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FINITE-SIZE EFFECTS IN THEORIES WITH FACTORIZABLE S-MATRICES

Michael LÄSSIG and Márcio J. MARTINS*

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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We study the energy spectrum of (1 + 1)-dimensional perturbed conformal field theories defined on the cylinder. The finite-size dependence of the two-particle levels allows a direct numerical measurement of the elastic S-matrix which we compare with the conjectured minimal S-matrix for several perturbations of minimal models. We discuss the simplifications that integrability imposes on the spectrum above threshold. In particular, the ultraviolet limit of the elastic phase shift of two lightest particles is related in a simple way to scaling dimensions of the conformal field theory.

1. Introduction

At a critical point, the universal long-distance behavior of a system is described by a scale-invariant field theory. Perturbing this theory by a generic combination of its relevant scaling fields introduces a finite correlation length ξ, and the resulting massive continuum field theory describes the universal off-critical behavior of the system. In two dimensions, often not only the critical theory can be solved exactly due to its infinite-dimensional conformal symmetry [1], but for some perturbations even the massive theory retains an infinite number of integrals of motion, and its mass spectrum and the factorizable S-matrix can be obtained exactly as well [2]. These encode in particular the thermodynamics of the system in the scaling region [3, 4]. For perturbations of minimal conformal theories, exact S-matrices and mass spectra have by now been predicted in many cases [2, 5–9].

The massive theory of the perturbed system also contains the massless critical theory as its ultraviolet limit, for distances much smaller than ξ. Using the thermodynamic Bethe ansatz, it is possible to reconstruct from the S-matrix the "effective central charge" of the ultraviolet conformal theory, i.e. the quantity $c - 12x_{\text{vac}}$, where c is the central charge and x_{vac} is the scaling dimension of the ground state [4, 10–12].

* On leave from Universidad Federal de São Carlos, São Carlos (S.P.)-13560, Brasil.

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The crossover from massless to massive behavior can be studied in the spectrum of the transfer matrix of the theory defined on a cylinder of finite circumference R. If R is much smaller than the correlation length ξ, it is predominantly the finite size of the system that destroys criticality, and the spectrum is determined by the ultraviolet critical theory [13]. In the thermodynamic limit, where R is much larger than ξ, the excitations become independent of the size of the system. This crossover can be described in the framework of conformal field theory. The conformal theory defines the basis of the Hilbert space and the unperturbed hamiltonian H_0, while the perturbation gives rise to an interaction term V. The matrix elements of H_0 and V between the conformal states are expressed in terms of the anomalous dimensions and the structure constants of the conformal theory [13]. The hamiltonian can be diagonalized numerically after truncating the conformal basis of the Hilbert space to a finite number of elements. Yurov and Al.B. Zamolodchikov proposed this "conformal truncation" method and applied it to the off-critical Yang–Lee model [14]; Lässig et al. studied the scaling region of the tricritical Ising model [15].

In this paper, we focus on the finite-size spectrum above the two-particle threshold. For states of two and more particles, the leading finite-size effect is the kinetic energy of the particles, which depends on R because the particle momenta are quantized. This dependence is modified by the elastic interactions between the particles, as expressed by the basic equation of the Bethe ansatz [3, 16]. Hence, as observed by Lüscher [17, 18], the multi-particle energy levels in a finite volume contain direct information about the elastic scattering matrix elements between those particles in the thermodynamic limit.

For integrable perturbations of minimal models, we can measure the elastic phase shifts in this way with remarkable accuracy over a wide range of momenta. In the cases we consider, we thus obtain a rather unambiguous confirmation of recently conjectured "minimal" S-matrices*. These include the Ising model in a magnetic field [2], the tricritical Ising model with leading thermal perturbation [6], and an integrable perturbation of the nonunitary model $M_{2,7}$ [7, 9, 20]. We discuss specifically the case of spontaneously broken $\mathbb{Z}_2$ symmetry, where the momentum quantization condition is altered.

For a generic theory, the finite-size spectrum above threshold is very difficult to disentangle. Integrability, however, imposes strong simplifications on the pattern of levels, which we discuss in detail. The eigenstates of the hamiltonian have a well-defined particle-content for any value of R, hence we can trace the behavior of each level in the conformal limit ($R \to 0$) where it is characterized by some scaling dimension x of the critical theory. The general features of this crossover are quite complicated. However, for the states of two lightest particles with zero

* We recall that the single-particle mass spectrum in the thermodynamic limit does not determine the S-matrix completely.

total momentum, the asymptotic behavior is given by the simple formula

$$x - x_{\text{vac}} = 2(\delta_\infty/\pi + n)$$

in terms of the ultraviolet limit δ_∞ of the corresponding two-particle phase shift (defined such that $\delta(0) = \pi/2$, see eq. (3.4) below). The quantum number n takes the values $n = 0, 1, 2\ldots$ or, in the case of spontaneously broken $\mathbb{Z}_2$ symmetry, the values $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Hence the elastic S-matrix of an integrable theory contains information about "effective scaling dimensions" $x - x_{\text{vac}}$ of its ultraviolet conformal theory.

This paper is organized as follows. In sect. 2, we recall generalities of finite-size scaling away from criticality and the conformal truncation method. In sect. 3, we discuss how multi-particle states in the finite volume are related to elastic scattering amplitudes and how integrability manifests itself in the spectrum. Sect. 4 contains numerical measurements of elastic phase shifts. In sect. 5, we consider the conformal limit of two-particle levels. The results are discussed in sect. 6.

2. Finite-size scaling away from criticality

Consider a massive euclidean quantum field theory in two dimensions which can be regarded as a conformal field theory perturbed by a relevant scaling field with angular momentum $\Delta - \bar{\Delta} = 0$ and scaling dimension $\Delta + \bar{\Delta} = x$. The euclidean action is

$$\mathscr{A}_\lambda = \mathscr{A}_0 + \lambda \int \Phi_{\Delta, \bar{\Delta}}(z, \bar{z})\, \mathrm{d}^2 z, \tag{2.1}$$

with a coupling constant λ of dimension $y = 2 - x > 0$. This theory can be defined on a cylinder of circumference R with complex coordinate $\omega = u + iv$, where v measures distances around the cylinder and u measures the euclidean time along the cylinder*. In these coordinates, the hamiltonian (the logarithm of the infinitesimal transfer matrix in u-direction) is

$$H_\lambda = H_0 + \lambda V. \tag{2.2}$$

The "unperturbed" part can be expressed in terms of the conformal operators L_0, $\bar{L}_0$, and the central charge c [13],

$$H_0 = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \tag{2.3}$$

* This is a preferred coordinate frame; Lorentz covariance emerges only in the limit $R \to \infty$.

while the interaction term V is given by

$$V = \int_0^R \Phi_{\Delta,\Delta}(\omega, \bar{\omega})\, \mathrm{d}v\,. \tag{2.4}$$

Since both H_0 and V commute with the momentum operator on the cylinder,

$$P = \frac{2\pi}{R}\left(L_0 - \bar{L}_0\right), \tag{2.5}$$

the dynamics factorizes into Hilbert space sectors of definite momentum $p = 2\pi n/R$, where n takes integer values for bosonic and half-integer values for fermionic sectors.

The eigenstates of H_0 ("conformal states") are assumed to form a basis of the Hilbert space. The matrix elements of V between these states are proportional to structure constants of the conformal theory [13],

$$\langle \Phi_j | V | \Phi_i \rangle = \left(\frac{2\pi}{R}\right)^{y} C_{\Phi_j \Phi \Phi_i}\,, \tag{2.6}$$

which in turn are given in terms of a finite number of primary structure constants [1]. The technicalities of the computation of these matrix elements are described in detail in ref. [21]. The conformal truncation method consists in reducing the Hilbert space to a finite number of conformal states; this is a viable way to find the off-critical spectrum by numerical matrix diagonalization [14].

In the finite geometry, the spectrum of the hamiltonian is quantized. The energy levels E_i ($i = 0, 1, 2, \dots$) have the scaling form

$$E_i(R, \lambda) = \frac{1}{R} f_i(\rho), \tag{2.7}$$

with the scaling variable $\rho = R/\xi$. The correlation length ξ is defined as the Compton wavelength of the lightest particle in the thermodynamic limit: $\xi = 1/m_1$. It is related to the coupling constant λ in eq. (2.2) by

$$\lambda = g \xi^{-y}\,. \tag{2.8}$$

If the massive system is integrable, the dimensionless coupling constant g can be calculated numerically with high accuracy by comparing the ground-state scaling function $f_0(\rho)$ obtained from the thermodynamic Bethe ansatz with conformal perturbation theory [4].

The asymptotic spectrum in the conformal regime ($\rho \ll 1$) is that of H_0,

$$E_i(R,\lambda) \simeq \frac{2\pi}{R}\left(x_i - \frac{c}{12}\right) \qquad (R \ll \xi), \tag{2.9}$$

where $x_i = \Delta_i + \bar{\Delta}_i$ (Δ_i and $\bar{\Delta}_i$ are eigenvalues of L_0 and $\bar{L}_0$, respectively).

In the thermodynamic regime ($\rho \gg 1$), an integrable theory is characterized by a set of stable particles A_a (labeled by the subscript a) with masses m_a and purely elastic interactions. Asymptotically, the energy levels,

$$E_i(R,\lambda) \simeq \frac{\epsilon_0}{\xi^2}R + M_i \qquad (R \gg \xi), \tag{2.10}$$

(where $M_i = \Sigma_a N_a m_a$ and N_a is the number of particles of type a in the state i) depend on R only through the ground-state term $E_0 = \epsilon_0 R/\xi^2$, which is infrared divergent in the limit $R \to \infty$. It can be computed exactly from the thermodynamic Bethe ansatz [4]*. The subtraction of this term,

$$\tilde{E}_i(R,\lambda) \equiv \frac{1}{R}\tilde{f}_i(\rho) = E_i(R,\lambda) - \frac{\epsilon_0}{\xi^2}R, \tag{2.11}$$

normalizes the partition function on a cylinder of length L,

$$Z(R,L,\lambda) = \sum_i \exp(LE_i(R,\lambda)), \tag{2.12}$$

to 1 in the thermodynamic limit $R, L \to \infty$. It is the subtracted ground-state scaling function $\tilde{f}_0$ that appears in the solution of the thermodynamic Bethe ansatz. For all massive perturbations of minimal models we studied numerically, this function appears to be monotonically increasing. Hence the function

$$\tilde{c}(\rho) \equiv -\frac{6}{\pi}\tilde{f}_0(\rho) \tag{2.13}$$

is a monotonically decreasing function interpolating between the limit values $\tilde{c}(0) = c - 12x_{\text{vac}}$ (the effective central charge of the conformal field theory), and $\tilde{c}(\infty) = 0$**. This is in agreement with the conjecture [10] that the effective central charge measures the degrees of freedom of a general conformal field theory, in the

* In most cases, the coefficient ϵ_0 is negative. The only counterexample known to us is the nonunitary model $M_{3,5}$ perturbed by the operator $\Phi_{1/5,1/5}$.

**An interesting series of examples are the nonunitary models $M_{2,q}$ ($q = 5,7,9,\dots$) perturbed by a primary field. These perturbations are super-relevant ($y > 2$) and hence the leading variation of $\tilde{c}(\rho)$ in the conformal limit is the infrared counterterm: $\tilde{c}(\rho) - \tilde{c}(0) = (6/\pi)\epsilon_0\rho^2 + o(\rho^2)$. This term is indeed negative since for these perturbations ϵ_0 is negative.

same way as the central charge c does for unitary theories. Unlike Zamolodchikov's c-function [22], however, the function (2.13) is not analytic in the coupling constant, because of the counterterm in eq. (2.11).

For the ground state and for zero-momentum single-particle states, the leading finite-size correction to the (subtracted) infinite-volume energy is due to virtual "interactions around the world" [17,23]. It decays exponentially:

$$\bar{E}_i(R,\lambda) - M_i \sim \exp(-b_i \rho) + \dots \tag{2.14}$$

for $i = 1, 2, \dots$, where the b_i are constants of order 1. For multi-particle states, the dominant correction is the kinetic energy of the particles, which depends algebraically on R in a way that is governed by the elastic interactions between them (see the detailed discussion in sect. 3).

At present, it is not known how to compute the exited levels $\bar{E}_i(R,\lambda)\,(i = 1, 2, \dots)$ from the thermodynamic Bethe ansatz. They may be calculated perturbatively in an expansion in λ about the conformal theory or in a large-volume expansion. In both cases, calculations are difficult beyond leading order. The conformal truncation method, however, gives a reliable spectrum in the entire region $0 \leqslant \rho \lesssim 20$. The levels obtained in this way become exact in the conformal limit; for large ρ they are distorted by truncation effects [15]. The low-lying masses can be extracted with a typical accuracy of about one percent, and the method is ideally suited to study the crossover region $\rho \sim 1$.

3. Multi-particle spectra in theories with factorizable scattering

3.1. MOMENTUM QUANTIZATION AND ELASTIC PHASE SHIFTS

Consider a system of noninteracting relativistic particles A_a on a cylinder of circumference R. Any eigenstate of the hamiltonian,

$$\Phi(R) = \left|(A_{a_1}, n_1), \dots, (A_{a_j}, n_j), \dots, (A_{a_N}, n_N)\right\rangle_R \tag{3.1}$$

is labeled by its particle content and the quantum numbers of the canonical momenta of the particles,

$$p_j = 2\pi n_j / R\,. \tag{3.2}$$

The n_j are integers for bosonic and half-integers for fermionic particles. The system is assumed to be Lorentz covariant in the infinite-volume limit. Hence in the thermodynamic regime, the energy of the states (3.1) should be given approximately by the relativistic dispersion relation

$$\bar{E}_{\Phi(R)} = \sum_{j=1}^{N} \sqrt{m_j^2 + k_j^2} \tag{3.3}$$

in terms of the kinetic momenta k_j, which equal the canonical momenta (3.2) for a free system. In particular, states of two bosons with total momentum $K = k_1 + k_2 = 0$ are labeled by a single momentum quantum number n which takes the values $n = 0, 1, 2, \dots$ if the particles are identical and the values $n = 0, \pm 1, \pm 2, \dots$ if they are different; we use the shorthand notation $|A_{a_1}, A_{a_2}; n\rangle$ for these states.

In $1+1$ dimensions, a theory with purely elastic short-ranged interactions is almost free. There is no particle production or annihilation, and the momenta of the particles are individually conserved in a scattering process. Thus the eigenstates of the hamiltonian (in the Heisenberg picture) can still be labeled as in eq. (3.1). At distances larger than the interaction length, the only effect on particle j of a scattering with particle l is a phase shift $2\delta_{a_j, a_l}$ between in- and out-wavefunction which depends only on the Lorentz-invariant difference of the particle velocities (defined by $\theta_j = \mathrm{Arsinh}(k_j/m_j)$) and is given in terms of the corresponding S-matrix element,

$$2\delta_{a_j, a_l}(\theta_j - \theta_l) = i \ln S_{a_j, a_l}(\theta_j - \theta_l)\,. \tag{3.4}$$

(The S-matrices considered in this paper satisfy $S_{a,a}(\theta = 0) = -1$ for the scattering of two identical particles, and the convention $\delta_{a,a}(\theta = 0) = \pi/2$ is convenient.) This phase-shift can be absorbed in a redefinition of the canonical momenta,

$$p_j = k_j - \frac{2}{R} \sum_{l=1}^{N} \delta_{a_j, a_l}(\theta_j - \theta_l)\,. \tag{3.5}$$

Eqs. (3.2), (3.3) and (3.5), which become exact in the thermodynamic limit, are the basis of the Bethe ansatz. It was observed [17, 18] by Lüscher and applied [19] by Lüscher and Wolff that they directly link the elastic two-particle phase shifts to the finite-size spectrum of multi-particle states. In fact, for a given state (3.1), these equations are just a parametric representation of the corresponding spectral line $\tilde{E}_{\Phi(R)}$. This is a good approximation in the thermodynamic regime, where off-mass-shell interactions around the world can be neglected. For $\rho \lesssim 1$, one expects significant deviations that are nonanalytic in ρ, from the conformal perturbation series. Surprisingly, at least for some two-particle states, we find that all corrections remain small, and eqs. (3.2), (3.3) and (3.5) give a good approximation to the levels for any value of ρ; this is discussed in sect. 5.

3.2. LEVEL CROSSINGS AND INTEGRABILITY

An integrable field theory is characterized by an infinite number of integrals of motion Q_s, labeled by a set of integers s (the first integral Q_1 is the hamiltonian itself). These integrals are a subset of the integrals of motion of the conformal theory describing it ultraviolet asymptotics. They lead to purely elastic interactions

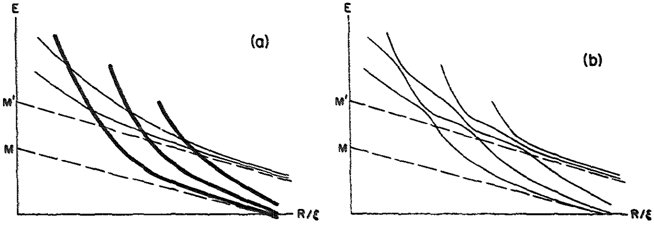


Fig. 1. (a) Crossing of multi-particle lines in an integrable system (schematic). All thick solid lines describe the same set of particles with combined infinite-volume mass $M = \sum_a N_a m_a$; the levels are distinguished by the momentum quantum numbers of the particles. All thin solid lines describe a set of particles with combined infinite-volume mass $M' = \sum_a N_a' m_a$. (b) A small perturbation of the system away from the integrable renormalization group trajectory shifts all lines by a small amount and removes all crossings. As a result, the ultraviolet limit of any line above threshold changes drastically. The infrared limit is now $2m_1$ for all such lines.

and the appearance of stable masses above the two-particle threshold $2m_1$. It has been suggested in ref. [14] that the existence of nontrivial integrals of motion (besides the hamiltonian) is connected to the crossings of levels observed in the finite-size spectrum. Indeed, in an integrable theory, there has to be an infinite number of level crossings, where the two levels that cross each other are distinguished by their particle content. To show this, consider a stable infinite-volume state Φ with energy M above threshold. In a finite volume, there is a state $\Phi(R)$ whose energy is shifted by a small amount. At a given $R_0 \gg \xi$, it is $\tilde{E}_{\Phi(R_0)} \equiv \tilde{E}_i(R) = M + \mathrm{O}(e^{-b_i r})$ for a single-particle state or $\tilde{E}_{\Phi(R_0)} \equiv \tilde{E}_i(R) = M + \mathrm{O}(1/MR^2)$ for a multi-particle state (the dependence of the levels on the coupling constant λ is suppressed in the notation of this section). At this value of R, there are $i-1$ levels with energy smaller than $\tilde{E}_{\Phi(R_0)}$. However, for sufficiently large R, any state $\Phi'(R) = |(A_{a_1}, n_1),(A_{a_2}, n_2),\ldots\rangle_R$ with combined infinite-volume mass $\sum_a N_a' m_a = M' < M$ has energy smaller than M. (In particular, all two-particle lines $|A_1, A_1; n\rangle$ accumulate at the threshold $2m_1$.) Hence any such level with $\tilde{E}_{\Phi'(R_0)} > \tilde{E}_{\Phi(R_0)}$ has to cross $\Phi(R)$ at some value $R' > R_0$. This is shown schematically in fig. 1a. There cannot be a finite gap between this pair of levels at $R = R'$ (as in fig. 1b) because this would imply a finite lifetime of the infinite-volume state Φ, by eqs. (3.2) and (3.5) [24].

At the crossing point, the states $\Phi(R')$ and $\Phi'(R')$ have the same energy, but they are distinguished by their different eigenvalues of the higher integrals of motion. Hence each continuous line of the crossover spectrum may be labeled by its asymptotic particle content and the momentum quantum numbers of the particles. This establishes a one-to-one correspondence between conformal states and the basis (3.1) of the Fock space. The lines interpolating between conformal

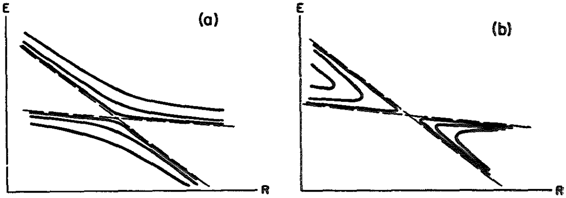


Fig. 2. Breaking of integrability by truncation of the Hilbert space (schematic). A line crossing in the infinite-dimensional Hilbert space (dashed lines) and its approximation for several levels of truncation (solid lines) (a) in the unitary case and (b) in the nonunitary case.

and thermodynamic limit follow a simple and beautiful pattern. The spectra shown in the next section (figs. 3, 5, 7, 8, 10, 13) have an abundance of level crossings, in some cases even multiple crossings (which we comment on in sect. 6). All of them involve states that differ in particle content; in a given sector of the Hilbert space, states of the same particles never cross.

A small perturbation of the system away from the integrable renormalization group trajectory distorts the spectrum slightly and in particular removes all crossings, because two generic lines, in the absence of any selection rules, cannot cross. For each level $\tilde{E}_i(R)$ above threshold, this drastically changes the conformal limit as well as the thermodynamic limit, which is now $2m_1$ for all lines (see fig. 1b). Any infinite-volume state Φ above threshold has a finite lifetime; therefore the finite-volume eigenstates $\Phi(R)$ cannot be assigned a definite particle content.

The truncation of the Hilbert space to a finite number of dimensions breaks the integrability of the system in a somewhat similar way. The charges Q_s commute in the infinite-dimensional Hilbert space. However, since the truncated space is not invariant under their action, the projections of the Q_s onto that subspace have a nonvanishing commutator. This commutator acting on a state in the truncated subspace involves the coupling of this state to the states in the orthogonal subspace; it is expected to vanish when the truncation threshold tends to infinity. Therefore in the truncated spectra the lines repel each other, with minimum gaps that decrease rapidly when the level of truncation is increased. This is shown in fig. 2a for a unitary theory and in fig. 2b for a nonunitary theory, where it was first observed in ref. [14].

4. Numerical measurements of the elastic phase shift

In this section, we use the spectrum obtained by the conformal truncation method to get, via eqs. (3.2), (3.3) and (3.5), a numerical prediction of the phase shift $\delta_{11}(\theta)$ for the elastic scattering of the two lightest particles. We compare this

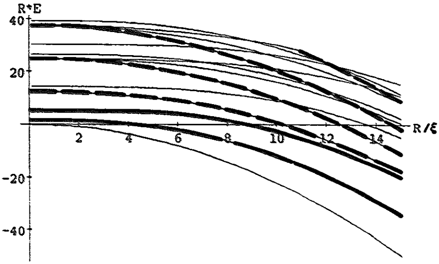


Fig. 3. The minimal model $M_{2,7}$ perturbed by the field $\Phi_{-2/7,-2/7}$ in the $K=0$ sector: the scaling function (2.7) for the lowest 16 levels. Identified lines: single-particle levels $|(A_1,0)\rangle$, $|(A_2,0)\rangle$ (thick solid lines), two-particle levels $|A_1, A_1; n\rangle$ for $n=0,1,2,3$ (long-dashed lines).

with the theoretical prediction from the conjectured "minimal" exact S-matrices for these models.

4.1. NONUNITARY MODEL $M_{2,7}$

Next to the Yang–Lee theory $M_{2,5}$, the nonunitary model $M_{2,7}$ is the second simplest conformal theory; its operator algebra does not possess any internal symmetries. It has central charge $c=-68/7$ and there are three primary scaling fields $\Phi_{-3/7,-3/7}, \Phi_{-2/7,-2/7}, 1$. On the cylinder, the ground state of the theory is $|\mathrm{vac}\rangle = |\Phi_{-3/7,-3/7}\rangle$, with energy $E_0(R) = (2\pi/R)(x_{\mathrm{vac}} - c/12) = (2\pi/R)(-1/21)$ as given by eq. (2.9). The perturbation of this theory by the field $\Phi_{-2/7,-2/7}$ yields a massive theory which, according to Smirnov [20], is related to the RSOS reduction of the Izerzin–Korepin model. He conjectured the minimal S-matrix for the scattering of two fundamental particles,

$$S_{11}(\theta) = \frac{\sinh\theta + i\sin(\pi/9)}{\sinh\theta - i\sin(\pi/9)} \frac{\sinh\theta + i\sin(2\pi/3)}{\sinh\theta - i\sin(2\pi/3)} \frac{\sinh\theta - i\sin(2\pi/9)}{\sinh\theta + i\sin(2\pi/9)}. \quad (4.1)$$

The theory has a bound state of mass $m_2 = 2\cos(\pi/18) = 1.9696$. Fig. 3 shows the scaling functions $f_i(\rho)$, as given by eq. (2.7), for the lowest 16 levels in the zero-momentum sector. One observes the single-particle states $|(A_1,0)\rangle$ and $|(A_2,0)\rangle$ and the four two-particles states $|A_1, A_1; n\rangle$ for $n=0,1,2,3$. In fig. 4, we plot the elastic phase shift obtained from the lowest two of these lines as a function of the scaled momentum $k\xi$ (where $k \equiv |k_1| = |k_2|$) in the massive relativistic region (i.e. for $k\xi$ of order 1) and compare with the theoretical phase shift given by eq. (4.1). We find good agreement for $k\xi \lesssim 1$. For larger values of

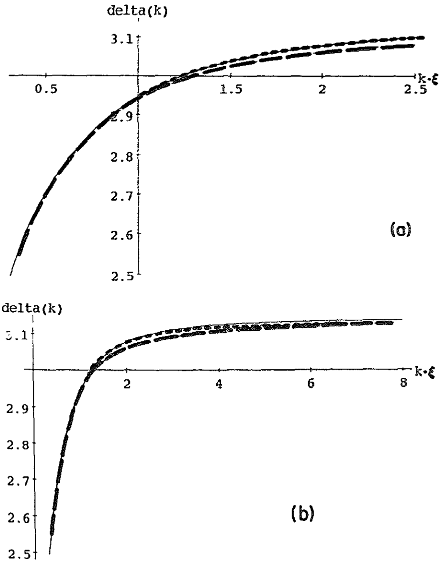


Fig. 4. Elastic scattering of two lightest particles in the minimal model $M_{2,7}$ perturbed by the field $\Phi_{-2/7,-2/7}$. Theoretical phase shift obtained from the minimal S-matrix $S_{11}(\theta)$ (solid line) and numerical data from the two-particle levels $|A_1, A_1; 0\rangle$ (long-dashed line) and $|A_1, A_1; 1\rangle$ (short-dashed line). (a) Massive relativistic region ($k\xi$ is of order 1). (b) Approach of the conformal limit.

$k\xi$, there are small deviations due to off-mass-shell corrections to the energies. These are much smaller for the second line ($n = 1$), because a given value of $k\xi$ corresponds to a larger value of ρ for this line. As fig. 4b shows, they decrease again in the ultra-relativistic region ($k\xi \gg 1$).

4.2. ISING MODEL

The magnetic perturbation $\Phi_{1/16,1/16}$ of the Ising model breaks the $\mathbb{Z}_2$ symmetry of the critical operator algebra and leads to a massive theory which is related to the Toda field theory based on the exceptional algebra e_8 [2]. The minimal S-matrix for the scattering of two lightest particles is

$$S_{11}(\theta) = \frac{\sinh\theta + i\sin(2\pi/5)}{\sinh\theta - i\sin(2\pi/5)}\,\frac{\sinh\theta + i\sin(2\pi/3)}{\sinh\theta - i\sin(2\pi/3)}\,\frac{\sinh\theta + i\sin(\pi/15)}{\sinh\theta - i\sin(\pi/15)}\,, \tag{4.2}$$

TABLE 1
The particles of the Ising model in a magnetic field

A_1	m_1	1
A_2	$m_2 = 2m_1\cos\left(\frac{\pi}{5}\right)$	1.6180
A_3	$m_3 = 2m_1\cos\left(\frac{\pi}{30}\right)$	1.9890
A_4	$m_4 = 4m_1\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{7\pi}{30}\right)$	2.4049
A_5	$m_5 = 4m_1\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{15}\right)$	2.9563
A_6	$m_6 = 4m_1\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{30}\right)$	3.2183
A_7	$m_7 = 8m_1\cos^2\left(\frac{\pi}{5}\right)\cos\left(\frac{7\pi}{30}\right)$	3.8911
A_8	$m_8 = 8m_1\cos^2\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{15}\right)$	4.7834

and the bootstrap closes with eight particles $A_1, A_2, \ldots, A_8$ whose masses are given in table 1 [2].

The first 5 one-particle states can be found in the spectrum of fig. 5, together with the lowest few two-particle levels $|A_1, A_1; n\rangle$ and $|A_1, A_2; n\rangle$. In fig. 6, we compare the elastic phase shift obtained from the lines $|A_1, A_1; n\rangle$ $(n = 0, 1)$ with the one predicted by eq. (4.2). Again, we find agreement in the region $k\xi \lesssim 1$, with small off-shell effects for large values of $k\xi$ that vanish again in the conformal limit.

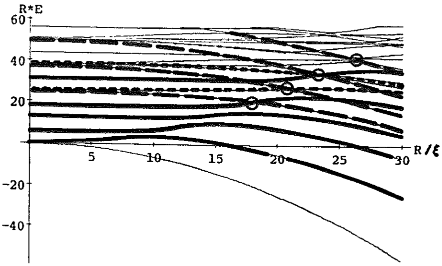


Fig. 5. The Ising model $M_{3,4}$ perturbed by a magnetic field, $K = 0$ sector: the scaling function (2.7) for the lowest 30 levels. Identified lines: single-particle levels $|(A_1,0)\rangle, |(A_2,0)\rangle, \ldots, |(A_5,0)\rangle$ (thick solid lines); two-particle levels $|A_1, A_1; n\rangle$ for $n = 0, 1, 2, 3$ (long-dashed lines), $|A_1, A_2; n\rangle$ for $n = 0, \pm 1$ (short-dashed lines). Focal points are marked by open circles.

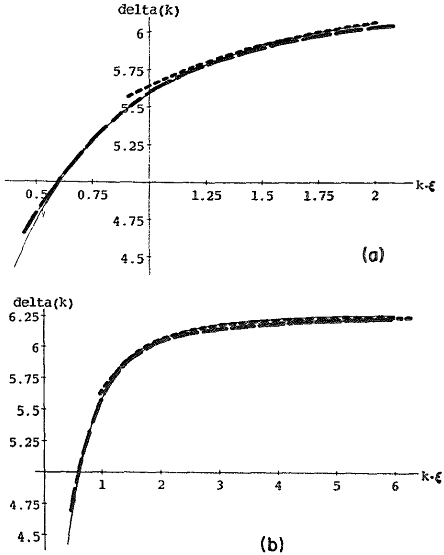


Fig. 6. Elastic scattering of two lightest particles in the Ising model perturbed by a magnetic field. Theoretical phase shift obtained from the minimal S-matrix $S_{11}(\theta)$ (solid line) and numerical data from the two-particle levels $|A_1, A_1; 0\rangle$ (long-dashed line) and $|A_1, A_1; 1\rangle$ (short-dashed line). (a) Massive relativistic region ($k\xi$ is of order 1). The deviation of the numerical data for small values of $k\xi$ marks the onset of truncation effects in the spectral lines for large ρ. (b) Approach of the conformal limit.

4.3. TRICRITICAL ISING MODEL

The operator algebra of the tricritical Ising model has the six primary scaling fields $\mathbf{1}$, $\Phi_{3/80,3/80}$, $\Phi_{1/10,1/10}$, $\Phi_{7/16,7/16}$, $\Phi_{6/10,6/10}$, $\Phi_{3/2,3/2}$. It is invariant under the $\mathbb{Z}_2$ spin-reversal transformation

$$\begin{aligned} Q:\quad & \Phi_{\frac{1}{10},\frac{1}{10}} \to \Phi_{\frac{1}{10},\frac{1}{10}}, \qquad \Phi_{\frac{6}{10},\frac{6}{10}} \to \Phi_{\frac{6}{10},\frac{6}{10}}, \qquad \Phi_{\frac{3}{2},\frac{3}{2}} \to \Phi_{\frac{3}{2},\frac{3}{2}}, \\ & \Phi_{\frac{3}{80},\frac{3}{80}} \to -\Phi_{\frac{3}{80},\frac{3}{80}}, \qquad \Phi_{\frac{7}{16},\frac{7}{16}} \to -\Phi_{\frac{7}{16},\frac{7}{16}}. \end{aligned} \tag{4.3}$$

The leading thermal perturbation $\Phi_{1/10,1/10}$ leads to a massive theory related to the Toda field theory based on e_7. The minimal S-matrix for two lightest particles is [6]

$$S_{11}(\theta) = -\frac{\sinh\theta + i\sin(\pi/9)}{\sinh\theta - i\sin(\pi/9)}\,\frac{\sinh\theta + i\sin(5\pi/9)}{\sinh\theta - i\sin(5\pi/9)}\,. \tag{4.4}$$

TABLE 2
The particles of the thermally perturbed tricritical Ising model and their $\mathbb{Z}_2$ symmetry

A_1	odd	m_1	1
A_2	even	$m_2 = 2m_1 \cos\left(\frac{5\pi}{18}\right)$	1.2856
A_3	odd	$m_3 = 2m_1 \cos\left(\frac{\pi}{9}\right)$	1.8794
A_4	even	$m_4 = 2m_1 \cos\left(\frac{\pi}{18}\right)$	1.9696
A_5	even	$m_5 = 4m_1 \cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)$	2.5321
A_6	odd	$m_6 = 4m_1 \cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{\pi}{9}\right)$	2.8794
A_7	even	$m_7 = 4m_1 \cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)$	3.7017

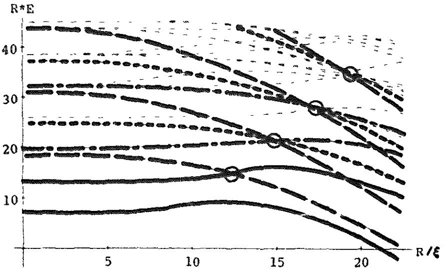
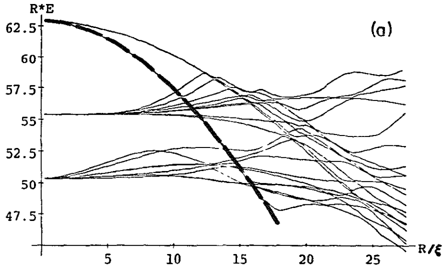


Fig. 7. Leading thermal perturbation of the tricritical Ising model, $K = 0$, even sector under spin reversal: the scaling function (2.7) for the levels 3 to 25. The spectrum in this sector is the same in the high-temperature ($\lambda > 0$) and low-temperature ($\lambda < 0$) phase. Identified lines: single-particle levels $|(A_4, 0)\rangle$, $|(A_5, 0)\rangle$ (solid lines); two-particle levels $|A_1, A_1; n\rangle$ for $n = 0, 1, 2, 3$ (long-dashed lines), $|A_2, A_2; n\rangle$ for $n = 0, 1, 2$ (short-dashed lines), $|A_1, A_3; n\rangle$ for $n = 0, \pm 1$ (dashed-dotted lines). Focal points are marked by open circles.

The $\mathbb{Z}_2$ symmetry of the critical theory is preserved under this perturbation and hence manifest in the massive Fock space. Table 2 lists the masses and $\mathbb{Z}_2$ symmetry of the seven particles $A_1, A_2, \ldots, A_7$ [6]. By duality, the spectrum in the even sector does not depend on the sign of λ*. In fig. 7, one recognizes the single-particle levels $|(A_4, 0)\rangle$ and $|(A_5, 0)\rangle$ (the lowest two levels $|0\rangle$ and $|(A_2, 0)\rangle$ are omitted), as well as two-particle levels $|A_1, A_1; n\rangle$, $|A_2, A_2; n\rangle$, $|A_1, A_3; n\rangle$.

* A detailed description of these spectra and their dependence on the boundary conditions can be found in ref. [15].



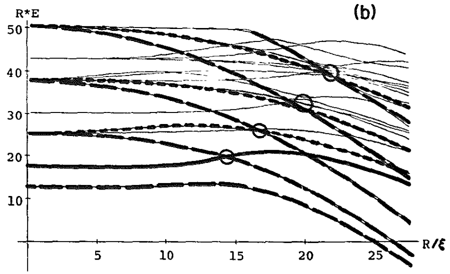


Fig. 8. Leading thermal perturbation of the tricritical Ising model, $K = 0$, odd sector under spin reversal: the scaling function (2.7) in the high-temperature phase ($\lambda > 0$) for the levels (a) 32 to 50 and (b) 3 to 31. Identified lines: single-particle level $|(A_6, 0)\rangle$ (solid line); two-particle levels $|A_1, A_2; n\rangle$ for $n = 0, \pm 1, \ldots, \pm 4$ (long-dashed lines), $|A_1, A_4; n\rangle$ for $n = 0, \pm 1, \pm 2$ (short-dashed lines). Focal points are marked by open circles.

The high-temperature spectrum in the odd sector is shown in fig. 8. There are the single-particle level $|(A_6,0)\rangle$ (the lowest two lines $|(A_1,0)\rangle$, $|(A_3,0)\rangle$ are omitted), and the two-particle levels $|A_1, A_2; n\rangle$ for $n = 0, \pm 1, \ldots, \pm 4$ and $|A_1, A_4; n\rangle$ for $n = 0, \pm 1, \pm 2$. In the low-temperature phase and for periodic boundary conditions on the cylinder, the levels in this sector become pairwise degenerate with the even levels as $\rho \to \infty$.

In the thermodynamic limit, the spin reversal symmetry is spontaneously broken. The two degenerate ground states,

$$|+\rangle, \qquad |-\rangle = Q|+\rangle, \tag{4.5}$$

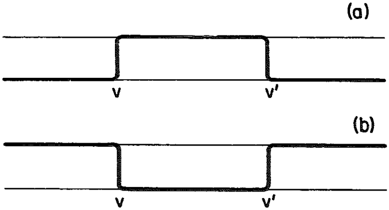


Fig. 9. Two-kink states in a low-temperature phase of spontaneously broken $\mathbb{Z}_2$ symmetry.

form a doublet under spin reversal. They can be constructed as linear combinations of the two lowest states in the finite volume (which are even and odd under Q, respectively) in the limit $R \to \infty$. The excited states can be constructed as linear combinations of states of definite symmetry as well, in particular the two-kink states

$$|K(v)\bar{K}(v')\rangle, \qquad |\bar{K}(v)K(v')\rangle = Q|K(v)\bar{K}(v')\rangle \tag{4.6}$$

shown in fig. 9. If in fig. 9a we move the kink at v' around the cylinder while keeping the kink at v fixed, we obtain the configuration of fig. 9b, and vice versa. Hence in a finite volume, the two-particle levels $|A_{a_1}, A_{a_2}; n\rangle$ are labeled by

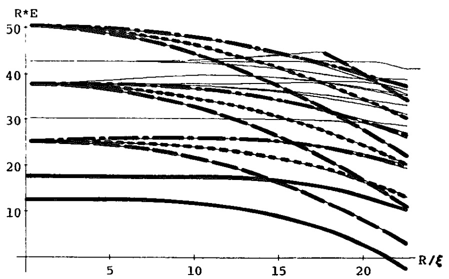


Fig. 10. Leading thermal perturbation of the tricritical Ising model, $K=0$, odd sector under spin reversal: the scaling function (2.7) in the low-temperature phase ($\lambda > 0$) for the levels 3 to 25. Identified lines: single-particle levels $|(A_4,0)\rangle$, $|(A_5,0)\rangle$ (thick solid lines); two-particle levels $|A_1, A_1; n\rangle$ for $n = 1/2, 3/2, 5/2, 7/2$ (long-dashed lines), $|A_2, A_2; n\rangle$ for $n = 1/2, 3/2, 5/2$ (short-dashed lines), $|A_1, A_3; n\rangle$ for $n = \pm 1/2, \pm 3/2, \pm 5/2$ (dashed–dotted lines).

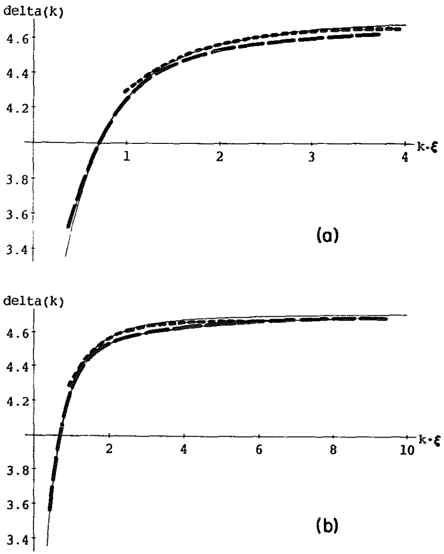


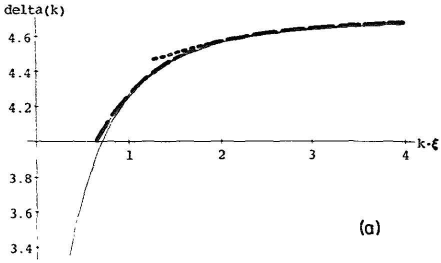
Fig. 11. Elastic scattering of two lightest particles in the tricritical Ising model with leading thermal perturbation. Theoretical phase shift obtained from the minimal S-matrix $S_{11}(\theta)$ (solid line) and numerical data from the two-particle levels $|A_1, A_1; 0\rangle$ (long-dashed line) and $|A_1, A_1; 1\rangle$ (short-dashed line) in the even sector under spin reversal. (a) Massive relativistic region ($k\xi$ is of order 1). The deviation of the dotted line for small values of $k\xi$ marks the onset of truncation effects in the corresponding spectral line for large ρ. (b) Approach of the conformal limit.

integer and half-integer momentum quantum numbers n. The integer levels come from the even sector (fig. 7), while the half-integer levels are part of the low-temperature spectrum in the odd sector, which is shown in fig. 10.

We compare the theoretical phase shift given by eq. (4.4) with the numerical results obtained from the two-particle levels $|A_1, A_1; n\rangle$ for $n = 0, 1$ in fig. 11 and for $n = \frac{1}{2}, \frac{3}{2}$ in fig. 12. Again, we find that off-mass-shell effects are small for all values of $k\xi$.

4.4. NONUNITARY MODEL $M_{3,5}$

The minimal model $M_{3,5}$ has four primary scaling fields $\Phi_{-1/20,-1/20}$, $\mathbf{1}$, $\Phi_{1/5,1/5}$, $\Phi_{3/4,3/4}$. It is the simplest nonunitary model whose operator algebra is invariant



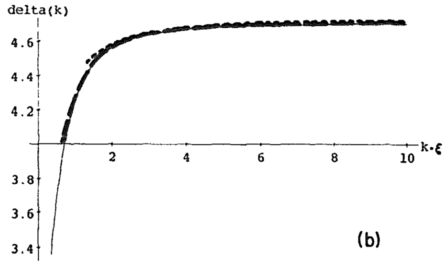


Fig. 12. Elastic scattering of two lightest particles in the tricritical Ising model with leading thermal perturbation. Theoretical phase shift obtained from the minimal S-matrix $S_{11}(\theta)$ (solid line) and numerical data from the two-particle levels $|A_1, A_1; 1/2\rangle$ (long-dashed line) and $|A_1, A_1; 3/2\rangle$ (short-dashed line) in the odd sector for $\lambda < 0$. (a) Massive relativistic region ($k\xi$ is of order 1). The deviation of the numerical data for small values of $k\xi$ marks the onset of truncation effects in the spectral lines for large ρ. (b) Approach of the conformal limit.

under a $\mathbb{Z}_2$ symmetry, namely

$$Q:\quad \Phi_{-\frac{1}{20},-\frac{1}{20}} \to -\Phi_{-\frac{1}{20},-\frac{1}{20}}, \qquad \Phi_{\frac{1}{5},\frac{1}{5}} \to \Phi_{\frac{1}{5},\frac{1}{5}}, \qquad \Phi_{\frac{3}{4},\frac{3}{4}} \to -\Phi_{\frac{3}{4},\frac{3}{4}}. \tag{4.7}$$

This symmetry can be broken spontaneously by perturbing with the field $\Phi_{1/5,1/5}$. Fig. 13a shows the lowest 12 levels $E_i(R)$ and fig. 13b the corresponding scaling functions $f_i(\rho)$ in this phase. This is another example of the quantization of the two-particle levels $|A_1, A_1; n\rangle$ in the case of spontaneously broken $\mathbb{Z}_2$ symmetry: the even levels have quantum numbers $n = 0, 1, 2, \ldots$ and the odd levels have $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The spectrum has no bound states below the two-particle threshold $2m_1$ [9]. The vacuum bulk energy density ϵ_0 is positive [12], in contrast to all the other examples.

5. The conformal limit

For an integrable theory, the set of equations (3.2), (3.3) and (3.5) describes the finite-size spectrum of all multi-particle levels in the thermodynamic region, where off-shell effects can be neglected. However, in all cases we have studied, we observe the surprising fact that for the lowest zero-momentum states $|A_1, A_1; n\rangle$, the off-shell corrections are small for all values of ρ and vanish for $\rho \to 0$ (see figs. 4b, 6b, 11b and 12b). This means that the S-matrix determines even the conformal limit of these levels. In this regime, the particles become ultra-relativistic, $\tilde{E}_1 = \tilde{E}_2 \simeq |k_1|$. Eqs. (3.2) and (3.5) then express the conformal limit of the scaling function (2.7) for the level $|A_1, A_1; n\rangle$ in terms of the phase shift $\delta_\infty = \lim_{\theta \to \infty} \delta_{11}(\theta)$,

$$\tilde{f}_n(0) = 2(2\delta_\infty + 2\pi n). \tag{5.1}$$

On the other hand, this limit of the scaling function is given by eq. (2.9) in terms of the scaling dimensions of the asymptotic conformal state and the ground state,

$$\tilde{f}_n(0) = 2\pi(x_n - x_{\mathrm{vac}}). \tag{5.2}$$

Thus, the ultraviolet limit of the elastic phase shift is related to effective scaling dimensions $x_n - x_{\mathrm{vac}}$ of the conformal theory,

$$x_n - x_{\mathrm{vac}} = 2(\delta_\infty/\pi + n). \tag{5.3}$$

If this holds true for all quantum numbers n, it follows that the levels of two lightest particles originate from a single conformal family in the cases where n takes integer values, and from two conformal families if n takes also half-integer values. The regularity of the observed spectra suggests that similar rules govern the ultraviolet limit of other multi-particle levels, but they need not be as simple.

The numerical spectra presented in the last section obey eq. (5.3). In the nonunitary model $M_{2,7}$, the ultraviolet limit of the phase shift given by eq. (4.1) is $\delta_\infty = \pi$, and eq. (5.3) then says that the two-particle line $|A_1, A_1; n\rangle$ originates from the conformal family of the ground state $|\Phi_{-3/7,-3/7}\rangle$ at level $n+1$; this can be seen in fig. 3 for $n = 0, 1, 2, 3$. The lowest levels $|A_1, A_2; n\rangle$ seem to originate from a single conformal family as well, but in general the situation is more complicated.

For the Ising model in a magnetic field, eq. (4.2) gives $\delta_\infty = 2\pi$; hence the line $|A_1, A_1; n\rangle$ should originate from the conformal family of the identity operator at level $n+2$. Fig. 5 shows this for $n = 0, 1, 2, 3$.

The thermally perturbed tricritical Ising model has $\delta_\infty = 3\pi/2$. When eq. (5.3) is applied to the two sectors of the Hilbert space separately, it predicts that the even two-particle levels $|A_1, A_1; n\rangle$ $(n = 0, 1, 2, \dots)$ originate from the conformal family $\Phi_{3/2,3/2}$ at level n, while the odd levels $|A_1, A_1; n\rangle$ $(n = 1/2, 3/2, \dots)$, that appear

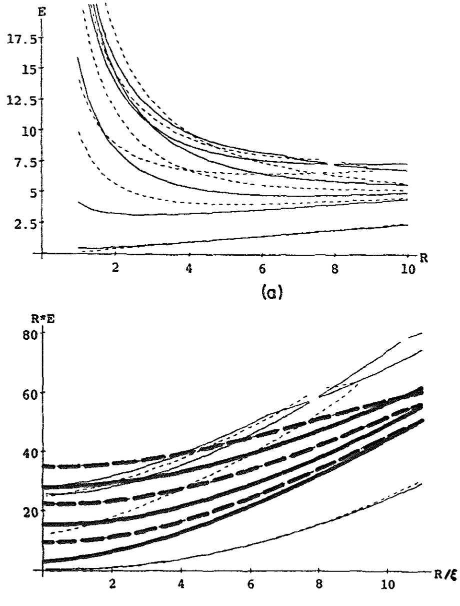


Fig. 13. The minimal model $M_{3,5}$ perturbed by the field $\Phi_{1/5,1/5}$ in the phase of spontaneously broken $\mathbb{Z}_2$ symmetry, $K=0$ sector. The coupling constant λ is chosen such that $\xi = 1$. (a) The lowest 6 even levels (solid lines) and the lowest six odd levels (dashed lines). For the higher levels, one observes "lacunae" indicating a pair of complex energy eigenvalues. (b) The scaling function (2.7) for the same levels. Identified lines: two-particle levels $|A_1, A_1; n\rangle$ for $n = 0, 1, 2$ (thick solid lines) and for $n = 1/2, 3/2, 5/2$ (long-dashed lines).

in the low-temperature phase only, originate from the family $\Phi_{3/80,3/80}$ at level $n + 3/2$ (see figs. 7 and 10, respectively)*.

In the model $M_{3,5}$, the ultraviolet limit of the phase shift [20] is $\delta_\infty = \pi/4$. Eq. (5.3) predicts the correct conformal limit of the two-particle levels if the rôles of the even and the odd ground state are exchanged: the even two-particle levels

* A similar case is the thermally perturbed Ising model, where $\delta_\infty = \pi/2$. Therefore the even two-particle levels $|A, A; n\rangle$ should originate from the family $\Phi_{1/2,1/2}$ at level n; the odd two-particle levels come from the family $\Phi_{1/16,1/16}$ at level $n + 1/2$.

$|A_1, A_1; n\rangle$ ($n = 0, 1, 2, \ldots$) in fig. 13 originate from the family $\Phi_{1/5,1/5}$ and satisfy $x_n - x_{\rm vac} = 2(1/4 + n)$, the odd levels $|A_1, A_1; n\rangle$ ($n = 1/2, 3/2, \ldots$) come from the family $\Phi_{3/4,3/4}$ and have $x_n - x_1 = x_n = 2(1/4 + n)$.

The asymptotic behavior of the two-particle states $|A_1, A_1; n\rangle$ has an interesting consequence for the correlation functions

$$G_\Phi(r, R, \lambda) = \langle \Phi(u, v) \Phi(u + r, v) \rangle \tag{5.4}$$

in the spectral representation

$$G_\Phi(r, R, \lambda) = \sum_i |F_i^2(R, \lambda)| \exp(-E_i(R, \lambda)) . \tag{5.5}$$

The form factors

$$F_i(R, \lambda) = \langle \mathbb{1} | \Phi | \Phi_i \rangle_{R, \lambda} \tag{5.6}$$

can be written in scaling form,

$$F_i(R, \lambda) = \left(\frac{2\pi}{R} \right)^x C_{\mathbb{1}\Phi\Phi_i}(\rho) , \tag{5.7}$$

where x is the scaling dimension of Φ. The scaling functions can be computed from the S-matrix in the thermodynamic limit [25], and they tend to the structure constants (2.6) in the conformal limit. Hence, in this limit, the field Φ decouples from the states of two lightest particles, unless it comes itself from the conformal family given by eq. (5.3).

6. Discussion

By analyzing the finite-size spectrum of the transfer matrix in the conformal truncation approximation, we have obtained, beyond the confirmation of the mass spectrum, a direct verification of the minimal elastic S-matrices for several integrable perturbations of minimal conformal theories.

The method of our analysis is, however, by no means restricted to integrable theories. An interesting physical system which is presumably not integrable and this method can be applied to is the tricritical Ising model in a magnetic field. This theory has three particles below threshold. Within the numerical accuracy, their masses equal those of the lowest three particles A_1, A_2, A_3 of the Ising model in a magnetic field [15, 26]. Thus the dynamics of the two systems are very similar at distance scales of the order of the correlation length, but this does not imply that the tricritical Ising model in a magnetic field is integrable as well [15]. Integrability requires fine-tuning of the lagrangian to a particular renormalization group trajectory. A small generic perturbation away from this trajectory destroys the factorizability of the S-matrix and forces the excitations above threshold to decay. Since the finite-size spectrum also contains information about the inelastic part of the S-matrix [24], this scenario can be tested: one can measure the lifetime of some of the e_8-resonances $A_4, A_5, \ldots, A_8$ which appear in the tricritical Ising model [27].

In an integrable theory, the additional conservation laws relate the thermodynamic limit and the conformal limit of the eigenstates of the transfer matrix. As an example which we hope can be generalized we have considered the states of two lightest particles. The thermodynamic limit of these states is the threshold $2m_1$; in the conformal limit, at least the lowest few of them originate from a single conformal family which is given by eq. (5.3) in terms of the ultraviolet limit of the S-matrix.

The multi-particle spectrum of an integrable theory has a high degree of order, imposed by the fact that in a given sector of the Hilbert space only levels that differ in particle content seem to cross each other. Much is yet to be learned from this pattern about the rôle that integrability plays for the universal behavior away from criticality. A second look at figs. 5, 7 and 8 reveals for instance that at least for the Ising model in a magnetic field and the tricritical Ising model perturbed by the thermal operator there is a series of "focal points" where in general more than two lines cross each other. All levels above threshold seem to go through one of these points. Even more surprisingly, the focal points lie very close to the parabola $-\epsilon_0\rho^2$ (which is the ground-state scaling function with reversed sign, up to exponentially small corrections).

To explain these observations, a better understanding of the representations of the higher symmetries away from criticality is needed. This has eluded us so far.

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