}) = F^{\text{dis}}_{\epsilon=0}(\zeta, \bar{\zeta}) + F_{\epsilon=0}(\zeta, \bar{\zeta}), \tag{A.7}$$

where

$$F^{\text{dis}}_{\epsilon=0}(\zeta, \bar{\zeta}) = \zeta^{-2}\bar{\zeta}^{-2} + (1-\zeta)^{-2}(1-\bar{\zeta})^{-2} + 1, \tag{A.8}$$

$$\begin{aligned} F_{\epsilon=0}(\zeta, \bar{\zeta}) = {} & \tfrac{1}{16} C^2_{\epsilon=0} 2\,\mathrm{Re}\Big[\zeta^{-2} + (1-\zeta)^{-2} + \zeta^{-2}(1-\bar{\zeta})^{-2}\Big] \\ & + \tfrac{1}{4} \check{C}^2_{\epsilon=0} 2\,\mathrm{Re}\Big[\zeta^{-1}(1-\bar{\zeta})^{-2} + \zeta^{-2}(1-\bar{\zeta})^{-1} - \zeta^{-1}(1-\zeta)^{-1}\Big] \\ & + \tfrac{1}{2} C^2_{\epsilon=0}\Big[\zeta^{-1}\bar{\zeta}^{-1} + (1-\zeta)^{-1}(1-\bar{\zeta})^{-1} + \zeta^{-1}\bar{\zeta}^{-1}(1-\zeta)^{-1}(1-\bar{\zeta})^{-1}\Big]. \end{aligned} \tag{A.9}$$

Appendix B

In this appendix, the Laurent expansion (3.16) for the integrals (3.15) is derived. Consider first the integral

$$I(a, \bar{a}, b, \bar{b}) \equiv \int_{\mathbb{R}^2} z^a \bar{z}^{\bar{a}} (z-1)^b (\bar{z}-1)^{\bar{b}}\, \mathrm{d}^2 r, \tag{B.1}$$

where a, $\bar{a}$, b, $\bar{b} \in \mathbb{R}$ (the bars do not denote complex conjugation) and, to ensure univaluedness of the integrand, $a - \bar{a}$ and $b - \bar{b}$ are integers. After a Wick contour rotation in the complex $\mathrm{Im}\, z$ plane, the integral factorizes into the product of two one-dimensional real integrals that can be solved in terms of gamma-functions,

$$I(a,\bar{a},b,\bar{b}) = \pi \frac{\Gamma(-1-a-b)\Gamma(1+\bar{a})\Gamma(1+\bar{b})}{\Gamma(-a)\Gamma(-b)\Gamma(2+\bar{a}+\bar{b})}. \tag{B.2}$$

It is easy to verify that this expression is real,

$$I(a,\bar{a},b,\bar{b}) = I(\bar{a},a,\bar{b},b), \tag{B.3}$$

and enjoys the symmetries

$$I(a,\bar{a},b,\bar{b}) = I(b,\bar{b},a,\bar{a}) = I(-2-a-b,-2-\bar{a}-\bar{b},b,\bar{b}). \tag{B.4}$$

They express the invariance of the integral under the group of conformal reparametrizations generated by $z \mapsto 1-z$ and $z \mapsto z^{-1}$, which permute the three poles at 0, 1, and ∞.

Thus one obtains for the integral over the three-point function, using (3.4) and (3.13),

$$\begin{aligned}\int \langle \phi^0(0)\phi^0(1)\phi^0(r)\rangle \, \mathrm{d}^2 r &= I(-1+\tfrac{1}{2}\varepsilon,-1+\tfrac{1}{2}\varepsilon,-1+\tfrac{1}{2}\varepsilon,-1+\tfrac{1}{2}\varepsilon)\\ &= 4\pi/\varepsilon + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{B.5}$$

To compute the integral over the four-point function

$$\begin{aligned}G_\varepsilon(r_1,r_2) &\equiv \langle \phi^0(0)\phi^0(1)\phi^0(r_1)\phi^0(r_2)\rangle\\ &= (z_1-1)^{-2+\varepsilon}(\bar{z}_1-1)^{-2+\varepsilon} z_2^{-2+\varepsilon}\bar{z}_2^{-2+\varepsilon} F_\varepsilon(\zeta,\bar{\zeta}),\end{aligned} \tag{B.6}$$

an auxiliary function

$$\tilde{G}_\varepsilon(r_1,r_2) = (z_1-1)^{-2+\varepsilon}(\bar{z}_1-1)^{-2+\varepsilon} z_2^{-2+\varepsilon}\bar{z}_2^{-2+\varepsilon} \cdot \tilde{F}_\varepsilon(\zeta,\bar{\zeta}) \tag{B.7}$$

is introduced with the following properties:

(i) the functions $F_\varepsilon - \tilde{F}_\varepsilon$ are bounded by an integrable function;
(ii) $\tilde{F}_\varepsilon \to F_\varepsilon$ as $\varepsilon \to 0$.

Using Lebesgue's theorem, one can then show that

$$\int G_\varepsilon(r_1,r_2)\,\mathrm{d}^2 r_1\,\mathrm{d}^2 r_2 = \int \tilde{G}_\varepsilon(r_1,r_2)\,\mathrm{d}^2 r_1\,\mathrm{d}^2 r_2 + \mathrm{O}(\varepsilon^0). \tag{B.8}$$

The function $\tilde{F}_\varepsilon$ to be used in the sequel is

$$\tilde{F}_\varepsilon(\zeta,\bar{\zeta}) = \left[\tfrac{1}{16}C^2 + \mathrm{O}(\varepsilon)\right] 2\,\mathrm{Re}\Big[\zeta^{-2+\varepsilon}\bar{\zeta}^{\varepsilon}(1-\zeta)^{-\varepsilon}(1-\bar{\zeta})^{-\varepsilon}$$
$$\times\zeta^{-\varepsilon}\bar{\zeta}^{-\varepsilon}(1-\zeta)^{-2+\varepsilon}(1-\bar{\zeta})^{\varepsilon} + \zeta^{-2+\varepsilon}\bar{\zeta}^{\varepsilon}(1-\zeta)^{\varepsilon}(1-\bar{\zeta})^{-2+\varepsilon}\Big] \tag{B.9}$$

$$+\left[\tfrac{1}{4}C^2 + \mathrm{O}(\varepsilon)\right] 2\,\mathrm{Re}\Big[\zeta^{-1+\varepsilon/2}\bar{\zeta}^{\varepsilon/2}(1-\zeta)^{\varepsilon}(1-\bar{\zeta})^{-2+\varepsilon}$$
$$+\zeta^{-2+\varepsilon}\bar{\zeta}^{\varepsilon}(1-\zeta)^{\varepsilon/2}(1-\bar{\zeta})^{-1+\varepsilon/2}$$
$$-\zeta^{-1+\varepsilon/2}\bar{\zeta}^{\varepsilon/2}(1-\zeta)^{-1-\varepsilon/2}(1-\bar{\zeta})^{-\varepsilon/2}\Big] \tag{B.10}$$

$$+\tfrac{1}{2}C^2\Big[\zeta^{-1+\varepsilon/2}\bar{\zeta}^{-1+\varepsilon/2} + (1-\zeta)^{-1+\varepsilon/2}(1-\bar{\zeta})^{-1+\varepsilon/2}$$
$$+\zeta^{-1+\varepsilon/2}\bar{\zeta}^{-1+\varepsilon/2}(1-\zeta)^{-1+\varepsilon/2}(1-\bar{\zeta})^{-1+\varepsilon/2}\Big]. \tag{B.11}$$

Property (i) may be checked by comparing the singularities of $\tilde{F}_\varepsilon$ with the ones of F_ε known from the operator product expansion. As $\varepsilon \to 0$, we have $F_\varepsilon \to F_{\varepsilon=0}$, and (A.9) shows that $\tilde{F}_\varepsilon \to F_{\varepsilon=0}$; this proves property (ii). Notice that $\tilde{F}_\varepsilon$ cannot be the true scaling function. This does not interfere with the argument. It is convenient to change the integration variables,

$$\int \tilde{G}_\varepsilon(r_1,r_2)\,\mathrm{d}^2r_1\,\mathrm{d}^2r_2 = \int \eta^{-1+\varepsilon}\bar{\eta}^{-1+\varepsilon}(\eta-1)^{-\varepsilon}(\bar{\eta}-1)^{-\varepsilon}(\eta-\zeta)^{-\varepsilon}(\bar{\eta}-\bar{\zeta})^{-\varepsilon}$$
$$\times\tilde{F}_\varepsilon(\zeta,\bar{\zeta})\frac{\mathrm{d}(\zeta+\bar{\zeta})}{2}\,\frac{\mathrm{d}(\zeta-\bar{\zeta})}{2}\,\frac{\mathrm{d}(\eta+\bar{\eta})}{2}\,\frac{\mathrm{d}(\eta-\bar{\eta})}{2}, \tag{B.12}$$

with $\eta \equiv (z_2-1)/z_2$. A look at the group of terms (B.11) shows that for the first two terms in this group the integral factorizes; the ζ- and η-integrals are both of the form (B.1). The third term does not factorize in these coordinates, but it is related to the second by the transformation $r_1 \leftrightarrow r_2$ and gives therefore the same contribution to the integral. In the group of terms (B.9) and (B.10), the η-integral

$$I_\eta \equiv \int \eta^{-1+\varepsilon}\bar{\eta}^{-1+\varepsilon}(\eta-1)^{-\varepsilon}(\bar{\eta}-1)^{-\varepsilon}(\eta-\zeta)^{-\varepsilon}(\bar{\eta}-\bar{\zeta})^{-\varepsilon}\frac{\mathrm{d}(\eta+\bar{\eta})}{2}\,\frac{\mathrm{d}(\eta-\bar{\eta})}{2} \tag{B.13}$$

may be performed first, by noting that the singularities as $\eta \to 1$ and $\eta \to \zeta$ are weak and the ε^{-1}-pole of the integral is determined only by the singularities as $\eta \to 0$ and

$\eta \to \infty$. Therefore,

$$I_\eta = I(-1+\varepsilon, -1+\varepsilon, -2\varepsilon, -2\varepsilon) + \mathrm{O}(\varepsilon^0). \tag{B.14}$$

The ζ-integrals are then again in their leading part of the form (B.1) and have a finite limit as $\varepsilon \to 0$. Taking everything together, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int \tilde{G}_\varepsilon(r_1, r_2)\,\mathrm{d}^2 r_1\,\mathrm{d}^2 r_2 \\
&\quad = \tfrac{1}{2}\left[\tfrac{1}{16}C^2 + \mathrm{O}(\varepsilon)\right] 2I_\eta\left[I(-2+\varepsilon, \varepsilon, -2\varepsilon, -2\varepsilon) + I(-2\varepsilon, -2\varepsilon, -2+\varepsilon, \varepsilon)\right. \\
&\qquad \left. + I(-2+\varepsilon, \varepsilon, \varepsilon, -2+\varepsilon) + \mathrm{O}(\varepsilon)\right] \\
&\quad + \tfrac{1}{2}\left[\tfrac{1}{4}C^2 + \mathrm{O}(\varepsilon)\right] 2I_\eta\left[I\left(-1-\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, \varepsilon, -2+\varepsilon\right) + I\left(-2+\varepsilon, \varepsilon, -\frac{\varepsilon}{2}, -1-\frac{\varepsilon}{2}\right)\right. \\
&\qquad \left. - I\left(-1-\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, -1-\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}\right) + \mathrm{O}(\varepsilon)\right] \\
&\quad + \frac{1}{2}\frac{C^2}{2}\left[I\left(-1+\frac{\varepsilon}{2}, -1+\frac{\varepsilon}{2}, -\varepsilon, -\varepsilon\right) I\left(-1+\frac{\varepsilon}{2}, -1+\frac{\varepsilon}{2}, -\varepsilon, -\varepsilon\right)\right. \\
&\qquad \left. + 2I\left(-\varepsilon, -\varepsilon, -1+\frac{\varepsilon}{2}, -1+\frac{\varepsilon}{2}\right) I\left(-1+\varepsilon, -1+\varepsilon, -\frac{3\varepsilon}{2}, -\frac{3\varepsilon}{2}\right)\right]
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
&= \tfrac{1}{2}\left[\tfrac{1}{16}C^2 + \mathrm{O}(\varepsilon)\right] 2\left(\frac{2\pi}{\varepsilon} + \mathrm{O}(\varepsilon^0)\right) \\
&\quad \times\left[\left(2\pi\varepsilon + \mathrm{O}(\varepsilon^2)\right) + \left(2\pi\varepsilon + \mathrm{O}(\varepsilon^2)\right) + \left(2\pi\varepsilon + \mathrm{O}(\varepsilon^2)\right) + \mathrm{O}(\varepsilon)\right] \\
&\quad + \tfrac{1}{2}\left[\tfrac{1}{4}C^2 + \mathrm{O}(\varepsilon)\right] 2\left(\frac{2\pi}{\varepsilon} + \mathrm{O}(\varepsilon^0)\right) \\
&\quad \times\left[\left(-\pi + \mathrm{O}(\varepsilon)\right) + \left(-\pi + \mathrm{O}(\varepsilon)\right) - \left(\pi + \mathrm{O}(\varepsilon)\right) + \mathrm{O}(\varepsilon)\right] \\
&\quad + \frac{1}{2}\frac{C^2}{2}\left[\left(\frac{4\pi}{\varepsilon} + \mathrm{O}(\varepsilon^2)\right)\left(\frac{4\pi}{\varepsilon} + \mathrm{O}(\varepsilon^2)\right) + 2\left(\frac{4\pi}{\varepsilon} + \mathrm{O}(\varepsilon^2)\right)\left(\frac{3\pi}{\varepsilon} + \mathrm{O}(\varepsilon^2)\right)\right]
\end{aligned} \tag{B.16}$$

$$= \frac{10\pi^2 C^2}{\varepsilon^2} - \frac{3\pi^2 C^2}{2\varepsilon} + \mathrm{O}(\varepsilon^0). \tag{B.17}$$

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