ELSEVIER

Nuclear Physics B 448 [FS] (1995) 559–574

NUCLEAR PHYSICS B [FS]

On the renormalization of the Kardar–Parisi–Zhang equation

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Received 23 January 1995; revised 18 May 1995; accepted 23 May 1995

Abstract

The Kardar–Parisi–Zhang (KPZ) equation of nonlinear stochastic growth in d dimensions is studied using the mapping onto a system of directed polymers in a quenched random medium. The polymer problem is renormalized exactly in a minimally subtracted perturbation expansion about $d = 2$. For the KPZ roughening transition in dimensions $d > 2$, this renormalization group yields the dynamic exponent $z^* = 2$ and the roughness exponent $\chi^* = 0$, which are exact to all orders in $\varepsilon \equiv (2 - d)/2$. The expansion becomes singular in $d = 4$. If this singularity persists in the strong-coupling phase, it indicates that $d = 4$ is the upper critical dimension of the KPZ equation. Further implications of this perturbation theory for the strong-coupling phase are discussed. In particular, it is shown that the correlation functions and the coupling constant defined in minimal subtraction develop an essential singularity at the strong-coupling fixed point.

1. Introduction

One focus of today's statistical mechanics is scale invariance far from equilibrium. Driven growth of surfaces is an example that widely occurs in nature; for a review, see e.g. Ref. [1]. On large scales of space and time, the effective growth dynamics may often be described by a stochastic evolution equation for a continuous "height field" $h(r, t)$. The Kardar–Parisi–Zhang (KPZ) equation [2]

$$\partial_t h = \nu \nabla^2 h + \tfrac{1}{2}\lambda (\nabla h)^2 + \eta \tag{1.1}$$

driven by gaussian white noise with

$$\begin{aligned} \overline{\eta(r,t)} &= 0\,, \\ \overline{\eta(r,t)\eta(r',t')} &= \sigma^2 \delta^d (r - r')\delta(t - t') \end{aligned} \tag{1.2}$$

SSDI 0550-3213(95)00268-5

has become the "standard model" for such processes since it represents the simplest universality class of *nonlinear* growth. Many realistic growth models are in other universality classes due to additional symmetries. Moreover, the KPZ equation has deep theoretical links with a number of more difficult nonequilibrium problems, notably fluid dynamics and turbulence. Phenomenologically, the link to turbulence is even more manifest for certain related growth models that show multiscaling [3].

The phenomenology of the KPZ equation is well known. In spatial dimensionalities $d \leqslant 2$, the nonlinearity $\frac{1}{2}\lambda(\nabla h)^2$ is a relevant perturbation of the gaussian dynamics ($\lambda = 0$). The strong-coupling regime is characterized by two basic exponents, the *roughness exponent* χ and the *dynamic exponent* z, which are defined e.g. by the asymptotic scaling on large scales

$$\langle (h(r_1,t_1) - h(r_2,t_2))^2 \rangle \sim |r_1 - r_2|^{2\chi} C(t\,|r_1 - r_2|^z) \tag{1.3}$$

of the height difference correlation function [2]. In the renormalization group, this is a crossover between two fixed points: the *gaussian fixed point*, which is (infrared-) unstable, and the *strong-coupling fixed point*, which is stable. In $d = 1$, the exponents $\chi = \frac{1}{2}$ and $z = \frac{3}{2}$ can be obtained exactly in several ways: by exploiting the symmetries of the system (namely Galilei invariance and a fluctuation–dissipation relation particular to $d = 1$), by a one-loop dynamic renormalization group analysis [4], or by mapping the KPZ dynamics onto an exactly solvable lattice model [5]. All of these tools fail in higher dimensions, and the properties of the strong-coupling fixed point are known only numerically. In $d = 2$, the KPZ equation is asymptotically free, and the crossover to the strong-coupling regime is exponentially slow [6]. Recent numerical values for the exponents are $\chi = 0.386$ and $z = 1.612$ [7]. For $d > 2$, the height profile is smooth in the gaussian theory. A small nonlinearity $\frac{1}{2}\lambda(\nabla h)^2$ does not alter this asymptotic scaling; there is now a *roughening transition* to the strong-coupling phase at finite critical values $\pm\lambda_c$ [8–10]. In the renormalization group, the transition is represented by a third fixed point. This *critical fixed point* is unstable and appears between the gaussian fixed point and the strong-coupling fixed point which are now both stable [6]. Numerical studies [11,7] indicate that a strong-coupling phase with $z < 2$ persists also in high dimensions; various theoretical arguments, on the other hand, predict the existence of a finite upper critical dimension $d_>$, above which $z = 2$ in both the weak- and strong-coupling regimes [9,12,13].

A satisfactory theory of stochastic growth should classify the different universality classes and the possible crossover phenomena between them, as well as give a way to calculate scaling indices exactly or in a controlled approximation. Despite considerable efforts, such a theory still seems far. In the framework of the renormalization group, the strong-coupling fixed point does not seem to be accessible by the methods of renormalized perturbation theory and the ε-expansion that have been so successful in equilibrium critical phenomena. This key difficulty is a further common feature of the KPZ equation and turbulence, and one may speculate that its eventual solution will be similar in both cases as well.

Recent progress has taken place mainly along a different avenue. Various groups have studied the so-called mode-coupling equations, a self-consistent approximation to the full problem. The mode-coupling equations have been shown to be exact in a certain large-N limit [14]. However, for two reasons it is not clear at present whether the mode-coupling solution can serve as the basis for a controlled $1/N$-expansion: (i) The numerical solution of the mode-coupling equations presents great difficulties since it relies so far on assumptions on the approximate scaling form of the propagator. While it incorporates the known exponents in $d = 1$ [15] and seems to produce exponents which are not too far from the best numerical estimates in $d = 2$ [16,14,17,18], there are considerable discrepancies in higher dimensions, which leaves the status of this approximation open. (ii) The corrections to mode coupling for finite N are not necessarily analytic as $N \to \infty$.

The aim of this paper is to compare the strong-coupling phase with the roughening transition under renormalization group aspects. The critical fixed point is perfectly accessible in an ε-expansion about the lower critical dimensionality $d_< = 2$, as I show in Sections 2 and 3 by dynamic renormalization and by exploiting the mapping of the KPZ equation onto a system of directed polymers with quenched disorder. The structure of the perturbative singularities is in fact very simple. In terms of the dimensionless coupling constant u_M^2 in a minimal subtraction scheme, the beta function reads exactly to all orders in perturbation theory

$$\beta(u_M^2) = 2\varepsilon u_M^2 + 2(u_M^2)^2 \tag{1.4}$$

with $\varepsilon = (2 - d)/2$. This yields the dimension-independent critical exponents

$$z^* = 2\,, \qquad \chi^* = 0 \tag{1.5}$$

at the roughening transition, which agree with a one-loop dynamic renormalization group calculation [6], that has recently been extended to two-loop order [19], and with a scaling argument by Doty and Kosterlitz [20].

The perturbation theory for the roughening transition is likely to be exact in the interval $2 \leqslant d \leqslant 4$; however, $d = 4$ is seen to be a singular point. It is hence tempting to identify $d_> = 4$; see the discussion below.

In Section 4, I turn to the consequences of this ε-expansion for the crossover to the strong-coupling fixed point in $d = 2$. It proves necessary to carefully distinguish between fields and couplings in *minimal subtraction* and properly *renormalized* fields and couplings defined by their finiteness at a renormalization point. In particular, it is shown that the functional dependence of the renormalized coupling constant u_R and the renormalized height field h_R on their minimal subtraction counterparts,

$$u_R(u_M)\,, \qquad h_R(h_M, u_M)\,, \tag{1.6}$$

has an essential singularity at $u_M = 0$. (In an ordinary ε-expansion, the mapping $u_R(u_M)$ is a diffeomorphism of which both the ultraviolet fixed point $u_R = u_M = 0$ and the infrared fixed point $u_R^*(u_M^*)$ are regular points.) This property allows one to pin down the reasons for the failure of perturbation theory for the strong-coupling fixed point.

The results are summarized and discussed in Section 5.

2. The roughening transition: dynamic renormalization

In this section, I will sketch the dynamic renormalization of the KPZ equation [4,6,21,19] in a formalism that facilitates comparison with the renormalization for the polymer system.

It is convenient to use the dynamic functional [22]

$$\int \mathcal{D}h\,\mathcal{D}\tilde{h}\,\exp\left[-\int \mathrm{d}^d r\,\mathrm{d}t_0\left(\tfrac{1}{2}\tilde{h}_0^2 + i\tilde{h}_0\left(\frac{\partial}{\partial t_0}h_0 - \tfrac{1}{2}\nabla^2 h_0 - \frac{\lambda_0}{2}(\nabla h_0)^2 - \rho\right)\right)\right] \tag{2.1}$$

in terms of the field h_0 and the "ghost" field $\tilde{h}_0$, which generates response functions,

$$\left\langle \prod_{j=1}^{\widetilde{N}} i\tilde{h}_0(r_j,t_{0j}) \prod_{j=\widetilde{N}+1}^{\widetilde{N}+N} h_0(r_j,t_{0j}) \right\rangle = \prod_{j=1}^{\widetilde{N}} \frac{\delta}{\delta\rho(r_j,t_{0j})} \left\langle \prod_{j=\widetilde{N}+1}^{\widetilde{N}+N} h_0(r_j,t_{0j}) \right\rangle . \tag{2.2}$$

Here the convention has been adopted to absorb all dimensionful constants of the linear theory into the "canonical" variables

$$t_0 = \nu t\,, \qquad h_0 = \left(\frac{\nu}{\sigma^2}\right)^{1/2} h\,, \qquad \tilde{h}\,, \tag{2.3}$$

which have dimensions

$$z_0 = 2\,, \qquad -\chi_0 = \frac{d-2}{2}\,, \qquad \chi_0 + d = \frac{d+2}{2}\,, \tag{2.4}$$

respectively. This convention is standard in field theory, but unfortunately is not generally used in the literature on dynamic renormalization, which tends to burden the calculations with redundant factors.

The linear theory has the response propagator

$$\begin{aligned} G_0(r_2 - r_1, t_{02} - t_{01}) &\equiv \langle i\tilde{h}_0(r_1,t_{01})\,h_0(r_2,t_{02})\rangle \\ &= \frac{\theta(t_{02}-t_{01})}{(4\pi(t_{02}-t_{01}))^{-d/z_0}} \exp\left[\frac{-(r_2-r_1)^{z_0}}{t_{02}-t_{01}}\right] . \end{aligned} \tag{2.5}$$

The formal expression for the height–height correlation function

$$\begin{aligned} C_0(r_2 - r_1, t_{02} - t_{01}) &\equiv \langle h_0(r_1,t_{01})\,h_0(r_2,t_{02})\rangle \\ &= \int \mathrm{d}^d r\,\mathrm{d}t_0\, G_0(t_{01}-t_0, r_{01}-r_0) G_0(t_{02}-t_0, r_{02}-r_0) \end{aligned} \tag{2.6}$$

requires for $d < 2$ (i.e. $\chi_0 > 0$) the introduction of an infrared cutoff. In a system of finite size L with periodic boundary conditions, the stationary correlation function at late times t_1, t_2 is translationally invariant; it has a singularity

$$C_0(r_1 - r_2, t_{01} - t_{02}, L) \sim L^{2\chi_0}\,. \tag{2.7}$$

Only the infrared-regularized correlation function remains well-defined in the thermodynamic limit $L \to \infty$:

$$C_0'(r_2 - r_1, t_{02} - t_{01}) \equiv \lim_{L\to\infty} [C_0(r_2 - r_1, t_{02} - t_{01}, L) - C_0(0,0,L)]$$
$$= |r_2 - r_1|^{2\chi_0} F(|t_{02} - t_{01}|/|r_2 - r_1|^{z_0}) . \tag{2.8}$$

In the interacting theory, the same subtraction is necessary to define the L-independent stationary two-point function $C'(r, t_0, u_0)$; the higher connected correlation functions require infrared regularizations as well. The scale L will also serve to generate the renormalization group flow below.

The canonical coupling constant

$$\lambda_0 = \left(\frac{\sigma^2}{\nu^3}\right)^{1/2} \lambda \tag{2.9}$$

has the dimension $\varepsilon \equiv (d-2)/2$. We define the dimensionless coupling constant

$$u_0 = \lambda_0 L^{\epsilon} . \tag{2.10}$$

The response and correlation functions of the nonlinear theory have the crossover scaling form

$$\left\langle \prod_{j=1}^{\widetilde{N}} i\tilde{h}_0(r_j, t_{0j}) \prod_{j=\widetilde{N}+1}^{\widetilde{N}+N} h_0(r_j, t_{0j}) \right\rangle$$
$$= L^{-N\chi_0 + \widetilde{N}(\chi_0+d)} F_{N\widetilde{N}}\left(\frac{r_j - r_k}{L}, \frac{t_{0j} - t_{0k}}{L^{z_0}}, u_0\right) , \tag{2.11}$$

which can be expressed as the "bare" Callan–Symanzik equation

$$\left(L \left.\frac{\partial}{\partial L}\right|_{\lambda_0} + \sum_j r_j \frac{\partial}{\partial r_j} + z_0 \sum_j t_{0j} \frac{\partial}{\partial t_{0j}} + \beta_0(u_0) \frac{\partial}{\partial u_0} - N\chi_0 + \widetilde{N}(\chi_0 + d) \right)$$
$$\times \left\langle \prod_{j=1}^{\widetilde{N}} i\tilde{h}_0(r_j, t_{0j}) \prod_{j=\widetilde{N}+1}^{\widetilde{N}+N} h_0(r_j, t_{0j}) \right\rangle = 0 \tag{2.12}$$

with

$$\beta_0(u_0) \equiv L\partial_L u_0 = \varepsilon u_0 . \tag{2.13}$$

For the infrared-regularized correlators, the explicit dependence on L vanishes in the thermodynamic limit.

In the strong-coupling limit $\lambda_0 \to \infty$, these correlation functions develop anomalous scaling and hence a singular dependence on the bare coupling constant λ_0. Renormalization consists in absorbing these singularities into new variables

$$
\begin{aligned}
h_{\mathrm{R}} &= Z_h(u_{\mathrm{R}})\, h_0\,, \\
\tilde{h}_{\mathrm{R}} &= Z_h^{-1}(u_{\mathrm{R}})\, \tilde{h}_0\,, \\
t_{\mathrm{R}} &= Z_t(u_{\mathrm{R}})\, t_0\,, \\
u_{\mathrm{R}} &= Z(u_{\mathrm{R}})\, u_0\,.
\end{aligned}
\tag{2.14}
$$

Under this change of variables, Eq. (2.12) transforms into the renormalized Callan–Symanzik equation

$$
\begin{aligned}
&\left(L \left.\frac{\partial}{\partial L}\right|_{\lambda_0} + \sum_j r_j \frac{\partial}{\partial r_j} + z(u_{\mathrm{R}}) \sum_j t_{Rj} \frac{\partial}{\partial t_{Rj}} \right. \\
&\quad \left. + \beta_{\mathrm{R}}(u_{\mathrm{R}}) \frac{\partial}{\partial u_{\mathrm{R}}} - N\chi(u_{\mathrm{R}}) + \tilde{N}(\chi(u_{\mathrm{R}}) + d) \right) \\
&\quad \times \left\langle \prod_{j=1}^{\tilde{N}} i\tilde{h}_{\mathrm{R}}(r_j, t_{Rj}) \prod_{j=\tilde{N}+1}^{\tilde{N}+N} h_{\mathrm{R}}(r_j, t_{Rj}) \right\rangle = 0
\end{aligned}
\tag{2.15}
$$

with

$$
\beta(u_{\mathrm{R}}) \equiv L \frac{\partial}{\partial L} u_{\mathrm{R}} = \frac{\varepsilon u_{\mathrm{R}}}{1 - u_{\mathrm{R}}\, \mathrm{d}Z/\,\mathrm{d}u_{\mathrm{R}}}\,, \tag{2.16}
$$

$$
z(u_{\mathrm{R}}) = z_0 - \beta \frac{\mathrm{d}}{\mathrm{d}u_{\mathrm{R}}} \log Z_t\,, \tag{2.17}
$$

$$
\chi(u_{\mathrm{R}}) = \chi_0 - \beta \frac{\mathrm{d}}{\mathrm{d}u_{\mathrm{R}}} \log Z_h\,. \tag{2.18}
$$

A different but equivalent Callan–Symanzik equation is derived in Ref. [19]. Notice that the renormalization of the ghost field is not independent since the response function

$$
G_{\mathrm{R}}(r_2 - r_1, t_{R2} - t_{R1}, u_{\mathrm{R}}) = \langle i\tilde{h}(r_1, t_{R1})\, h(r_2, t_{R2}) \rangle
$$

always has dimension d by its definition. Furthermore, a Ward identity due to Galilei invariance [19] enforces the following relation between the Z-factors:

$$
Z = Z_t^{-1} Z_h^{-1}\,. \tag{2.19}
$$

By inserting this relation into Eqs. (2.16), (2.17), and (2.18), one obtains

$$
\beta(u_{\mathrm{R}}) = [-2 + z(u_{\mathrm{R}}) + \chi(u_{\mathrm{R}})]\, u_{\mathrm{R}} \tag{2.20}
$$

and hence at any nontrivial fixed point $u_{\mathrm{R}}^* \neq 0$ the exponent identity

$$
z(u_{\mathrm{R}}^*) + \chi(u_{\mathrm{R}}^*) = 2\,. \tag{2.21}
$$

The renormalized variables (2.14) can be defined in a nonperturbative way by imposing two independent normalization conditions e.g. on the infrared-regularized correlators in an infinite system,

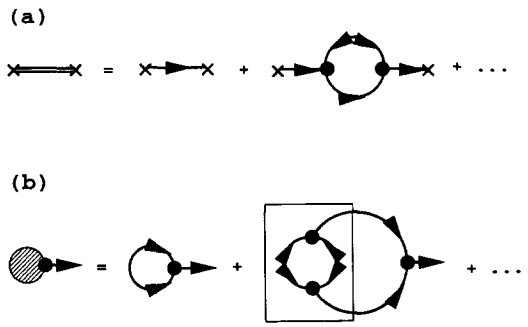

Fig. 1. Diagrammatic expansions generated by the dynamic functional (2.1). The lines with one and two arrows denote the unperturbed response function (2.5) and the unperturbed correlation function (2.8), respectively. Dots represent the vertices $i\tilde{h}(\nabla h)^2$; each incoming line to a vertex has to be differentiated with respect to r. (a) Response function $G_0(r, t_0, u_0)$. The one-loop diagram is regular at $\varepsilon = 0$. (b) Stationary growth rate $\langle \partial_{t_0} h_0 \rangle (L, u_0)$. The boxed subdiagram contains a simple pole at $\varepsilon = 0$. A further diagram with a loop as in (a) is regular at $\varepsilon = 0$ and has been omitted.

$$L^{2\chi_0} C'_{\mathrm{R}}(L, t_{\mathrm{R}} = 0, u_{\mathrm{R}}) = L^{2\chi_0} C'_0(L, t_0 = 0) , \tag{2.22}$$

$$L^d G_{\mathrm{R}}(0, t_{\mathrm{R}} = L^2, u_{\mathrm{R}}) = L^d G_0(0, t_0 = L^2) . \tag{2.23}$$

L is now an arbitrary normalization scale. An alternative to (2.22) is the normalization condition

$$L^{2-\chi_0} \langle \partial_{t_{\mathrm{R}}} h_{\mathrm{R}} \rangle (L, u_{\mathrm{R}}) = -b u_{\mathrm{R}} \tag{2.24}$$

on the universal finite-size correction to the stationary growth velocity in a system of size L, where $b > 0$ is a constant independent of u_{R} that is defined in (2.26) below.

In a perturbative ε-expansion, there is an alternative way of constructing the Z-factors, namely order by order through a minimal subtraction prescription. This makes the r.h.s. of Eqs. (2.22) and (2.23) analytic functions of the minimally renormalized coupling constant u_{M} with coefficients that remain finite as $\varepsilon \to 0$. As long as the coupling constant is small, this scheme is clearly equivalent to normalization conditions. As we shall see in Section 4, this is no longer the case for large values of u_{R}. I will therefore denote all quantities in the minimal subtraction scheme by the subscript M, and reserve the term "renormalized" and the subscript R to quantities defined by normalization conditions.

I will discuss the perturbative renormalization not for the momentum-space response and correlation functions in the infinite system as it is customarily done but for the position space response function $G_0(r, t_0, u_0)$ and the stationary finite-size amplitude $\langle \partial_{t_0} h_0 \rangle (L, u_0)$. The calculation is in close analogy to the polymer renormalization group of the next section.

The response function has the diagrammatic expansion shown in Fig. 1a. To order u_0^2, the expansion reads

$$L^d G_0(0, t_0 = L^2, u_0) = L^d G_0(0, t_0 = L^2) + \mathrm{O}(u_0^2 \varepsilon^0, u_0^4) . \tag{2.25}$$

The one-loop diagram does not have a pole at $d = 2$ since its short-distance singularity cancels with a geometric factor $2 - d$ [21,19]. As Frey and Täuber have shown, constructing the strong-coupling fixed point in $d = 1$ requires taking into account this "hidden" pole of the response function [19]. However, this finite renormalization can be ignored for the critical fixed point above $d = 2$, which is the focus of this section.

The expansion for the growth velocity is shown in Fig. 1b. The tadpole diagram at order u_0 consists of a nonuniversal ultraviolet-divergent part and of the universal finite-size correction

$$\int_{\text{reg}} \mathrm{d}^d r' \, \mathrm{d}t_0' \, [\partial_{r'} G_0(r', t_0', L)]^2 = -bL^{-d} , \tag{2.26}$$

At order u_0^3, the boxed subdiagram contributes a pole,

$$L^{2-\chi_0} \langle \partial_{t_0} h_0 \rangle (L, u_0) = -bu_0 \left(1 + \frac{c}{\epsilon} u_0^2\right) + \mathrm{O}(u_0^3 \varepsilon^0, u_0^5) \tag{2.27}$$

with $c = 1/32\pi$. This pole originates from the integration region where the two vertices approach each other. It can be absorbed into the definition of the variables $h_{\mathrm{M}} = Z_{Mh} h_0$, $\tilde{h}_{\mathrm{M}} = Z_{Mh}^{-1} \tilde{h}_0$, $t_{\mathrm{M}} = Z_{Mt} t_0$, $u_{\mathrm{M}} = Z_{\mathrm{M}} u_0$, with

$$Z_{Mh}(u_{\mathrm{M}}) = 1 - \frac{c}{2\varepsilon} u_{\mathrm{M}}^2 + \mathrm{O}(u_{\mathrm{M}}^4) , \tag{2.28}$$

$$Z_{Mt}(u_{\mathrm{M}}) = 1 + \mathrm{O}(u_{\mathrm{M}}^4) , \tag{2.29}$$

$$Z_{\mathrm{M}}(u_{\mathrm{M}}) = 1 + \frac{c}{2\varepsilon} u_{\mathrm{M}}^2 + \mathrm{O}(u_{\mathrm{M}}^4) . \tag{2.30}$$

This reparametrization respects (2.19) and renders both the response function and the growth rate regular as $\varepsilon \to 0$,

$$L^d G_{\mathrm{M}}(0, t_{\mathrm{M}} = L^2, u_{\mathrm{M}}) = L^d G_0(0, t_0 = L^2) + \mathrm{O}(u_{\mathrm{M}}^2 \varepsilon^0, u_{\mathrm{M}}^4) , \tag{2.31}$$

$$L^{2-\chi_0} \langle \partial_{t_{\mathrm{M}}} h_{\mathrm{M}} \rangle (L, u_{\mathrm{M}}) = -bu_{\mathrm{M}} + \mathrm{O}(u_{\mathrm{M}}^3 \varepsilon^0, u_{\mathrm{M}}^5) . \tag{2.32}$$

From (2.28) and (2.16), one obtains the beta function

$$\beta_{\mathrm{M}}(u_{\mathrm{M}}) = \varepsilon u_{\mathrm{M}} + c u_{\mathrm{M}}^3 + \mathrm{O}(u_{\mathrm{M}}^5) . \tag{2.33}$$

For $d > 2$, it has a pair of real-valued unstable fixed points

$$u_{\mathrm{M}}^{\star 2} = -\frac{\epsilon}{c} + \mathrm{O}(\varepsilon^2) \tag{2.34}$$

that describe the roughening transition from the weak coupling to the strong-coupling phase. Relations (2.17) and (2.18) then give the critical exponents [6]

$$z^\star = 2 + \mathrm{O}(\varepsilon^2) , \qquad \chi^\star = 0 + \mathrm{O}(\varepsilon^2) \tag{2.35}$$

satisfying (2.21). While higher-order calculations are cumbersome in the dynamic framework, we will see in the next section that these values are exact to all orders in perturbation theory.

3. The roughening transition: replica renormalization

It is well known that the KPZ equation can be mapped onto a system of *directed polymers* given by the partition function

$$Z = \int \mathcal{D}r \exp\left[-\frac{1}{2\nu}\int \mathrm{d}t\left(\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \lambda\eta\right)\right]. \tag{3.1}$$

Here $r(t)$ denotes the polymer displacement field in d transversal dimensions as a function of the longitudinal "timelike" coordinate t. The polymer is subject to the *quenched* random potential $\lambda\eta(r,t)$ that has the statistics (1.2) and describes point impurities with short-ranged correlations.

The height field of Eq. (1.1) is related to the restricted partition sum $Z(r,t)$ of all paths ending at a given point (r,t) by the Hopf–Cole transformation

$$h(r,t) = \frac{2\nu}{\lambda}\log Z(r,t)\,. \tag{3.2}$$

Hence the *disorder-averaged* free energy per unit longitudinal length

$$\overline{f}(L) \equiv -2\nu \lim_{T\to\infty} \partial_T \overline{\log Z}(T,L) \tag{3.3}$$

in a system of size $T \times L$ (with periodic boundary conditions in transversal direction) is proportional to the stationary growth velocity,

$$\overline{f} = -\lambda\langle \partial_t h\rangle\,. \tag{3.4}$$

A convenient way to set up perturbation theory is the replicated partition function

$$Z_p = \int \prod_{\alpha=1}^{p} \mathcal{D}r_\alpha \exp\left[-\tfrac{1}{4}\int \mathrm{d}t_0 \left(\sum_\alpha \left(\frac{\mathrm{d}r_\alpha}{\mathrm{d}t_0}\right)^2 - \lambda_0^2 \sum_{\alpha<\beta}\delta(r_\alpha - r_\beta)\right)\right], \tag{3.5}$$

where the parameters ν and σ^2 are again absorbed into the definition of the canonical variables t_0 and h_0. Since we are interested in arbitrary particle numbers p, we rewrite the partition function in second quantization,

$$Z = \int \mathcal{D}\phi\,\mathcal{D}\bar{\phi} \exp\left[-\tfrac{1}{4}\int \mathrm{d}t_0\,\mathrm{d}^d r \left(\bar{\phi}(\partial_{t_0} - \partial_r^2)\phi - \lambda_0^2\bar{\phi}^2\phi^2\right)\right], \tag{3.6}$$

where $\phi(r,t_0)$ is a complex field. The normal-ordered interaction term $-\lambda_0^2\bar{\phi}^2\phi^2$ is an *attractive* pair contact potential. More generally, we define the normal-ordered m-line contact fields

$$\Phi_m(t_0) \equiv \int \mathrm{d}^d r\,[\bar{\phi}(r,t)]^m[\phi(r,t)]^m\,. \tag{3.7}$$

The perturbation series for the free energy

$$f_p(L,u_0^2) = -L^{-2}\sum_{N=1}^{\infty}\frac{u_0^{2N}}{4^N N!}L^{-2N\varepsilon}\int \mathrm{d}t_2 \ldots \mathrm{d}t_N \langle \Phi_2(0)\Phi_2(t_2)\ldots\Phi_2(t_N)\rangle_p \tag{3.8}$$

is a sum involving connected pair field correlations in the p-line sector of the unperturbed theory ($u_0 = 0$). The integrals in Eq. (3.8) are infrared-regularized by the system size L; their ultraviolet singularities are determined by the short-distance structure of the pair field correlations and have to be absorbed into the coupling constant renormalization. Hence consider the asymptotic scaling of the N-point function $\langle \Phi_2(t_{01})\ldots\Phi_2(t_{0N})\rangle$ as the points $t_{01},\ldots,t_{0N}$ approach each other,

$$t_{0j} - t_{0k} = t\tau_{jk} \quad \text{and} \quad t/L^2 \to 0 \tag{3.9}$$

with τ_{jk} and the "center of mass" $t_0' = N^{-1}\sum_{j=1}^{N} t_{0j}$ remaining fixed. This is given by the p-independent short-distance algebra

$$\Phi_2(t_{01})\ldots\Phi_2(t_{0N}) = \sum_{m=2}^{N+1} t^{-(N-m+1)d/2}\left[C_N^m(\tau_1,\ldots,\tau_{N-2})\Phi_m(t_0') + \ldots\right], \tag{3.10}$$

where C_N^m are scaling functions of the $N-2$ linearly independent distance ratios τ_{jk} and the dots denote terms that are subleading by positive integer powers of t/L^2. The integration over the relative distances then yields

$$\begin{aligned}&\int \mathrm{d}t\, t^{N-2}\prod_{l=1}^{N-2}\mathrm{d}\tau_l \langle \Phi_2(t_{01})\ldots\Phi_2(t_{0N})\rangle \\ &= \sum_{m=2}^{N+1}\int J_N^m\, t^{m-3+\varepsilon(N-m+1)}\,\mathrm{d}t\,\langle\Phi_m(t_0')\rangle + \ldots\end{aligned} \tag{3.11}$$

with

$$J_N^m = \int\prod_{l=1}^{N-2}\mathrm{d}\tau_l\, C_N^m(\tau_1,\ldots,\tau_{N-2})\,. \tag{3.12}$$

Hence we obtain to one-loop order

$$L^2 f_p(L,u_0^2) = -L^d\langle\Phi_2\rangle_p\cdot\tfrac{1}{4}u_0^2\left(1+\frac{c}{\epsilon}u_0^2\right) + \mathrm{O}(u_0^4\varepsilon^0, u_0^6) \tag{3.13}$$

with $L^d\langle\Phi_2\rangle_p = \frac{1}{2}p(p-1)$. The pole in (3.13) originates from the term in (3.11) with $m = N = 2$, which is shown diagrammatically in Fig. 2a. This pole can be absorbed into the definition $u_{\mathrm{M}}^2 = Z_{\mathrm{M}}^2 u_0^2$ with Z_{M} given by (2.30). To this order, we hence recover the beta function (2.33) and the fixed points (2.34) of the dynamic calculation.

In the polymer framework, however, it is not difficult to discuss higher orders. The ultraviolet singularities of the Nth order integral (3.11) are contained in the coefficients J_N^m or arise from the integration over t. In the first case, they are due to a *proper*

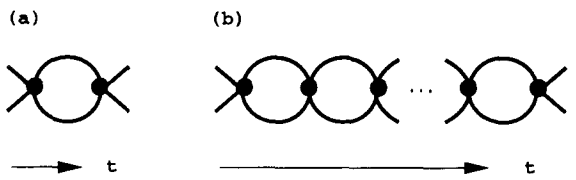

Fig. 2. Singular diagrams contributing to the finite-size free energy $f_p(L, u_0^2)$ in the expansion (3.8). The lines denote unperturbed single-line propagators $\langle \tilde{\phi}(r_1, t_{01}) \phi(r_2, t_{02}) \rangle$, the dots pair contact vertices $\Phi_2(t_0)$. (a) One-loop diagram containing a primitive pole at $\varepsilon = 0$. (b) N-loop diagram containing a pole of order $N - 1$ at $\varepsilon = 0$.

subdiagram and hence already absorbed into the renormalized coupling constant at lower order. Only the divergences from the integration over t, with J_N^m denoting the regular part of (3.12), may contribute to the *primitive singularity* at order N. Inspection of (3.11) then shows that a pole at $\varepsilon = 0$ only appears for $m = 2$. However, there is only one diagram per order of this kind, which is shown in Fig. 2b. Since this diagram factorizes into loops of the kind of Fig. 2a, it contributes a pole in ε of order $N - 1$. Therefore, the pole at order $N = 2$ is the only primitive singularity in the series (3.8) for the free energy; analogous arguments apply to the expansions of the contact field correlation functions. It follows that the one-loop equations (2.30), (2.33) and (2.34) are exact to all orders in perturbation theory (see also Refs. [24,25]).

The replica trick is unproblematic within perturbation theory, since it reduces to convenient bookkeeping of the averaging over disorder. Indeed, the random limit $p \to 0$ is trivial in Eqs. (2.30), (2.33) and (2.34) which are independent of p. The crossover scaling function of the disorder-averaged free energy

$$\mathcal{C}(u_{\mathrm{M}}^2) \equiv L^2 \overline{f}(L, u_{\mathrm{M}}^2) = L^2 \lim_{p \to 0} \frac{1}{p} f_p(L, u_{\mathrm{M}}^2) \tag{3.14}$$

is a regular function of the minimally subtracted coupling constant u_{M}^2, as follows by inserting (2.30) into (3.13). By (3.4), (2.28), (2.29), and (2.30), the function $-\mathcal{C}(u_{\mathrm{M}}^2)/u_{\mathrm{M}}$ equals the scaling function (2.32) of the minimally subtracted growth rate, and hence $b = \frac{1}{8}$. If the renormalization point condition (2.24) is chosen, the universal function $\mathcal{C}(u_{\mathrm{M}}^2)$ is directly related to the renormalized coupling constant, $\mathcal{C}(u_{\mathrm{M}}^2) = b u_{\mathrm{R}}^2(u_{\mathrm{M}}^2)$. Its finite fixed point value

$$\mathcal{C}^* = \mathcal{C}(u_{\mathrm{M}}^{*2}) = -\frac{b}{c}\varepsilon + \mathrm{O}(\varepsilon^2)\,, \tag{3.15}$$

the *Casimir amplitude* at the roughening transition, is the analog of the central charge in conformally invariant field theories [23]. The function $\mathcal{C}(u_{\mathrm{M}}^2)$ has the finite limit (3.15) since the free energy does not develop an anomalous dimension at the roughening transition, i.e. hyperscaling is preserved. By (3.4), this implies that $\langle \partial_t h \rangle \sim L^{-2}$ at the transition, and hence $\chi^* - z^* = 2$. The exponents (1.5) then follow from this relation together with (2.21).

The dynamic exponent $z^* = 2$ can also be verified independently. Consider the two-point function of the normal-ordered density field $\Phi_1(t_0) \equiv \bar{\phi}\phi(r = 0, t_0)$,

$$\langle \Phi_1(t_0)\Phi_1(t_0')\rangle(L, u_0^2) = \sum_{N=0}^{\infty} \frac{u_0^{2N}}{4^N N!} L^{-2N\varepsilon} \int \mathrm{d}t_{01} \ldots \mathrm{d}t_{0N} \times \langle \Phi_1(t_0)\Phi_1(t_0')\Phi_2(t_{01}) \ldots \Phi_2(t_{0N})\rangle . \tag{3.16}$$

Its short-distance asymptotics gives the *return probability* of a single line to the origin $r = 0$. In the linear theory,

$$\langle \Phi_1(t_0)\Phi_1(t_0')\rangle(L, 0) \sim |t_0 - t_0'|^{-d/z_0} \tag{3.17}$$

for $|t_0 - t_0'|/L^2 \ll 1$. Any anomalous contribution to this exponent arises from the renormalization of the fields $\Phi_1(t_0)$ and $\Phi_1(t_0')$. The renormalization of $\Phi_1(t_0)$ is due to a short-distance coupling of the form

$$\Phi_1(t_0)\Phi_2(t_{01}) \ldots \Phi_2(t_{0N}) = t^{-Nd/2} C_N^1(\tau_1, \ldots, \tau_{N-1})\Phi_1(t_0) + \ldots \tag{3.18}$$

for $t/L^2 \to 0$ (with $t_{0j} - t_0 = t\tau_{0j}$, $t_{0j} - t_{0k} = t\tau_{jk}$ for $j, k = 1, \ldots, N$, and $\tau_1, \ldots, \tau_{N-1}$ denoting a basis of the fixed ratios τ_{0j}, τ_{jk}), and there is a corresponding expression for $\Phi_1(t_0')$. However, it is obvious that the product on the l.h.s. couples only to contact fields of at least two lines, and therefore $C_N^1 = 0$ at all orders N.

4. The strong-coupling fixed point

As discussed in the previous sections, the crossover from the critical fixed point to the gaussian fixed point in $d > 2$ can be parametrized in terms of the coupling constant u_{M} of a minimal subtraction scheme. In the framework of the ε-expansion, u_{M} is completely equivalent to the coupling constant u_{R} defined by normalization conditions, e.g. (2.22) and (2.23): the two couplings are related by a diffeomorphism that remains regular in the limit $\varepsilon \to 0$, defined on a domain of interaction space that contains the fixed points $u_{\mathrm{M}} = u_{\mathrm{R}} = 0$ and $u_{\mathrm{R}}^*(u_{\mathrm{M}}^*)$. This equivalence is lost in the crossover to the strong-coupling fixed point, as I will now show.

Integration of the flow equation (1.4) for u_{M}^2 in two dimensions with the initial condition $u_{\mathrm{M}}^2(L_0) = u_1^2$ yields

$$u_{\mathrm{M}}^2(L) = \frac{u_1^2}{1 - u_1^2 \log(L/L_0)} , \tag{4.1}$$

an expression that diverges at a finite value

$$L = L_1 \equiv L_0 \exp(1/u_1^2) . \tag{4.2}$$

It follows immediately that $u_{\mathrm{M}}(L)$ is not well suited to describe the crossover to the strong-coupling fixed point since it divides the crossover into an ultraviolet regime $L < L_1$ and an infrared regime $L > L_1$ where $u_{\mathrm{M}}^2(L)$ is defined by analytic continuation

of the solution (4.1). However, the pole of $u_M^2(L)$ at $L = L_1$ is only a "coordinate singularity" [26] of the minimal subtraction scheme; any renormalized coupling $u_R(L)$ is expected to remain regular at $L = L_1$. Hence the function $u_M^2(u_R^2)$ also has a pole at the value $u_R^2(L = L_1)$ between the fixed points $u_R^2 = 0$ and u_R^{*2}.

For small $u_M^2 > 0$, the correlation functions (2.22) and (2.23) at the normalization point can be calculated perturbatively, which results in an analytic renormalization

$$h_R = [1 + a_1 u_R^2 + a_2 u_R^4 + \ldots] h_M , \tag{4.3}$$

$$t_R = [1 + b_1 u_R^2 + b_2 u_R^4 + \ldots] t_M , \tag{4.4}$$

and hence by (2.21)

$$u_M^2 = [1 + (a_1 + b_1) u_R^2 + (a_2 + a_1 b_1 + b_2) u_R^4 + \ldots] u_R^2 . \tag{4.5}$$

with finite coefficients a_N, b_N. Hence u_R^2 has the beta function

$$\beta_R(u_R^2) = u_R^4 + (a_1^2 + a_1 b_1 + b_1^2 - a_2 - b_2) u_R^8 + O(u_R^{10}) . \tag{4.6}$$

In perturbation theory, one would calculate these power series up to some finite order in u_R^2 and look for a fixed point u_R^{*2} of Eq. (4.6). However, a low-order calculation does not yield nontrivial exponents at the strong-coupling fixed point. Consider e.g. the expression (2.18) for the roughness exponent in $d = 2$,

$$\chi(u_R^{*2}) = -\beta_R(u_R^2) \frac{d}{du_R^2} \log Z_h(u_R^2) \Big|_{u_R^2 = u_R^{*2}} . \tag{4.7}$$

It is zero if $Z_h(u_R^2)$ is a regular function of u_R^2 since $\beta_R(u_R^{*2}) = 0$. In an ordinary ε-expansion, the beta function and the Z-factor are treated as power series; a finite result would then arise from the linear part of $\beta_R(u_R^2)$ together with the simple poles of $Z_h(u_R^2)$. Here, if we assume the existence of an expansion parameter $\tilde{\varepsilon}$ and naively take $u_R^* \sim \tilde{\varepsilon}$, we obtain $\chi(u_R^{*2}) = O(\tilde{\varepsilon}^{10})$ from (4.3), (4.6), and (4.7). In fact, we expect a perturbative fixed point of Eq. (4.6) to be spurious at any order, i.e. not to reflect properties of the long-distance asymptotic regime. The reason is that all coefficients a_N, b_N are finite for $\varepsilon = 0$ and thus depend on details of the infrared regularization, unlike the residues of the poles in ε which determine the Z-factors in an ordinary ε-expansion. It is difficult to see how this dependence could cancel out to produce universal exponents.

In the infrared regime $L > L_1$, the relationship between u_M^2 and u_R^2 is no longer given by Eq. (4.5). As $L \to \infty$, the asymptotic behavior of the renormalized quantities is

$$u_R^{*2} - u_R^2 \sim L^{-y'} = \exp[y'/u_M^2] , \tag{4.8}$$

$$t_R = Z_t t_M \sim L^{z - z_0} = \exp[-(z - z_0)/u_M^2] , \tag{4.9}$$

$$h_R = (u_M^2 u_R^{-2} Z_t^{-1}) h_M \sim u_M^2 \exp[(z - z_0)/u_M^2] h_M , \tag{4.10}$$

in terms of their minimal subtraction counterparts, where the exponent y' is defined by

$$\beta_R(u_R^2) = -y'(u_R^2 - u_R^{*2}) + O(u_R^2 - u_R^{*2})^2 .$$

Hence all of these quantities have essential singularities at $u_M = 0$, which are tied to an essential singularity

$$\langle h_M(r_1,t)h_M(r_2,t)\rangle \sim u_M^{-4}\exp[-2(z-z_0)/u_M^2] \tag{4.11}$$

of the correlation function in minimal subtraction in the same limit, for fixed distance $|r_1 - r_2|$. This shows the inequivalence of the two renormalization schemes.

5. Discussion

In this paper, it has been shown that the perturbation theory for the KPZ roughening transition can be developed consistently in the dynamic and in the directed polymer framework. To all orders, it predicts the exponents

$$z^* = 2\,, \qquad \zeta^* = 0 \tag{5.1}$$

resulting from the exact beta function (1.4) in minimal subtraction.

It should be emphasized that this calculation can only be trusted for $d < 4$. The minimally subtracted pair contact field $\Phi_{2,M}$ has the *exact* dimension $2 + 2\varepsilon$ at the critical fixed point. This dimension would turn negative for $d > 4$, which is obviously unphysical. This exhibits the singular rôle of dimension 4 for the roughening transition. Indeed, the regularization of the perturbation series (3.8) by minimal subtraction breaks down at $d = 4$ since already the two-loop integral $\int dt_{02}\,\langle\Phi_2(t_{01})\Phi_2(t_{02})\rangle$ develops a new pole that is not absorbed into the coupling constant renormalization (2.30).

This singularity also gives some indications on the existence of an upper critical dimension $d_>$ for the strong-coupling phase of the KPZ equation. At the critical fixed point u_M^{*2}, the nonlinearity is a relevant perturbation

$$\beta(u_M^2) = -2\varepsilon(u_M^2 - u_M^{*2}) + \ldots . \tag{5.2}$$

Hence beyond this fixed point, the scale-dependent interface roughness $\chi(L)$ defined in (2.18) increases to positive values to leading order in perturbation theory,

$$\chi(u_M^2(L)) = (u_M^2 - u_M^{*2}) + \ldots . \tag{5.3}$$

With the plausible assumption that $\chi(L)$ is a monotonic function of L over the entire crossover, this implies $\chi = \lim_{L\to\infty}\chi(L) > 0$ for the strong-coupling infrared limit. Again, this argument breaks down at $d = 4$, which implies $d_> \geqslant 4$ and in fact suggests the speculation that $d_> = 4$. This is in agreement with the most recent mode-coupling results [18]. It should be possible to corroborate these results by extending the analysis of the roughening transition to dimensions $d > 4$.

The form (1.4) of the flow equation has an interesting consequence for the description of the crossover to strong-coupling behavior in $d = 2$. The minimal subtraction coupling constant $u_M^2(u_R^2)$ develops a pole at a finite value $u_R^2 < u_R^{*2}$ of the renormalized coupling constant, a singularity not taken into account by existing renormalization group

treatments of the KPZ equation. That singularity divides the crossover into an ultraviolet regime $u_R^2 < u_I^2$ and an infrared regime $u_R^2 > u_I^2$. Only the former is accessible in perturbation theory: even the summation of an infinite number of terms e.g. in the series (4.3) and (4.4) can only lead to a relation $u_R^2(u_M^2)$ for $u_M^2 > 0$. Whether information on the infrared regime can be gained by inverting this relation and then analytically continuing it to values $u_R^2 > u_I^2$ remains to be seen.

The precise form (1.4) of the beta function is clearly due to the fact that the intrinsic dimension of the polymer, or the dimension of time, is one. The disorder problem, however, has a natural generalization from directed lines to D-dimensional directed surfaces. For $D \neq 1$, higher-order perturbative singularities are expected, which generate further terms in (1.4). This generalization has an interest in its own right. It is not clear, however, if it is useful to solve the original problem: any perturbative fixed point will tend to infinity as $D \to 1$, but the exponents may still be well-behaved in that limit.

Acknowledgements

Useful discussions with H. Kinzelbach are gratefully acknowledged.

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