

Multiple crossover phenomena and scale hopping in two dimensions

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We study the renormalization group for nearly marginal perturbations of a minimal conformal field theory M_p with $p \gg 1$. To leading order in perturbation theory, we find a unique one-parameter family of “hopping trajectories” that is characterized by a staircase-like renormalization group flow of the C -function and the anomalous dimensions and that is related to a factorizable scattering theory recently solved by Al. B. Zamolodchikov. We argue that this system is described by interactions of the form $t\phi_{(1,3)} - \bar{t}\phi_{(3,1)}$. As a function of the *relevant* parameter t , it undergoes a phase transition with new critical exponents simultaneously governed by all fixed points M_p, M_{p-1}, \dots, M_3 . Integrable lattice models represent different phases of the same integrable system that are distinguished by the sign of the *irrelevant* parameter \bar{t} .

1. Introduction

The simplest scale-invariant field theories in two dimensions are the series of minimal models M_p ($p = 3, 4, \dots$) [2], which describe the universal $(p - 1)$ -critical behavior of Landau–Ginzburg theories with a single bosonic field and polynomial interactions [3]. It is a difficult and widely open problem to reveal the embedding renormalization group (RG) scenario of these fixed points, which determines the universal behavior off criticality as well as crossover phenomena. An important aspect of this problem is that in two dimensions the theory has an infinite number of integrals of motion not only at the RG fixed points, where it is conformally invariant [2], but on a larger submanifold of theory space. The precise extent of this manifold of integrability is unknown, but it does contain some perturbations of a critical point M_p by a single scaling field [4].

If such a perturbation is *relevant*, it will either induce a crossover to another critical point of lower criticality or lead to purely massive infrared behavior. In the latter case, the exact factorizable S -matrix can be conjectured [4], which in principle determines all scaling functions associated to that RG trajectory. At least some properties of this scaling regime have indeed been predicted, such as universal finite-size effects [5–12] and amplitude relations [13]. The only known

example of an integrable crossover from the model M_p to another critical point is generated by the perturbation

$$\mathcal{L} = \mathcal{L}_p^* + t_p \phi_{p(1,3)}, \quad (1.1)$$

where $\phi_{p(1,3)}$ is the weakest relevant scaling field, i.e. the field with the smallest positive RG eigenvalue, and the (dimensionful) coupling constant t_p is positive. This crossover changes the order of criticality by one to M_{p-1} ; this has been shown perturbatively for large values of p [14], by supersymmetry arguments for the crossover from the tricritical to the critical Ising model (the case $p = 4$) [15], and more recently by thermodynamic Bethe ansatz methods for general values of p [16]. Hence the manifold \mathcal{E}_p of $(p - 1)$ -criticality is nested into all manifolds of lower criticality, as one would expect from a mean-field analysis of the Landau–Ginzburg picture: $\mathcal{E}_p \subset \mathcal{E}_{p-1} \subset \dots \subset \mathcal{E}_3$. In contrast to mean-field arguments, however, crossovers changing the order of criticality by more than one are induced by fine-tuned linear combinations of all scaling fields that are even under spin reversal, and it is not clear if any of the interpolating field theories are integrable.

The manifold of integrability also contains the leading *irrelevant* scaling perturbation of the critical theories M_p *

$$\mathcal{L} = \mathcal{L}_p^* - \bar{t}_p \phi_{p(3,1)}, \quad (1.2)$$

at least to first order in perturbation theory [4,9]. While generically a linear combination of two integrable perturbations does not generate an integrable field theory off criticality **, the perturbations (1.1) and (1.2) share infinitely many integrals of motion so that even an arbitrary linear combination

$$\mathcal{L} = \mathcal{L}_p^* + t_p \phi_{p(1,3)} - \bar{t}_p \phi_{p(3,1)} \quad (1.3)$$

should still be integrable [9]. As will be argued below, this fact is connected to the existence of non-trivial integrable lattice models in two dimensions. The presence of non-zero irrelevant coupling constants in lattice models can drastically alter their crossover behavior: since the $(p - 1)$ -critical lattice model is characterized by a point on \mathcal{E}_p different from the fixed point M_p , the variation of a thermodynamic parameter causing the continuum theory M_p to cross over to $\mathcal{E}_{p'}$ ($p' < p$) need not be tangent to $\mathcal{E}_{p'}$ at that point, which leaves the perturbed lattice model on a less critical manifold $\mathcal{E}_{p''}$ ($p'' < p'$) or in a massive phase.

* The bar does not denote complex conjugation.

** The two-dimensional Ising model was studied recently [17] in the entire scaling region $\mathcal{L} = \mathcal{L}^* + h\phi_{(1,2)} + t\phi_{(1,3)}$, which is spanned by two integrable perturbations. The finite-size spectrum of the transfer matrix obtained by the conformal truncation method [10–12] did not show any sign of integrability except for $h = 0$ or $t = 0$.

This paper studies such crossover phenomena by analyzing the renormalization group flow in the neighborhood of a minimal model M_p for $p \ll 1$, where a subset of the scaling fields (including $\phi_{p(1,3)}$ and $\phi_{p(3,1)}$) becomes nearly marginal. A perturbation of M_p can then be described by the lagrangian

$$\mathcal{L} = \mathcal{L}_p^* + \sum_i U_p^i \Phi_{pi}; \tag{1.4}$$

the running coupling constants U_p^i and their conjugate fields Φ_{pi} are defined by an expansion in the parameter $\varepsilon = 4/(p + 1)$ (which is the RG eigenvalue of $\phi_{p(1,3)}$) [14]. The ε -expansion can be trusted in a neighborhood $U_p^i = O(\varepsilon)$ of M_p , which contains infinitely many other fixed points $M_{p'}$. This is an important difference to the usual ε -expansion about the upper critical dimension, where only two fixed points are at a distance of $O(\varepsilon)$. But at least on the trajectory linking M_p and M_{p-1} , this ε -expansion has been shown to be a consistent RG scheme in minimal subtraction to $O(\varepsilon^2)$ [18].

To leading order in perturbation theory, we find in particular a unique one-parameter family of *hopping trajectories* $\check{U}_p^i(\theta, \theta_0)$ (where θ is the RG “time” varying along each trajectory and θ_0 labels the trajectories). They come close to each fixed point M_p and are *self-similar* in the following sense:

$$\check{U}_{p-1}^i(\theta + \theta_0, \theta_0) = \check{U}_p^i(\theta, \theta_0). \tag{1.5}$$

We identify these trajectories with the one-parameter family of integrable trajectories that A.I.B. Zamolodchikov recently found by solving the thermodynamic Bethe ansatz for a simple factorizable scattering theory containing a single type of massive particles [1]. The flow of the C -function $C(\theta, \theta_0)$ along these trajectories is computed and seen to follow the characteristic staircase pattern that interpolates between the central charges c_p . A similar pattern is found for the flow of the anomalous dimensions $x^{(i)}(\theta, \theta_0)$. We shall argue that the one-parameter family $M(\theta_0)$ of integrable field theories defined by these S -matrices is described by a lagrangian of the form (1.3) where both coupling constants t_p and \bar{t}_p are positive.

The same family of field theories can be considered for negative values of \bar{t}_p , where the RG trajectories behave very differently: they come close to only two neighboring fixed points M_p and M_{p-1} and should describe an integrable system in a $(p - 2)$ -phase coexistence region. This is very likely to be the eight-vertex solid-on-solid model of Andrews, Baxter and Forrester (ABF) [19] in the scaling region of the so-called regime IV. The RG analysis thus establishes an intimate connection between this model and Zamolodchikov’s system.

As a function of the relevant temperature-like parameter t_p , the system undergoes a second-order phase transition with a rather intricate critical behavior. For $t_p < 0$ and any value of \bar{t}_p , it is governed by the *single* fixed point M_p . For $t_p > 0$ and $\bar{t}_p < 0$, *two* neighboring fixed points determine the exponents; the RG con-

firmes the scaling ansatz proposed by Huse [20] to explain the exponents in regime IV of the ABF-model. For $t_p > 0$ and $\tilde{t}_p > 0$, they are determined by *all* fixed points $M_p, M_{p-1}, M_{p-2}, \dots, M_3$ visited by the hopping trajectories.

This paper is organized as follows. In sect. 2, we write down the RG equations and determine some useful symmetry properties. Sect. 3 discusses the hopping trajectories. Sect. 4 describes the various phase coexistence regions and the critical behavior as a function of t_p . Sect. 5 contains a discussion of the results.

2. First-order renormalization about a minimal model M_p

To leading order in perturbation theory, the RG equations about the fixed point M_p can be written in the form [14,21] *

$$\frac{d}{d\theta} U^i = y_j^i U^j - \pi C_{jk}^i U^j U^k, \tag{2.1}$$

where $x^{(i)} = 2 - y^{(i)}$ are the anomalous dimensions and C_{jk}^i the structure constants of the scaling operators $\phi_i = \Phi_i(U = 0)$, and $y_j^i = y^{(i)}\delta_j^i$. In eq. (2.1), the indices run over all scaling fields that transform as scalars under rotations, including pure derivative fields $\partial_z \partial_{\bar{z}} \phi_i$. Hence the RG acts on a space of coupling constants whose dimensionality exceeds that of the thermodynamic space of the system. At any point U , there are linear combinations of the fields $\Phi_i(U)$ that are proportional to the pure derivative fields $\partial_z \partial_{\bar{z}} \Phi_i(U)$ and hence not conjugate to any thermodynamic parameter; these fields generate redundant [22] directions in coupling constant space.

The RG equations can be restricted to the “thermal” couplings that preserve the \mathbb{Z}_2 -symmetry of M_p under spin reversal. Further simplifications arise in an expansion in the parameter $\varepsilon \equiv y^{(1,3)} = 4/(p + 1)$. Such an expansion is possible since both the structure constants and the scaling dimensions are analytic in ε . For $\varepsilon \rightarrow 0$, the scaling fields ϕ_i in the lower-left corner of the Kac table (shown in fig. 1) are spectrally separated:

- (i) the primary fields $\phi_{(m,n)}$ with $|m - n| \leq 1$ have dimension $x^{(m,n)} \leq 1/2 + O(\varepsilon)$,
- (ii) the primary fields $\phi_{(n,n \pm 2)}$ and the (conveniently normalized [14]) descendant fields $\tilde{\phi}_{(n,n)} \equiv (x^{(n,n)})^{-1} \partial_z \partial_{\bar{z}} \phi_{(n,n)}$ have dimension $2 \mp O(\varepsilon)$ and $2 - O(\varepsilon^2)$, respectively, and
- (iii) all other fields have dimension $\geq 5/2 + O(\varepsilon)$ **.

* From now on, the index p will be suppressed where no ambiguities can arise.

** The spectral separation breaks down for $n = O(p)$, but for the solutions of the RG equations to be discussed in the sequel, the couplings of these fields are exponentially suppressed.

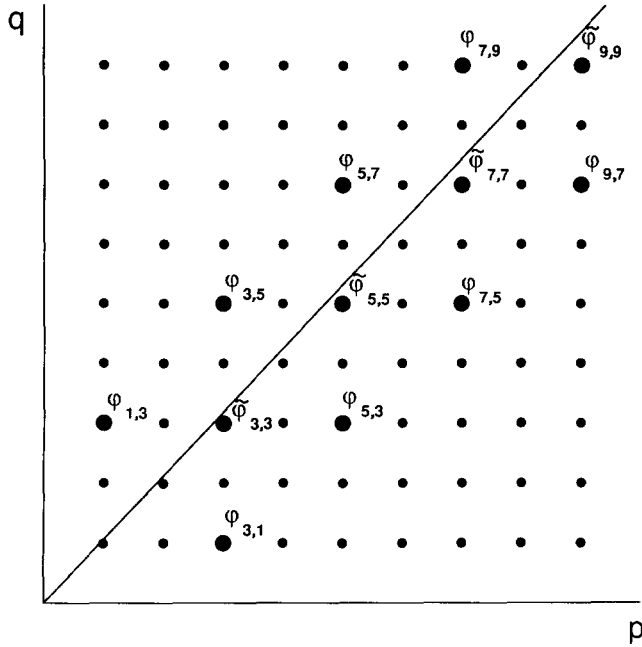


Fig. 1. Positions of the nearly marginal thermal operators in the Kac table of a unitary minimal model M_p for $p \gg 1$. The operators $\phi_{(n,n+2)}$ are relevant, while the operators $\phi_{(n,n-2)}$ and $\tilde{\phi}_{(n,n)}$ are irrelevant.

Hence the couplings $U^{(n,n \pm 2)}$ and $\tilde{U}^{(n,n)}$ become marginal in this limit, while all other couplings remain strictly relevant or irrelevant. To leading order in ε , the system of equations (2.1) can be truncated consistently to the nearly marginal couplings $U^i = O(\varepsilon)$, the other couplings remain of $O(\varepsilon^2)$. A convenient rescaling $U^i(\theta) = \varepsilon u^i(\varepsilon\tau) / (\pi C_{(1,3)(1,3)}^{(1,3)})$ then brings the RG equations into the form

$$\frac{d}{d\tau} u^i = \gamma_j^i u^j - c_{jk}^i u^j u^k + O(\varepsilon), \tag{2.2}$$

with $\gamma_j^i = \lim_{\varepsilon \rightarrow 0} (y_j^i / \varepsilon)$ and $c_{jk}^i = \lim_{\varepsilon \rightarrow 0} (C_{jk}^i / C_{(1,3)(1,3)}^{(1,3)})$.

These equations determine in particular the *renormalizable manifold* \mathcal{R}_p of M_p , i.e. the set of all trajectories

$$u_p^i(\tau) \quad \text{with} \quad u_p^i(\tau) \rightarrow 0 \quad \text{for} \quad \tau \rightarrow -\infty, \tag{2.3}$$

and the *critical manifold* \mathcal{E}_p of M_p , i.e. the set of all trajectories

$$u_p^i(\tau) \quad \text{with} \quad u_p^i(\tau) \rightarrow 0 \quad \text{for} \quad \tau \rightarrow +\infty, \tag{2.4}$$

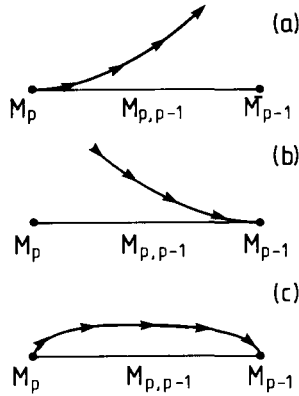


Fig. 2. Special solutions of the RG equations in the vicinity of $M_{p,p-1}$ (schematic). (a) A trajectory in \mathcal{R}_p . (b) A trajectory in \mathcal{E}_{p-1} . (c) A redundant trajectory.

modulo the redundant couplings (see fig. 2). The $(p - p')$ -dimensional *crossover manifold*

$$M_{p,p'} = \mathcal{R}_p \cap \mathcal{E}_{p'} \tag{2.5}$$

describes the $(p - p')$ -parameter family of field theories whose ultraviolet asymptotics is determined by M_p and whose infrared behavior is determined by $M_{p'}$. The simplest such solution is the unique trajectory $M_{p,p-1}$ [14]

$$u^{(1,3)}(\tau) = \frac{\exp(\tau - \tau_m)}{1 + \exp(\tau - \tau_m)}, \quad u^i(\tau) = 0 \quad \text{for } i \neq (1, 3), \tag{2.6}$$

where τ_m is a free parameter. This trajectory interpolates between M_p and the infrared fixed point $u_*^{(1,3)} = 1$ associated to M_{p-1} .

Under a simultaneous RG time reversal and basis change involving a reflection about the diagonal of the Kac table

$$\tau \rightarrow -\tau, \quad \phi_{(m,n)} = \phi_i \rightarrow \phi_i = -\phi_{(n,m)}, \tag{2.7}$$

the equations (2.2) remain invariant since $\gamma_j^i = -\gamma_j^i$ and $c_{jk}^i = c_{jk}^i$. Hence to every RG trajectory $u^i(\tau)$, there is a conjugate trajectory $\bar{u}^i(\tau) = -u^i(-\tau)$, and to every fixed point u_*^i , there is a conjugate fixed point $\bar{u}_*^i = -u_*^i$. For example, the trajectory conjugate to $M_{p,p-1}$ interpolates between M_p and the ultraviolet fixed point $u_*^{(3,1)} = -1$ associated to M_{p+1} . Of particular importance in the sequel will be the self-conjugate trajectories, which satisfy

$$u^i(\tau) = -u^i(-\tau + \tau_1) \tag{2.8}$$

for some value of τ_1 .

In a small neighborhood of the trajectory $M_{p,p-1}$, the RG equations can be linearized in the other couplings $v^i \equiv u^i \ll 1$ ($i \neq (1, 3)$). The equation for $u \equiv u^{(1,3)}$ then decouples and $u(\tau)$ is given by eq. (2.6); the equations for v^i take the form

$$\frac{d}{d\tau} v^i = \gamma_j^i(u(\tau)) v^j, \tag{2.9}$$

where $\gamma_j^i(u)$ factorizes into (3×3) -matrices

$$\gamma^{(n)} = \begin{pmatrix} \frac{n+1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{n-1}{2} \end{pmatrix} - 2u \begin{pmatrix} \frac{n+3}{n+1} & \frac{n-1}{n+1} \left(\frac{n+2}{n}\right)^{1/2} & 0 \\ \frac{n-1}{n+1} \left(\frac{n+2}{n}\right)^{1/2} & \frac{4}{n^2-1} & \frac{n+1}{n-1} \left(\frac{n-2}{n}\right)^{1/2} \\ 0 & \frac{n+1}{n-1} \left(\frac{n-2}{n}\right)^{1/2} & \frac{n-3}{n-1} \end{pmatrix} \tag{2.10}$$

acting on the triplets of couplings

$$v^{(n)} = \begin{pmatrix} u^{(n,n+2)} \\ \tilde{u}^{(n,n)} \\ u^{(n,n-2)} \end{pmatrix} \tag{2.11}$$

with $n = 3, 5, 7, \dots$

A basis of solutions of eq. (2.9) is given by the trajectories with the definite RG time reversal symmetry

$$(u_p(\tau_m - \tau), v_p(\tau_m - \tau)) = (\bar{u}_{p-1}(\tau_m + \tau), \pm \bar{v}_{p-1}(\tau_m + \tau)). \tag{2.12}$$

where $u_p(\tau)$ is given by eq. (2.6). For each value of n , there is precisely one linearly independent even solution $v_p^{(n)+}$ and two linearly independent odd solutions $v_p^{(n)-}$ and $v_p^{(n)0}$. The couplings $v_p^{(n)0}$ are conjugate to the pure derivative fields $\partial_z \partial_{\bar{z}} \Phi_{(n,n)}(u)$ of $M_{p,p-1}$ and play a redundant rôle.

For the asymptotic behavior of the trajectories as $\tau \rightarrow -\infty$ (i.e. $u_p \equiv u \rightarrow 0$), there are three possibilities. For every value of n , there is one linearly independent solution of (2.9),

$$v_p^{(n)r}(u) = u^{(n+1)/2} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -\frac{2(n+3)}{n+1} \\ -\frac{4(n-1)}{(n+3)(n+1)} \left(\frac{n+2}{n}\right)^{1/2} \\ 0 \end{pmatrix} + \mathcal{O}(u^2) \right], \quad (2.13)$$

defining a renormalizable trajectory $(u_p(\tau), v_p^{(n)r}(u(\tau)))$ and one solution

$$v_p^{(n)0}(u) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} \frac{4}{n+1} \left(\frac{n+2}{n}\right)^{1/2} \\ -\frac{8}{n^2-1} \\ -\frac{4}{n-1} \left(\frac{n-2}{n}\right)^{1/2} \end{pmatrix} + \mathcal{O}(u^2), \quad (2.14)$$

defining a redundant trajectory. Any solution that is linearly independent from (2.13) and (2.14) describes a theory that is non-renormalizable about M_p . Conversely, there is one linearly independent solution $v_p^{(n)c}(u)$ that defines a trajectory in \mathcal{E}_{p-1} ; any solution that is linearly independent of $v_p^{(n)c}(u)$ and $v_p^{(n)0}(u)$ is of lower criticality.

3. Self-similar hopping trajectories

In this section, we study the RG flow of a self-conjugate perturbation of the fixed point M_p ,

$$u_p^i(\tau = 0) = \bar{u}_p^i(\tau = 0) \ll 1, \quad (3.1)$$

corresponding to a point in theory space that is much closer to M_p than any of the other fixed points $M_{p'}$. We define the parameter

$$s \equiv u_p^{(1,3)} \bar{u}_p^{(1,3)} > 0. \quad (3.2)$$

It is easy to verify that there is a unique one-parameter family $\hat{u}_p^i(\tau, s)$ of trajectories that satisfy the conditions (3.1) and $\hat{u}_p^i(\tau, s) \subset \mathcal{E}_{p-1}$, i.e. \hat{v}_p can be

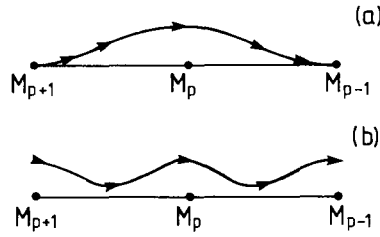


Fig. 3. Self-conjugate trajectories in the vicinity of the fixed point M_p (schematic). (a) A trajectory in $M_{p+1,p-1}$. (b) A self-similar trajectory.

expanded in the basis of triplets $v_p^{(n)c}$ and $v_p^{(n)0}$ *. Self-conjugacy then dictates $\hat{u}_p^i(\tau, s) \subset \mathcal{R}_{p+1}$ as well and therefore

$$\hat{u}_p^i(\tau, s) \subset M_{p+1,p-1}. \tag{3.3}$$

Any trajectory in $M_{p+1,p-1}$ with a given value of s differs from $\hat{u}_p^i(\tau, s)$ only by spurious couplings $v_p^{(n)0}$, hence these trajectories (shown in fig. 3a) span the two-dimensional crossover manifold $M_{p+1,p-1}$.

Consider now the one-parameter family $\check{u}_p^i(\tau, s)$ of trajectories that satisfy the condition (3.1) and are even in the sense of eq. (2.12) up to spurious couplings, i.e. v_p can be expanded in the basis of triplets $v_p^{(n)+}$ and $v_p^{(n)0}$. The trajectory $\check{u}_p^i(\tau, s)$ is self-similar (see fig. 3b) after a scaled RG time $\tau_0 = \varepsilon\theta_0 \simeq \log(1/s)$ for small s :

$$\check{u}_{p-1}^i(\tau_0, s) = \bar{\check{u}}_p^i(0, s) = \check{u}_p^i(0, s). \tag{3.4}$$

It comes close to each fixed point $M_{p'}$ in the time interval

$$(p - p' - \frac{1}{2})\theta_0 \leq \theta \leq (p - p' + \frac{1}{2})\theta_0 \tag{3.5}$$

and up to a minimum distance given by the parameter $s_{p'} = s(1 + O(\varepsilon))$, whereafter it hops to the next-lower fixed point. It is again easy to check that up to

* For a given value of s , one has $\hat{u}_p^{(1,3)} = s^{1/2} = -\hat{u}_p^{(3,1)}$ by eqs. (3.1) and (3.2). The remaining couplings are recursively determined by the equations

$$\hat{v}_p^{(n)}(0, s) = a_{(n)c}(s)v_p^{(n)c}(u = s^{1/2}) + a_{(n)0}(s)v_p^{(n)0}(u = s^{1/2})$$

and the self-conjugacy conditions

$$\hat{v}_p^{(n,n+2)}(0, s) = -\hat{v}_p^{(n+2,n)}(0, s) \quad \text{and} \quad \hat{u}_p^{(n,n)}(0, s) = 0.$$

Analogous recursion relations hold for the trajectories $\check{u}_p^i(\tau, s)$ below. Notice that both families and every value of n , the ratio of the relevant and the irrelevant coupling $u_p^{(n,n+2)}(0, s)/u_p^{(n,n-2)}(0, s)$ in the n th triplet goes to 0 as $s \rightarrow 0$, hence the couplings with higher n are strongly suppressed for small s and τ .

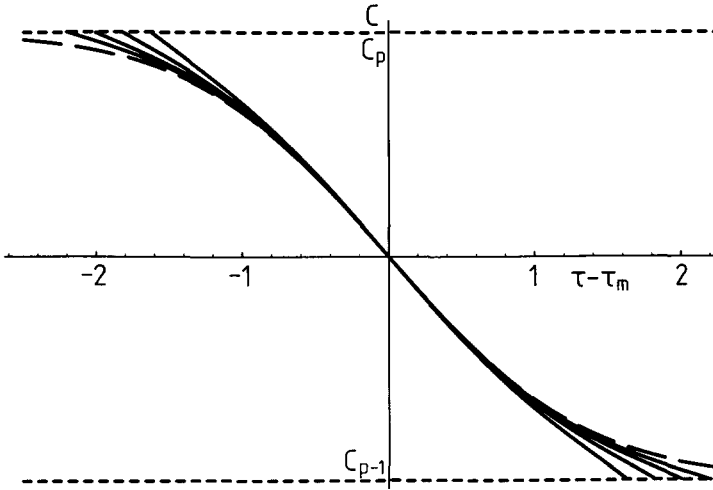


Fig. 4. The C -function $C(\tau, \tau_0)$ of the unique self-similar trajectory $\check{u}_p^i(\tau, s(\tau_0))$: a step in the staircase pattern for $\tau_0 = 3.2, 3.6, 4.0$ and 4.4 (solid lines). For larger values of τ_0 , the steps get more pronounced as the solutions tend towards the limit trajectory $M_{p,p-1}$ (long-dashed line).

spurious couplings, this is the only self-similar trajectory for that value of s . Thus the trajectories $\check{u}_p^i(\tau, s = s(\theta_0))$ define a unique one-parameter family of field theories $M(\theta_0)$.

The RG flow of the C -function [14]

$$C(u_p^i) = c_p + \frac{3\varepsilon^3}{16} \left(-3\gamma_{ij}u_p^i u_p^j + 2c_{ijl}u_p^i u_p^j u_p^l \right) + O(\varepsilon^4) \tag{3.6}$$

for the theory $M(\theta_0)$ satisfies

$$C(\theta, \theta_0) - c_p = C(\theta + \theta_0, \theta_0) - c_{p-1} + O(\varepsilon^4), \tag{3.7}$$

and in particular for integer k

$$C(k\theta_0, \theta_0) = c_{p-k} + O(\varepsilon^4),$$

$$C\left(\left(k + \frac{1}{2}\right)\theta_0, \theta_0\right) = c_{p-k} - \frac{1}{2}(c_{p-k} - c_{p-k-1}) + O(\varepsilon^4), \tag{3.8}$$

by eq. (3.4). A step of this self-repeating staircase pattern for several values of τ_0 is shown in fig. 4, which was obtained by numerical integration of eq. (2.9) *.

* Eqs. (3.3) and (3.4) also indicate the possibility that there exists a two-parameter family of trajectories $u_p^i(\tau, \hat{s}, \check{s})$ with $u_p^i(\tau, \hat{s}, \check{s}) \rightarrow \check{u}_p^i(\tau, \check{s})$ as $\hat{s} \rightarrow 0$ and $u_p^i(\tau, \hat{s}, \check{s}) \rightarrow \hat{u}_p^i(\tau, \hat{s})$ as $\check{s} \rightarrow 0$, their C -function being a staircase pattern where all steps have approximately the same length except the step at c_p , which is shorter.

The anomalous dimensions $x^{(i)}$, i.e. the eigenvalues of the matrix

$$2\delta_j^i - y_j^i(u_p^l) = 2\delta_j^i - y_{j(p)}^i + 2\epsilon c_{jl}^i u_p^l + O(\epsilon^2) \tag{3.9}$$

show a very similar pattern. For example, the spectral flow associated to the second subdiagonals of the Kac table satisfies

$$\begin{aligned} x^{(n,n+2)}(\theta + \theta_0, \theta_0) &= x^{(n-2,n)}(\theta, \theta_0) + O(\epsilon^2), \\ x^{(n,n-2)}(\theta + \theta_0, \theta_0) &= x^{(n+2,n)}(\theta, \theta_0) + O(\epsilon^2), \end{aligned} \tag{3.10}$$

and in particular for integer k

$$\begin{aligned} x^{(n,n+2)}(k\theta_0, \theta_0) &= x_p^{(n-2k,n+2-2k)} + O(\epsilon^2) \quad (2k < n), \\ x^{(n,n-2)}(k\theta_0, \theta_0) &= x_p^{(n+2k,n-2+2k)} + O(\epsilon^2) \quad (2k > -n), \end{aligned} \tag{3.11}$$

and similar equations hold for the other fields.

Thus the field theories $M(\theta_0)$ behave under RG transformations in a strikingly similar way to the one-parameter family of integrable systems with a single type of massive particles characterized by the factorizable S -matrix [1]

$$S(\rho, \theta_0) = \frac{\sinh \rho - i \cosh 2\theta_0}{\sinh \rho + i \cosh 2\theta_0}, \tag{3.12}$$

written in terms of the Lorentz-invariant rapidity difference ρ . Since the self-similarity (3.4) is unique to the theories $M(\theta_0)$, we are lead to identify them with this type of integrable system. It is plausible that the hopping trajectories $\check{u}_p^i(\tau, s)$ describe integrable systems since the bare lagrangian $(\partial/\partial s)\Sigma_i(\check{U}_p^i\Phi_{pi})(0, s)|_{s=0}$ is of the form (1.3) with $t_p > 0$ and $\bar{t}_p > 0$, but it is difficult to make such a statement precise within the ϵ -expansion since these trajectories are non-renormalizable about any minimal model $M_{p'}$.

The following scaling argument indicates, however, that the lagrangian

$$\mathcal{L}_p = \mathcal{L}_p^* + \sum_i t_p^i \phi_{pi} \tag{3.13}$$

for the theories $M(\theta_0)$ is precisely (1.3) for any value of p . We define the dimensionless scaling variables

$$s_p^i \equiv t_p^i t_p^{\omega_p^i}, \quad \bar{s}_p^i \equiv t_p^i \bar{t}_p^{\bar{\omega}_p^i}, \tag{3.14}$$

where ω_p^i and $\bar{\omega}_p^i$ are the crossover exponents

$$\omega_p^i \equiv -\frac{y_p^{(i)}}{y_p}, \quad \bar{\omega}_p^i \equiv -\frac{y_p^{(i)}}{\bar{y}_p}, \quad (3.15)$$

with

$$y_p \equiv y_p^{(1,3)} = \frac{4}{p+1}, \quad \bar{y}_p \equiv y_p^{(3,1)} = -\frac{4}{p}. \quad (3.16)$$

The theory $M_{p,p-1}$ has $s_p^i = 0$ for $i \neq (1, 3)$ and $\bar{s}_{p-1}^i = 0$ for $i \neq (3, 1)$, while for any finite value of θ_0 , the theory $M(\theta_0)$ must have irrelevant couplings

$$s_p^i \ll 1 \quad \text{for } \theta_0 \gg 1, \quad (3.17)$$

since its ultraviolet behavior differs from M_p , and relevant couplings

$$s_{p-1}^i \ll 1 \quad \text{for } \theta_0 \gg 1, \quad (3.18)$$

since its infrared behavior differs from M_{p-1} . There is an analytic mapping between the two sets of couplings (3.17) and (3.18), which is just a coordinate transformation on theory space [18]. The most relevant coupling s_{p-1}^{rel} of the set (3.18) and the most irrelevant coupling $s_p^{\text{irr}} \sim s_{p-1}^{\text{rel}}$ of the set (3.17) determine the logarithmic scale intervals (the RG time intervals) in which the theory $M(\theta_0)$ is governed by the fixed points M_p and M_{p-1} , respectively:

$$\exp(\Delta\theta_p) \sim (s_p^{\text{irr}})^{1/y_p^{\text{irr}}} \gg 1, \quad \exp(\Delta\theta_{p-1}) \sim (s_{p-1}^{\text{rel}})^{-1/y_{p-1}^{\text{rel}}} \gg 1. \quad (3.19)$$

Zamolodchikov's solution [1] of the thermodynamic Bethe ansatz equations for the S-matrix (3.12) says that eq. (3.5) is valid beyond perturbation theory, i.e. $\Delta\theta_p = \Delta\theta_{p-1} = \theta_0$. This dictates

$$\frac{1}{y_p^{\text{irr}}} = -\frac{1}{y_{p-1}^{\text{rel}}}, \quad (3.20)$$

which can be satisfied only if $t_p^{\text{irr}} \sim t_p^{(3,1)}$ and $t_{p-1}^{\text{rel}} \sim t_{p-1}^{(1,3)}$, and hence $y_p^{\text{irr}} = -4/p = -y_{p-1}^{\text{rel}}$ by eq. (3.16). Repeating the argument for $p' = p + 1$ then fixes the form of the lagrangian (1.3). Hence from the lagrangian point of view, the scale hopping of the theories $M(\theta_0)$ is caused by an intricate interplay of the relevant field $\phi_{(1,3)}$ and the irrelevant field $\phi_{(3,1)}$ under the renormalization group.

The scaling parameter $s_p \equiv \bar{t}_p t_p^{\omega_p}$ (with $\omega_p \equiv \omega_p^{(3,1)} = (p+1)/p$) can be expressed by

$$s_p = g_p^{\omega_p} \bar{g}_p \exp(\bar{y}_p \theta_0) \quad (3.21)$$

in terms of θ_0 and the dimensionless coupling constants

$$g_p = t_p \xi_{p,p-1}^{y_p}, \quad \bar{g}_p = \bar{t}_p \xi_{p+1,p}^{y_p}. \tag{3.22}$$

Here $\xi_{p,p-1}$ and $\xi_{p+1,p}$ denote the crossover length scales of $M_{p,p-1}$ and $M_{p+1,p}$, respectively. Unlike the running couplings, g_p and \bar{g}_p are measurable parameters related to universal amplitude relations [13]. By comparing the solution of the thermodynamic Bethe ansatz with conformal perturbation theory, they can be computed to arbitrarily high accuracy [6,16]. To leading order in the ε -expansion, one obtains *

$$g_p = \bar{g}_p = \frac{\varepsilon}{\pi C_{(1,3)(1,3)}^{(1,3)}} + O(\varepsilon^2). \tag{3.23}$$

4. Phase coexistence and critical behavior

The integrability of the theories given by the lagrangian (1.3) should not depend on the sign of the two coupling constants t_p and \bar{t}_p . However, the behavior of the RG trajectories and hence the long-distance structure crucially depends on these signs: the four one-parameter families of field theories

$$M_p^{++}(s_p) = M(\theta_0(s_p)), \quad M_p^{+-}(s_p), \quad M_p^{-+}(s_p), \quad M_p^{--}(s_p), \tag{4.1}$$

labeled by the scaling parameter s_p and the signs of t_p and \bar{t}_p , describe the system in different thermodynamic phases which we discuss below. The qualitative RG scenario and the resulting phase diagram in the (t_p, \bar{t}_p) -plane are shown in fig. 5. As a function of the relevant parameter t_p , the system undergoes a second order phase transition whose exponents depend on the phase.

4.1. THE THEORIES M_p^{++}

For $t_p > 0$ and $\bar{t}_p > 0$, the solutions of the RG equations are the self-similar hopping trajectories \check{u}_p^i . Following such a trajectory down to M_3 shows that also $t_3 > 0$; the system is in a disordered high-temperature phase. As $t_p \rightarrow 0$, these trajectories come arbitrarily close to all fixed points $M_{p'}$. This implies that the leading thermodynamic singularities are governed by the fixed points $M_{p'}$ with $p' \leq p$, while the other fixed points contribute corrections to scaling. Exact critical exponents for these theories will be reported in a forthcoming publication [23].

* Hence to this order, s_p coincides with the parameter s defined in eq. (3.2).

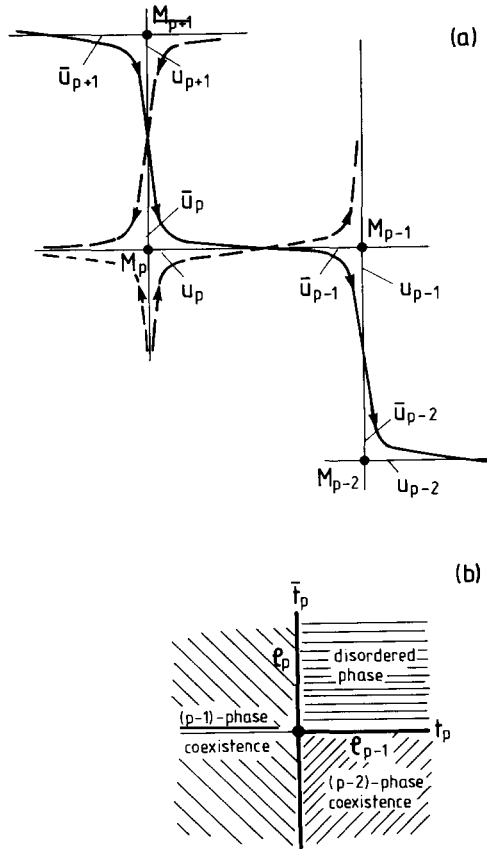


Fig. 5. (a) The RG flow in the vicinity of M_p . A self-similar trajectory representing M_p^{++} (solid line) visits all fixed points, trajectories representing M_p^{+-} or M_p^{-+} (long-dashed lines) visit two fixed points, and a trajectory representing M_p^{-+} (short-dashed line) visits only one fixed point. (b) The resulting phase diagram in the (t_p, \bar{t}_p) -plane.

4.2. THE THEORIES M_p^{+-}

The trajectories for $t_p > 0$ and $\bar{t}_p < 0$ are obtained by analytically continuing the solutions $\check{u}_p^i = (\check{u}_p, \check{v}_p)$ of regime I to $(\check{u}_p, -\check{v}_p)$ in the neighborhood of $M_{p,p-1}$, i.e. at times $0 \leq \theta \leq \theta_0(s)$; these solutions are still even under RG time reversal according to eq. (2.12) up to spurious couplings. At times $\theta \geq \theta_0(s)$, they approach the trajectory $M_{p-1}^{(-)}$ generated by the integrable perturbation (1.1) of M_{p-1} with $t_{p-1} < 0$ which describes the system in a low-temperature region of $p - 2$ coexisting phases. It is likely that this one-parameter family of solutions shares the same infrared behavior. Hence they are very different from the self-similar trajectories:

they come close to only two fixed points M_p and M_{p-1} and run away in both time limits,

$$\bar{u}_p(\theta, s) \rightarrow -\infty \quad \text{as } \theta \rightarrow -\infty, \quad u_{p-1}(\theta, s) \rightarrow -\infty \quad \text{as } \theta \rightarrow +\infty. \quad (4.2)$$

An ordered phase above the critical “temperature” $t_p = 0$ is not to be expected for the continuum theory, but it does occur in regime IV of the ABF lattice model. It is easy to show that the above renormalization-group picture indeed reproduces the correct order parameter exponents known from the exact solution. The behavior of the trajectories indicates that the theories $M_p^{+-}(s_p)$ are characterized by two length scales, the crossover scale $\xi_{p,p-1}$ and the inverse mass ξ . Their asymptotic temperature dependence is given by

$$\xi_{p,p-1} \sim t_p^{-1/y_p}, \quad (4.3)$$

and

$$\frac{\xi}{\xi_{p,p-1}} \sim \exp \theta_0 \sim t_p^{\omega_p/\bar{y}_p} = t_p^{-\omega_p/y_{p-1}}, \quad (4.4)$$

as follows from eqs. (3.21) and (3.16). For the leading singular behavior of the order parameters

$$\langle \phi_{p(n,n)} \rangle \sim \xi_{p,p-1}^{-x_p^{(n,n)}} \left(\frac{\xi}{\xi_{p,p-1}} \right)^{-x_p^{(n,n)}}, \quad (4.5)$$

we obtain therefore

$$\langle \phi_{p(n,n)} \rangle \sim t_p^{\beta_p^{(n,n)} + \omega_p \beta_{p-1}^{(n,n)}}, \quad (4.6)$$

with $\beta_p^{(n,n)} = x_p^{(n,n)}/y_p$, which is precisely Huse’s result [20]. We conclude that the theories M_p^{+-} describe the ABF model in the scaling region of regime IV. This explains the phase structure as a consequence of the same interplay of $\phi_{1,3}$ and $\phi_{3,1}$ that causes the scale hopping of the theories M_p^{++} .

4.3. THE THEORIES M_p^{-+}

For $t_p < 0$ and $\bar{t}_p > 0$, the trajectories are conjugate to those of M_p^{+-} and in fact just those of M_{p+1}^{+-} ; they describe the system in the $(p-1)$ -phase coexistence region. At times $-\theta_0(s) \leq \theta \leq 0$, they come close to the fixed point M_{p+1} , the trajectory $M_{p+1,p}$, and the fixed point M_p ; at large times, they run away,

$$\bar{u}_{p+1} \rightarrow -\infty \quad \text{as } \theta \rightarrow -\infty, \quad u_p \rightarrow -\infty \quad \text{as } \theta \rightarrow +\infty. \quad (4.7)$$

The critical behavior as $t_p \rightarrow 0$ is governed by the fixed point M_p , with corrections to scaling due to the irrelevant operator $\phi_{p(3,1)}$.

4.4. THE THEORIES M_p^-

For $t_p < 0$ and $\bar{t}_p < 0$ (hence $s > 0$), one expects solutions that are again self-conjugate, $u_p^i(\theta, s) = -\bar{u}_p^i(-\theta, s)$, and describe the system in the $(p-1)$ -phase coexistence region. They come close only to one fixed point M_p , and run away at large times,

$$\bar{u}_p^{(1,3)} \rightarrow -\infty \quad \text{as } \theta \rightarrow -\infty, \quad u_p^{(1,3)} \rightarrow -\infty \quad \text{as } \theta \rightarrow +\infty. \quad (4.8)$$

The theories M_p^- should describe the ABF models in the scaling region of regime III. The critical behavior is governed by M_p , but the corrections to scaling are of opposite sign compared to the theories M_p^+ .

5. Discussion

We have studied perturbations of a minimal conformal field theory M_p by a linear combination of the scaling fields $\phi_{(1,3)}$ and $\phi_{(3,1)}$. This generates four one-parameter families of massive integrable field theories $M_p^{++}(s_p)$, $M_p^{+-}(s_p)$, $M_p^{-+}(s_p)$ and $M_p^{--}(s_p)$, which are labeled by the signs of the two coupling constants and the dimensionless scaling parameter s_p , and describe the system in different phases off criticality.

The disordered high-temperature phase corresponds to the theories $M_p^{++}(s_p)$, which are related to Zamolodchikov's scattering theory (3.12), and show a novel behavior under the renormalization group: the trajectories come close to many fixed points $M_{p'}$ for a certain RG time interval $\theta_0(s_p)$, whereafter they hop to the next fixed point $M_{p'-1}$. The correlation functions of these theories are characterized by a multitude of crossover length scales $\xi_{p,p-1}$; any two subsequent such scales have the same ratio $\xi_{p-1,p-2}/\xi_{p,p-1} = e^{\theta_0}$.

To leading order in an ε -expansion, we have shown that the RG equations have indeed a unique one-parameter family of solutions with this behavior, which is tied to the simultaneous presence of relevant and irrelevant coupling constants with scaling dimensions of $O(\varepsilon)$.

The theories $M_p^{+-}(s_p)$ and $M_p^{-+}(s_p)$ are argued to describe the scaling region of the ABF lattice models in regime III and IV, respectively. It would be interesting to study the corrections to scaling in these exactly solved models. Are all non-analytic corrections due to irrelevant operators in the family of $\phi_{(3,1)}$? This would severely restrict the possible lattice effects. And is it possible to find lattice models

with the leading irrelevant coupling of opposite sign, that would hence be in Zamolodchikov's phase?

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Note added

After the draft of this paper had been completed, I received a copy of ref. [24], where Zamolodchikov's S -matrix for antiperiodic boundary conditions is associated to the D series of minimal models. This issue deserves further study. The lagrangian description suggests that integrable systems with scale hopping trajectories should exist in the A and D series.

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