

Quantum Game Theory

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A systematic theory is introduced that describes stochastic effects in game theory. In a biological context, such effects are relevant for the evolution of finite populations with frequency-dependent selection. They are characterized by *quantum Nash equilibria*, a generalization of the well-known Nash equilibrium points in classical game theory. The implications of this theory for biological systems are discussed in detail.

I. INTRODUCTION

Classical game theory is a well-known mathematical formalization of competitions with rational rules and rational players [1]. The strength of this theory lies in the abstraction from the detailed scenario. In its simplest form, a game is reduced to a set of basic *strategies* $i = 1, \dots, s$ and a matrix $\mathbf{A} = (A_{ij})$ whose elements denote the *payoff* or relative success of strategy i played against strategy j . A *mixed strategy* $\mathbf{x} = (x_1, \dots, x_s)$ is defined to be a probability distribution over basic strategies (i.e., $x_i \geq 0$ for all $i = 1, \dots, s$ and $\sum_i x_i = 1$). Rational playing of mixed strategies will lead to a *Nash equilibrium*, that is, to a strategy \mathbf{x}^* that maximizes the payoff against itself [2]. This is the central concept of classical game theory. A more precise formulation will be given below.

Game theory has been widely applied to explain the distribution of different phenotypes in biological populations. These applications are based on a dynamical extension called *evolutionary game theory*, which is due to Maynard Smith [3]. In the simplest case, the different phenotypes in a population are associated with the basic strategies of a game. The time-dependent phenotype frequency distribution is then a mixed strategy $\mathbf{x}(t)$. It is assumed that the phenotypes are hereditary and are preserved under the reproduction process. The payoff of a basic strategy i played against the population strategy $\mathbf{x}(t)$ enters the *fitness* f_i of the corresponding phenotype, that is, the expected number of viable offspring per individual and per unit of time. This game-based population dynamics can be written as an evolution equation for the frequency distribution $\mathbf{x}(t)$. It can be shown that *every stable fixed point of this dynamics is a Nash equilibrium* [4]. This result is conceptually important since it shows how strategic optimization is reached in biological systems through reproductive success alone, without the need for rational thinking.

Classical game theory is a deterministic theory. A Nash equilibrium is the outcome of fully rational playing, without any effects of chance. Biological game dynamics describes an equally deterministic course of evolution. Formally, this corresponds to the limit of infinite populations obtained by identifying the reproduction rate of

a phenotype i with its expectation value f_i .

In this paper, we extend game theory to a stochastic theory including the effects of fluctuations. This is based on a probabilistic game dynamics suitable to describe the evolution of finite biological populations. A different form of stochastic game dynamics has been discussed by Marsili and Zhang in the context of economical systems [5].

In biological populations, fluctuations arise since the actual number of viable offspring of a given individual in a given time interval differs from its expectation value determined by the fitness of the individual's phenotype. This may be caused by a number of different biological reasons. It will become clear that mechanisms producing frequency-dependent fitness values – which are the subject of evolutionary game theory – also give rise to frequency-dependent fluctuations and thereby determine the probabilistic dynamics. In this formalism, the time-dependent population state defines a probability distribution over mixed strategies, $p(\mathbf{x}, t)$, and evolution takes the form of a Schrödinger equation in imaginary time, $\partial_t p(\mathbf{x}, t) = Hp(\mathbf{x}, t)$. The analogy with quantum physics allows for a systematic inclusion of fluctuations and suggests the name *quantum game theory*.

Stochastic effects have been studied extensively in Kimura and Ohta's theory of *neutral evolution*, which describes the dynamics of populations whose phenotypes have little or no fitness difference [6,7]. Fluctuations are then the dominant force of evolution. In particular, it has been shown that an initially small fraction $x_m(t)$ of mutants in an otherwise homogeneous resident population can be driven either to extinction or to fixation, i.e., $x_m(t) = 0$ or $x_m(t) = 1$ for late t . The probabilities for these processes depend on the fitness gap between mutants and residents and on the overall population size, and they determine the rate of evolution of the population as a whole.

Fluctuation effects in game theory turn out to be more involved. Quantum game theory shows a nontrivial interplay between deterministic fitness and stochastic forces, both of which depend on the phenotype frequencies $\mathbf{x}(t)$. Hence, a stable stationary state $p^*(\mathbf{x})$ – called a *quantum Nash equilibrium* – differs from its classical counterpart. It depends strongly on two parameters defined below,

the characteristic population size n^* and the game coupling λ . We find a crossover between neutral evolution for $\lambda n^* \ll 1$ and classical game theory for $\lambda n^* \gg 1$. Fluctuations can lead to extinction or fixation; the corresponding probabilities now depend not only on the population size but also on the phenotype considered. We discuss this in detail for the simplest game with a mixed classical Nash equilibrium, the so-called *hawk-dove game*.

The paper is organized as follows. In the next section, we define a game-based classical dynamics suitable for finite populations. In section 3, we discuss the ‘quantization’ of this dynamics. Section 4 contains a detailed analysis of the quantum theory for the hawk-dove game, and the results are discussed in section 5.

II. CLASSICAL GAME THEORY AND POPULATION DYNAMICS

We start by recalling a few well-known definitions and results of classical game theory. In a game with basic strategies $i = 1, \dots, s$ and relative payoffs A_{ij} , the payoff of a mixed strategy \mathbf{x} played against another mixed strategy \mathbf{x}' is taken to be bilinear, $\sum_{i,j=1}^s x_i A_{ij} x'_j \equiv \mathbf{x} \mathbf{A} \mathbf{x}'$. We can now define a (symmetrical) Nash equilibrium \mathbf{x}^* as an optimal strategy against itself, i.e.,

$$\mathbf{x}^* \mathbf{A} \mathbf{x}^* \geq \mathbf{x} \mathbf{A} \mathbf{x}^* \quad \text{for all strategies } \mathbf{x}. \quad (1)$$

Consider a population that contains the phenotypes $i = 1, \dots, s$ with time-dependent population numbers $\mathbf{n}(t) = (n_1(t), \dots, n_s(t))$ and has total size $n(t) = \sum_i n_i(t)$; these numbers are positive integers. The evolution of this system can be described deterministically by an equation of the form

$$\frac{1}{n_i} \frac{dn_i}{dt} = f_i(\mathbf{n}); \quad (2)$$

the r.h.s. is called the fitness of the phenotype i . We assume its frequency dependent part is proportional to the payoff of the basic strategy i played against the mixed strategy $\mathbf{x} \equiv \mathbf{n}/n$ in a game with payoff matrix \mathbf{A} , i.e., $f_i(\mathbf{n}) = \lambda(\mathbf{A}\mathbf{x})_i + \lambda'$, where λ and λ' are coefficients independent of \mathbf{x} . The phenotype frequency distribution then obeys the closed evolution equation

$$\frac{1}{x_i} \frac{dx_i}{dt} = \lambda f_i^{\text{game}}(\mathbf{x}) \quad (3)$$

with

$$f_i^{\text{game}}(\mathbf{x}) = (\mathbf{A}\mathbf{x})_i - \mathbf{x} \mathbf{A} \mathbf{x}, \quad (4)$$

which is well known in evolutionary game theory [4]. The coupling constant $\lambda < 1$ describes the strength of game-based contributions to the fitness. Classically, it determines only the time scale in eq. (3) but it will have a

crucial role in the quantum theory. The second term on the r.h.s. of eq. (4) ensures probability conservation, i.e., $\sum_i dx_i/dt = 0$.

This frequency dynamics is independent of population size. In order to include fluctuations, which depend on absolute population numbers, we start from the full dynamics (2). We choose

$$f_i(\mathbf{n}) = \lambda f_i^{\text{game}}(\mathbf{x}) + f^{\text{size}}(n) \quad (5)$$

so that the frequencies $x_i(t)$ follow the evolutionary game dynamics (3) and decouple from the dynamics of the total population size,

$$\frac{1}{n} \frac{dn}{dt} = f^{\text{size}}(n). \quad (6)$$

With

$$f^{\text{size}}(n) = \frac{n^* - n}{n^*}, \quad (7)$$

the latter describes standard logistic growth to a stable population size n^* .

This population dynamics is similar to the well-known Lotka-Volterra equations. It can be rewritten in terms of the scaled population sizes $\mathbf{y} = (y_1, \dots, y_s) = \mathbf{n}/n^*$,

$$\frac{1}{y_i} \frac{dy_i}{dt} = \lambda f_i^{\text{game}}(\mathbf{y}/y) + f^{\text{size}}(y) \quad (8)$$

with $y = \sum_i y_i = n/n^*$. For every symmetric Nash equilibrium \mathbf{x}^* of the underlying game, eq. (8) has a stable fixed point $\mathbf{y}^* = \mathbf{x}^*$.

III. QUANTUM GAME DYNAMICS

As discussed above, the actual number of viable offspring produced by an individual of phenotype i in a given time interval is an integer which fluctuates around its expectation value given by the fitness f_i . In any finite population, these fluctuations produce deviations from the deterministic dynamics (2) and lead us to a stochastic description of evolution. The population state then becomes a probability distribution $P(\mathbf{n}, t)$. It is convenient to write this in the form of a quantum state,

$$|P(t)\rangle = \sum_{\mathbf{n}} P(\mathbf{n}, t) |\mathbf{n}\rangle \quad (9)$$

with $|\mathbf{n}\rangle = a_1^{\dagger n_1} \dots a_s^{\dagger n_s} |0\rangle$.

The dynamics takes the form of an (imaginary-time) Schrödinger equation,

$$\frac{d}{dt} |P(t)\rangle = H |P(t)\rangle. \quad (10)$$

The Hamilton operator H contains the creation and annihilation operators a_i^{\dagger} and a_i , which obey canonical commutation relations and describe birth and death of an

individual of phenotype i , respectively. This formalism has been widely applied to related dynamical problems such as reaction-diffusion models; see, e.g., refs. [8,9].

In order to construct the Hamiltonian in a systematic way, recall the biological meaning of fitness in the classical theory (8). The coefficient function $f_i(\mathbf{y})$ is not a simple birth rate but the effective rate of reproductive success, taking into account birth and death processes. It can be written as the difference between a birth rate $b_i(\mathbf{y})$ and a death rate $d_i(\mathbf{y})$ which are both positive,

$$f_i(\mathbf{y}) = b_i(\mathbf{y}) - d_i(\mathbf{y}) = \lambda(b_i^{\text{game}}(\mathbf{x}) - d_i^{\text{game}}(\mathbf{x})) + b^{\text{size}}(n) - d^{\text{size}}(n). \quad (11)$$

Clearly, this decomposition requires additional biological input as will be discussed below for the hawk-dove game. In the quantum theory, it becomes important since birth and death processes have individual fluctuations. As operators, these rates determine the reproduction current,

$$J_i = a_i^\dagger a_i b_i(\hat{\mathbf{n}}) - a_i d_i(\hat{\mathbf{n}}). \quad (12)$$

with $\hat{\mathbf{n}} = (\hat{n}_1, \dots, \hat{n}_s) = (a_1^\dagger a_1, \dots, a_s^\dagger a_s)$. We have $J_i |P(t)\rangle = \sum_{\mathbf{n}} j_i(\mathbf{n}, t) |\mathbf{n}\rangle$, where

$$j_i(\mathbf{n}, t) = n_i b_i(\mathbf{n}) P(\mathbf{n}, t) - (n_i + 1) d_i(\mathbf{n} + i) P(\mathbf{n} + i, t) \quad (13)$$

is the probability current between the states $|\mathbf{n}\rangle$ and $|\mathbf{n} + i\rangle$. Here we use the shorthand

$$\mathbf{n} \pm i \equiv (n_1, \dots, n_{i-1}, n_i \pm 1, n_{i+1}, \dots, n_s). \quad (14)$$

The Hamiltonian then takes the form

$$H = \sum_{i=1}^s (a_i^* - 1) J_i \quad (15)$$

so that eq. (10) is equivalent to the Master equation

$$\begin{aligned} (\partial/\partial t) P(\mathbf{n}, t) &= (n_i - 1) b_i(\mathbf{n} - i) P(\mathbf{n} - i, t) \\ &\quad - n_i b_i(\mathbf{n}) P(\mathbf{n}, t) \\ &\quad - n_i d_i(\mathbf{n}) P(\mathbf{n}, t) \\ &\quad + (n_i + 1) d_i(\mathbf{n} + i) P(\mathbf{n} + i, t). \end{aligned} \quad (16)$$

Probability conservation implies that H only has eigenvalues with non-positive real parts. We are interested in particular in the leading eigenstate $|\Phi^*(t)\rangle$ with support in the coexistence region of all phenotypes, $n_1 \geq 1, \dots, n_s \geq 1$. This eigenstate decays due to extinction processes $n_i = 1 \rightarrow 0$. The eigenvalue $E^* < 0$ can be written in terms of the extinction current given by eq. (13),

$$\frac{E^*}{\mathcal{N}(t)} = \sum_{i=1}^s \sum_{\mathbf{n}|n_i=0} j_i^*(\mathbf{n}, t) = \sum_{i=1}^s \sum_{\mathbf{n}|n_i=1} d_i(\mathbf{n}) \Phi^*(\mathbf{n}, t) \quad (17)$$

with $\mathcal{N}(t) = \sum_{n_1 \geq 1, \dots, n_s \geq 1} \Phi^*(\mathbf{n}, t)$.

For further analysis, let us approximate the Master equation (16) by a continuous diffusion equation, which is conveniently written for a population state depending on the scaled variables \mathbf{y} ,

$$\begin{aligned} \frac{\partial}{\partial t} \Phi(\mathbf{y}, t) &= \frac{1}{2n^*} \sum_{i=1}^s \frac{\partial^2}{\partial y_i^2} G_i(\mathbf{y}) \Phi(\mathbf{y}, t) \\ &\quad - \lambda \sum_{i=1}^s \frac{\partial}{\partial y_i} V_i(\mathbf{y}) \Phi(\mathbf{y}, t) \end{aligned} \quad (18)$$

with

$$G_i(\mathbf{y}) = y_i [b_i(\mathbf{y}) + d_i(\mathbf{y})], \quad (19)$$

$$\lambda V_i(\mathbf{y}) = y_i [b_i(\mathbf{y}) - d_i(\mathbf{y})]. \quad (20)$$

Biological populations may be sufficiently large so that their variation in relative size, which is of order

$$\langle (y - 1)^2 \rangle \sim 1/n^*, \quad (21)$$

can be neglected. The fluctuations in phenotype composition may still be significant, depending on the scaled game coupling constant λn^* . They are described by a projected diffusion equation for the phenotype state

$$\phi(\mathbf{x}, t) \equiv \int d\mathbf{y} \Phi(\mathbf{y}, t) \delta(\mathbf{x} - \mathbf{y}/y). \quad (22)$$

Long-term fluctuations are characterized by the leading eigenstate $\phi^*(\mathbf{x}, t)$ with support in the coexistence region $0 < x_i < 1$ and the corresponding eigenvalue $e^* < 0$. Removing the time-dependence by normalization defines a stationary phenotype probability distribution $p^*(\mathbf{x})$, the quantum Nash equilibrium. For game-based evolution, the diffusion coefficients and the drift fields have a non-trivial dependence on the phenotype composition \mathbf{x} , preventing a solution in closed form. Therefore, we will now turn to a specific example.

IV. THE QUANTUM HAWK-DOVE GAME

The hawk-dove game [3] is one of the simplest classical games. ‘Hawks’ ($i = 1$) and ‘doves’ ($i = 2$) are the two phenotypes of this game, and there is a single independent frequency variable $x \equiv x_1 = 1 - x_2$. The game is defined by the (suitably normalized) payoff matrix

$$\mathbf{A} = \begin{pmatrix} 1 - c & 2 \\ 0 & 1 \end{pmatrix} \quad (23)$$

with a constant $c > 1$, leading to a classical population dynamics of the form (3), (4),

$$\frac{dx}{dt} = \lambda x(1 - x)(1 - cx). \quad (24)$$

The unique stable fixed point is the Nash equilibrium $x^* = 1/c$.

The hawk-dove game dynamics has been used to explain instinctive aggression control in animal species. Individuals of the same species may fight for territory, mating partners, etc. ‘Hawks’ are aggressive individuals who escalate fights, while ‘doves’ avoid escalation and retreat. Hence, a hawk will always win against a dove ($A_{12} = 2, A_{21} = 0$), compared to the payoff of doves against doves ($A_{22} = 1$). However, hawk-hawk encounters involve a fitness decrease due to mutually inflicted injuries ($A_{11} = 1 - c$). In species with heavy weaponry ($c \gg 1$), fights are usually limited to ritual display of force and truly aggressive individuals are rare, as indicated by the Nash equilibrium $x^* \ll 1$. Needless to say, this is but the simplest approximation to the complexities of animal behavior.

The biological interpretation also suggests a decomposition (11) of the fitness into birth and death processes. Here we choose

$$\begin{aligned} b_i^{\text{game}}(\mathbf{x}) &= (\mathbf{B}\mathbf{x})_i - \mathbf{x}\mathbf{D}\mathbf{x}, \\ d_i^{\text{game}}(\mathbf{x}) &= (\mathbf{D}\mathbf{x})_i - \mathbf{x}\mathbf{B}\mathbf{x} \end{aligned} \quad (25)$$

with

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}. \quad (26)$$

The matrix \mathbf{B} describes game-related variations of the birth rates, while \mathbf{D} contains the deaths due to fights. The offset terms $\mathbf{x}\mathbf{B}\mathbf{x}$ and $\mathbf{x}\mathbf{D}\mathbf{x}$ required by eq. (4) are associated with the other type of process, respectively. The population size is assumed to be controlled by a bare birth rate and a variable death rate,

$$b^{\text{size}} = 1, \quad d^{\text{size}} = y. \quad (27)$$

Of course, this decomposition is not unique, and a different biological context may suggest a different choice. The qualitative results described below do not depend on these details.

Using the decomposition (25)–(27), we obtain the one-dimensional phenotype diffusion equation

$$\frac{\partial}{\partial t}\phi(x, t) = \frac{1}{2n^*} \frac{\partial^2}{\partial x^2} g(x)\phi(x, t) - \lambda \frac{\partial}{\partial x} v(x)\phi(x, t) \quad (28)$$

with

$$\begin{aligned} g(x) &= x(1-x)[(1-x)(b_1(x) + d_1(x)) \\ &\quad + x(b_2(x) + d_2(x))] \end{aligned} \quad (29)$$

and

$$v(x) = x(1-x)(1-x/x^*). \quad (30)$$

The quantum Nash equilibrium $p^*(x)$ and the eigenvalue e^* can be computed approximately. For $\lambda = 0$,

the solution is $p^*(x) = 1$ and $e^* = 1/2n^*$, describing the well-known case of neutral evolution. Corrections for $\lambda n^* \ll 1$ have the form of a power series in λn^* . For $\lambda n^* \gg 1$, obtain the approximate functional form of the solution by neglecting extinctions, i.e., by setting $e^* \approx 0$. The current in the Nash state, $j^*(x) \equiv (1/2n^*)(d/dx)(g(x)p^*(x)) - \lambda v(x)p^*(x)$, is also zero in this approximation, yielding

$$p^*(x) = p^*(x^*) \frac{g(x^*)}{g(x)} \exp \left[2\lambda n^* \int_{x^*}^x dx' \frac{v(x')}{g(x')} \right]. \quad (31)$$

For very large populations, $p^*(x)$ is seen to be a Gaussian of width $\sim 1/\sqrt{\lambda n^*}$ centered around the classical Nash equilibrium x^* . The approximation (31) is seen to be self-consistent for

$$x_{\min} \simeq \frac{1}{2n^*\lambda} \frac{g'(0)}{v'(0)} \lesssim x \lesssim x_{\max} \simeq 1 - \frac{1}{2n^*\lambda} \frac{g'(1)}{v'(1)}. \quad (32)$$

Of course, the current cannot be neglected close to the boundary, where we have $j^*(x) = (-1/2n^*)g'(0)\phi(0) + O(x)$ and $j^*(x) = (-1/2n^*)g'(1)\phi(1) + O(1-x)$. Matching the two regimes and using (17) gives an asymptotic expression for e^* . The ratio of the extinction currents takes the simple form

$$\frac{j^*(1)}{-j^*(0)} \simeq \exp \left[2\lambda n^* \int_{x_{\min}}^{x_{\max}} dx' \frac{v(x')}{g(x')} \right]. \quad (33)$$

Fig. 1 shows the quantum Nash state $p^*(x)$ obtained numerically for various values of the scaled game coupling λn^* . The solution of the full Master equation (16) is projected onto the frequency variable x using eq. (22). The crossover from neutral to game-dominated behavior is seen already at small equilibrium population sizes (here $n^* = 70$). The strong-coupling approximation (31), also shown in fig. 1, turns out to be an excellent approximation in the regime $\lambda n^* \gtrsim 1$, $x_{\min}(\lambda, n^*) \lesssim x \lesssim x_{\max}(\lambda, n^*)$.

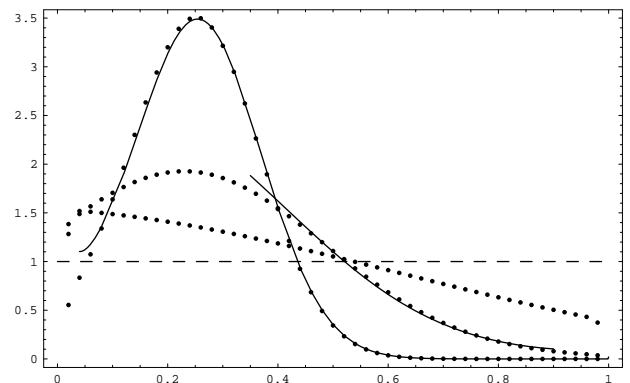


Fig. 1. The quantum Nash equilibrium $p^*(x)$ for $n^* = 70$ and $x^* = 0.28$. Numerical solution of the Master equation (16) for $\lambda = 1, 0.1, 0.01$ (dots), the limit $\lambda = 0$ of neutral evolution (dashed line), and the strong-coupling solution (31) for $x_{\min}(\lambda, n^*) \lesssim x \lesssim x_{\max}(\lambda, n^*)$ (solid lines).

A comprehensive analytical treatment of the quantum hawk-dove game is beyond the scope of this paper. Instead we now discuss properties of the solution which are valid beyond this specific example.

V. RESULTS AND DISCUSSION

The quantum Nash equilibrium $p^*(x)$ describes the likelihood of finding phenotype frequencies \mathbf{x} in long-term observations of biological sample populations. The fluctuations in phenotype composition depend on the scaled game coupling constant λn^* and may thus be significant even in large populations where the relative size fluctuations can be neglected. There are two modes of evolution.

In the *weak-coupling* regime $\lambda n^* \ll 1$, the system is dominated by stochastic forces. These produce a broad quantum Nash equilibrium, which implies large frequency variations,

$$\langle (\mathbf{x} - \langle \mathbf{x} \rangle)^2 \rangle \sim 1. \quad (34)$$

In the limit $\lambda = 0$, the evolution becomes strictly neutral.

In the *strong-coupling* regime $\lambda n^* \gg 1$, the game-related deterministic forces become relevant. These systems have only small frequency variations around a classical Nash equilibrium,

$$\langle (\mathbf{x} - \mathbf{x}^*)^2 \rangle \sim 1/\lambda n^*, \quad (35)$$

as shown by the quantum Nash state (31) for the hawk-dove game. Classical game theory is recovered in the limit $n^* \rightarrow \infty$.

At any finite λn^* , the mixed Nash equilibrium can be altered drastically by the extinction of phenotypes. The probability of extinction depends strongly on the phenotype considered. In the hawk-dove game, hawks face a much higher risk of extinction than doves, as indicated by the ratio of the extinction currents in the quantum Nash state given by (33). For $x^* < 1/2$, the current $j^*(0)$ is exponentially larger in magnitude than $j^*(1)$. This result also implies that an initially small hawk mutant population in a dove resident population is less likely to grow to its classical equilibrium frequency x^* than a dove mutant in a hawk resident population.

In a similar way, stochastic effects may influence the internal evolution of phenotypes even in the strong-coupling regime, where the overall frequencies are close to the classical Nash equilibrium \mathbf{x}^* . In the hawk-dove game, consider a mutant phenotype with a linkage disequilibrium such that it can invade only the hawk subpopulation. According to the standard theory of neutral evolution, the fixation probability of this mutant depends on the fitness difference to the resident hawks and on the effective hawk population size n_1^{eff} . It is easy to show that this may be much smaller than the effective population size n_2^{eff} for doves, as given by the ratio

$$\frac{n_1^{\text{eff}}}{n_2^{\text{eff}}} = \frac{x^* b_2(x^*) + d_2(x^*)}{1 - x^* b_1(x^*) + d_1(x^*)}. \quad (36)$$

Hence, hawks face a larger mutation load and a lesser chance of fixation for adaptive mutations than doves.

To summarize: In classical game theory, the basic strategies represented in a mixed Nash equilibrium \mathbf{x}^* are equivalent in the sense that their payoffs $(\mathbf{A}\mathbf{x})_i$ are all equal to the average payoff $\mathbf{x}\mathbf{A}\mathbf{x}$, see eq. (3). Stochastic effects break this equivalence and alter the equilibrium state. In biological systems, stochasticity is caused by frequency-dependent birth and death rates, that is, by the same mechanism that gives rise to frequency-dependent fitness values and underlies the application of classical game theory. Stochastic evolution creates a bias against phenotypes with larger fluctuations in their birth and death rates.

Quantum game theory opens a systematic way to quantify these stochastic effects. It contains Maynard-Smith's evolutionary game theory and Kimura's theory of neutral evolution as the limit cases of weak and strong stochasticity, respectively. We have introduced here the relevant concepts and applied them to the simplest kind of game. Obvious extensions include more complicated games such as bimatrix games.

Another avenue for future work is cooperative fluctuations in phenotype and *space*. In biological systems, population states $p(\mathbf{x}, \mathbf{r})$ now depend also on spatial coordinates, and stochastic game theory becomes a non-equilibrium quantum field theory. Population dynamical problems involving diffusion and migration have been treated by field-theoretic methods [10], and more recently the process of biological speciation has been identified as a coupled phase separation in phenotype and real space [11].

Finally, it is tempting to speculate about a different kind of randomness in game dynamics. The stochastic effects discussed so far originate from events random in time. However, we may also consider games with a payoff random for individual pairings of players but independent of time. This *quenched disorder* may affect Nash equilibria in new ways.

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