Handout – renormalisation of the quartic term in ϕ^4 theory

We start from

$$Z = \int \mathcal{D}\phi_s \, e^{-\mathcal{H}_s(\phi_s)} \int \mathcal{D}\phi_F \, e^{-\frac{1}{2}\int d\underline{x} \left((\nabla\phi_F)^2 + t\phi_F^2\right)} \Big[1 - \tfrac{g}{4!}\int d\underline{x}\, \phi_F^4(\underline{x}) + \ldots\Big]$$

$$\Big[1 - \tfrac{g}{12}\int d\underline{x} \left(2\phi_s^3\phi_F + 3\phi_s^2\phi_F^2 + 2\phi_s\phi_F^3\right)$$

$$+ \tfrac{g^2}{288}\int d\underline{x} \big(\underset{b)}{2\phi_s^3(\underline{x})\phi_F(\underline{x})} + \underset{a)}{3\phi_s^2(\underline{x})\phi_F^2(\underline{x})} + \underset{c)}{2\phi_s(\underline{x})\phi_F^3}\big) \int d\underline{y} \big(\underset{c)}{2\phi_s^3(\underline{y})\phi_F(\underline{y})} + \underset{a)}{3\phi_s^2(\underline{y})\phi_F^2(\underline{y})}$$

$$+ \underset{b)}{2\phi_s(\underline{y})\phi_F^3}\big)$$

$$+ O(g^3)\Big]$$

The term of order g^2 contains three terms which are quartic in ϕ_s and will thus contribute to a new $\int d\underline{x}\, \phi_s^4(x)$ term, namely

a) $\frac{g^2}{288}\, 1 \cdot 3^2 \int dx\, \phi_s^2(\underline{x})\, \phi_F^2(\underline{x}) \int d\underline{y}\, \phi_s^2(\underline{y})\, \phi_F^2(\underline{y})$

$$= g^2/32 \prod_{i=1}^{4} \int \frac{dk_i\, dq_i}{(2\pi)^{2d}} \cdot (2\pi)^d \delta(\underline{k}_1 + \underline{k}_2 + q_1 + q_2)(2\pi)^d (\underline{k}_3 + \underline{k}_4 + q_1 + q_2)$$

$$\phi_s(\underline{k}_1)\phi_s(\underline{k}_2)\phi_s(\underline{k}_3)\phi_s(k_4) \quad \phi_F(q_1)\phi_F(q_2)\phi_F(q_3)\phi_F(q_4)$$

Averaging over ϕ_F leads to two terms $\langle\phi_F(q_1)\phi_F(q_2)\rangle\langle\phi_F(q_3)\phi_F(q_4)\rangle + 2\langle\phi_F(q_1)\phi_F(q_3)\rangle\langle\phi_F(q_{\overset{2}{\nearrow}})\phi_F(q_{\overset{4}{\uparrow}})\rangle$

The second term is crucial and will be treated below. The first is the square of the term found in first order of g. This term is cancelled exactly by the second-order change to Z caused when we re-exponentiate the first order term ~~X~~3. (It turns out this cancellation holds to all orders of g; due to the so-called linked-cluster theorem only linked diagrams contribute to $\ln Z$, the others are generated by exponentiating the unlinked ones.)

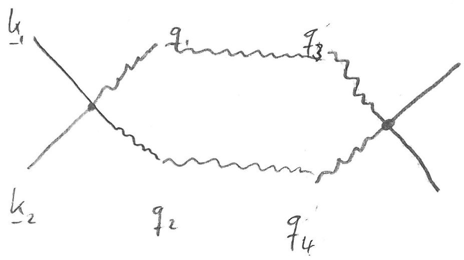
b) and c) each give the same term, taken together they yield

$$2\,\frac{g^2}{288}\, 2^2 \int d\underline{x}\, \phi_s^3(\underline{x})\, \phi_F(\underline{x}) \int d\underline{y}\, \phi_s(\underline{y})\, \phi_F^3(\underline{y})$$

It turns out this term does not contribute due to "momentum conservation", see last page.

From the second term in a) we thus have

$$\frac{g^2}{16}\prod_{i=1}^{4}\int_0^{\Lambda-\delta\Lambda}\frac{d\underline{k}_i}{(2\pi)^d}\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{dq_i}{(2\pi)^d}(2\pi)^d\delta\left(\underline{k}_1+\underline{k}_2+q_1+q_2\right)(2\pi)^d\delta\left(\underline{k}_3+\underline{k}_4+q_3+q_4\right)$$



$$\phi_s(\underline{k}_1)\phi_s(\underline{k}_2)\phi_s(\underline{k}_3)\phi_s(\underline{k}_4)$$

$$\underbrace{\langle\phi_F(q_1)\phi_F(q_3)\rangle}_{(2\pi)^d\delta(q_1+q_3)/q_1^2+t}\underbrace{\langle\phi_F(q_2)\phi_F(q_4)\rangle}_{(2\pi)^d\delta(q_2+q_4)/q_2^2+t}$$

(the factor of two comes from the alternative pairing of q_1 with q_4; q_2 with q_3)

The expectation values produce δ-functions setting $q_1+q_3=0$, $q_2+q_4=0$

With the other two δ-functions this gives

$$\underline{k}_1+\underline{k}_2+q_1+q_2=0 \qquad \Rightarrow \qquad \underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4=0$$

$$\underline{k}_3+\underline{k}_4-(q_1+q_2)=0 \qquad\qquad -q_2=\underline{k}_1+\underline{k}_2+q_1$$

leading to

$$\frac{g^2}{16}\prod_{i=1}^{4}\int_0^{\Lambda-\delta\Lambda}\frac{dk_i}{(2\pi)^d}(2\pi)^d\delta\left(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4\right)\phi_s(\underline{k}_1)\phi_s(\underline{k}_2)\phi_s(\underline{k}_3)\phi_s(\underline{k}_4)$$

$$\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{dq_1}{(2\pi)^d}\frac{1}{q_1^2+t}\frac{1}{(\underline{k}_1+\underline{k}_2+q_1)^2+t}$$

The last integral can be evaluated by an expansion around $q_1 \equiv q$

$$\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{dq}{(2\pi)^d}\frac{1}{q^2+t}\frac{1}{((\underline{k}_1+\underline{k}_2)+q)^2+t}=\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{dq}{(2\pi)^d}\frac{1}{(q^2+t)^2}\left[1-\frac{2q\cdot(\underline{k}_1+\underline{k}_2)+(\underline{k}_1+\underline{k}_2)^2}{q^2+t}+\dots\right]$$

Combining the terms containing k_1, k_2 with their field $\phi_s(k_1)$, $\phi_s(k_2)$ and performing inverse FT, the $O(\underline{k})$ terms generate terms like $\nabla(\phi_1+\phi_2)$ and higher orders of derivatives (terms whose coupling can in painstaking work be shown to be irrelevant).

The $\phi^4(x)$ term is generated by the $O(\underline{k}^0)$ term

$$\frac{g^2}{16} \prod_{i=1}^{4} \int_0^{\Lambda - \delta\Lambda} \frac{dk_i}{(2\pi)^d} (2\pi)^d \, \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4) \, \phi_s(\underline{k}_1) \phi_s(\underline{k}_2) \phi_s(\underline{k}_3) \phi_s(\underline{k}_4)$$

$$\int_{\Lambda - \delta\Lambda}^{\Lambda} \frac{dq}{2\pi} \frac{1}{(q^2 + t)^2}$$

Comparison with the ϕ^4-term in $\underline{k}$-space gives

$$g(\Lambda - \delta\Lambda) = g(\Lambda) - \frac{4!}{16} g^2(\Lambda) \int_{\Lambda - \delta\Lambda}^{\Lambda} \frac{dq}{(2\pi)^d} \frac{1}{(q^2+t)^2}$$

$$\frac{4!}{16} = \frac{3}{2}$$

$4!$ from coupling $g/4! = u$

Rescaling lengths and fields leads to a term $b^{4\left(\frac{d+2}{2}\right)} b^{-3d} = b^{-d+4} = b^{\epsilon}$

└ 4 integrals over $\underline{k}$ minus one from δ-function

└ rescaling 4 fields by $b^{\frac{d+2}{2}} = z$

caveat: we assumed the same scaling as for the Gaussian theory suffices to keep the $(\nabla\phi)^2$-term invariant. This is true to leading order in ϵ, at higher orders terms from the Taylor expansion on the previous page contribute.

$$g' = b^{\epsilon} g - \frac{3}{2} g^2 b^{\epsilon} \int_{\Lambda/b}^{\Lambda} \frac{dq}{(2\pi)^d} \frac{1}{(q^2+t)^2}$$

As before, $b = 1 + \delta b$ gives

$$\frac{dg}{db} = \epsilon g - \frac{3}{2} g^2 \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^d}{(\Lambda^2 + t)^2}$$

Defining $\lambda = \Lambda^{-\epsilon}$, $u = t/\Lambda^2$ yields $\quad (\Omega_{d=4} = 2\pi^2)$

$$\frac{d\lambda}{db} = \epsilon\lambda - \frac{3}{2} \lambda^2 \frac{1}{8\pi^2} \frac{1}{(1+u)^2}$$

We dropped the term b)+c) without any justification. So let's look at that

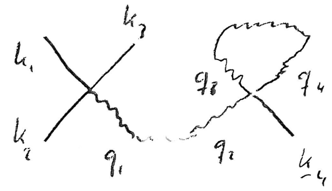
$$3\,\frac{g^2}{36}\prod_{i=1}^{4}\int\frac{dk_i\,dq_i}{(2\pi)^{2d}}\,(2\pi)^d\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+q_1)\,(2\pi)^d\delta(\underline{k}_4+q_2+q_3+q_4)$$

$$\frac{2+2^2}{288}=\frac{1}{36}$$

$$\phi_S(\underline{k}_1)\,\phi_S(\underline{k}_2)\,\phi_S(\underline{k}_3)\,\phi_S(\underline{k}_4)$$

$$\langle\phi_F(q_1)\,\phi_F(q_2)\rangle\,\langle\phi_F(q_3)\,\phi_F(q_4)\rangle$$

$$(2\pi)^d\delta(q_1+q_2)/_{q_1^2+t} \qquad (2\pi)^d\delta(q_3+q_4)/_{q_3^2+t}$$



The factor of three comes from equivalent terms

$$\underline{k}_1+\underline{k}_2+\underline{k}_3+q_1=0 \qquad k_4-q_1=0$$
$$\Rightarrow \underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4=0\,,\quad q_1=\underline{k}_4$$

Wavevector conservation and wavevector constraints of fast/slow fields make this term exactly zero: $q_3=-q_4$ and $\underline{k}_4+q_2+q_3+q_4=0$ imply $\underline{k}_4=-q_2$

However the integrals over $\underline{k}_4$ (slow) and q_2 (fast) have no overlapping support:

By definition $0\leq|\underline{k}_4|\leq\Lambda-\delta\Lambda$ and $\Lambda-\delta\Lambda<|q_2|\leq\Lambda$.

Hence the integral of q_2 over $\delta(q_2+\underline{k}_4)$ gives zero, and the entire diagram does not contribute.