Upper Critical Dimension of the Kardar-Parisi-Zhang Equation

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The strong-coupling regime of Kardar-Parisi-Zhang surface growth driven by short-ranged noise is shown to have an upper critical dimension $d_>$ less than or equal to four [where the dynamic exponent z takes the value $z(d_>) = 2$]. To derive this, we use the mapping onto directed polymers with quenched disorder. Two such polymers coupled by a small contact attraction of strength u are shown to form a bound state at all temperatures $\beta^{-1} \leq \beta_c^{-1}$, the roughening temperature of a single polymer. Comparing singularities of the (de-)localization transition at $u = 0$ below β_c^{-1} and at β_c^{-1} then yields $d_> \leq 4$. [S0031-9007(97)02298-9]

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The Kardar-Parisi-Zhang equation [1] has been introduced as the simplest nonlinear evolution equation

$$\partial_t h(\mathbf{r},t) = \nu \nabla^2 h(\mathbf{r},t) + \frac{\lambda}{2}[\nabla h(\mathbf{r},t)]^2 + \eta(\mathbf{r},t) \quad (1)$$

for a continuous "height field" $h(\mathbf{r},t)$ driven by Gaussian noise $\eta(\mathbf{r},t)$ with $\langle \eta(\mathbf{r},t)\rangle = 0$ and $\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = \sigma^2\delta^d(\mathbf{r}-\mathbf{r}')\delta(t-t')$. It appears ubiquitously in a number of nonequilibrium statistical problems such as fluid dynamics or dissipative transport, as well as in systems with quenched disorder; for example, flux lines in a dirty type-II superconductor (see [2] for reviews).

The morphology of a growing surface governed by (1) is well known in low spatial dimensions. The asymptotic scaling of the spatiotemporal height correlations

$$\langle [h(\mathbf{r}_1,t_1) - h(\mathbf{r}_2,t_2)]^2\rangle \sim r^{2\chi} C(t/r^z), \quad (2)$$

(with $r \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ and $t \equiv |t_1 - t_2|$) defines the *roughness exponent* χ and the *dynamic exponent* z. In the Gaussian dynamics ($\lambda = 0$), one has $\chi = (2-d)/2$ and $z = 2$. For $\lambda \neq 0$, Galilei invariance imposes the relation $\chi + z = 2$ [3]. In dimensions $d \leq 2$, any small nonlinearity $(\lambda/2)(\nabla h)^2$ is a relevant perturbation of the Gaussian theory and leads to new values of the exponents ($\chi = \frac{1}{2}$, $z = \frac{3}{2}$ in $d = 1$ [4] and $\chi \approx 0.386$, $z \approx 1.612$ in $d = 2$ [5]). In the renormalization group, there is a crossover between the Gaussian fixed point, which is (infrared-)unstable, and the *strong-coupling* fixed point, which is stable. For $d > 2$, a Gaussian surface is smooth. A small nonlinearity $(\lambda/2)(\nabla h)^2$ does not alter this asymptotic scaling. There is now a roughening transition to the strong coupling regime at finite values $\pm\lambda_c$ [6–8]. In the renormalization group, the transition is represented by a third fixed point. This *critical* fixed point is unstable and appears between the Gaussian fixed point and the strong-coupling fixed point which are now both stable [9,10]. The critical point is characterized by the dimension-independent critical exponents $\chi^\star = 0$ and $z^\star = 2$ [11,12].

It is a notorious difficulty of the strong-coupling regime that above dimension one its properties are inaccessible to any known systematic approximation, let alone to an exact solution. In particular, renormalized perturbation theory with an ε expansion about $d = 2$ fails to produce a strong-coupling fixed point [12]. This feature, which the Kardar-Parisi-Zhang equation shares with other driven systems such as fully developed turbulence, is a major open problem in nonequilibrium statistical physics. In this Letter we discuss a (modest) step towards its eventual solution. We find that the Kardar-Parisi-Zhang equation has an *upper critical dimension* $d_> \leq 4$, where the exponents in the strong-coupling regime take the values $z = 2$ and $\chi = 0$, equal to those governing the limit $d \longrightarrow \infty$ [13,14] (and thus presumably the whole interval $d_> \leq d < \infty$). As $d_>$ is approached from below, the exponents tend to these values continuously. Hence the upper critical dimension could serve as the starting point for a controlled expansion. However, the reader should be cautioned that the name "upper critical dimension" may be misleading since $d_>$ does not mark the borderline to simple mean-field behavior as in the standard theory of critical phenomena, but instead to an even more complicated state in high dimensions with presumed glassy characteristics [14,15].

Even the existence of a finite upper critical dimension has been very controversial. Numerical work seems to indicate that a strong coupling phase with nontrivial exponents $z < 2$, $\chi > 0$ persists in dimensions $d = 3, 4, \ldots, 7$ [5]. However, since the available system sizes rapidly decrease with increasing d, it becomes difficult to distinguish the apparent power laws from logarithmic prefactors or other corrections to scaling that mask the true exponents. Various theoretical arguments favor the existence of a finite upper critical dimension $d_>$ but all of them rest on approximation schemes whose status is not very well understood. This is the case for functional renormalization group calculations [16], which are supported by a Flory-type argument [17], and for a

$1/d$ expansion based on a solution on the Cayley-tree [7] (criticized, however, in [3]). Recent work treating the Kardar-Parisi-Zhang dynamics in a mode-coupling approach gives contradictory results, predicting values of $d_>$ between 3 and 4 [15,18] or $d_> = \infty$ [19].

The arguments of this Letter are of a different kind since they are not tied to any approximation scheme for the strong-coupling regime. We use the known equivalence of the Kardar-Parisi-Zhang equation to a system of directed polymers and compare *exact* physical properties of this system in the strong-coupling phase and at the roughening transition. Individual numerical tests of these properties are possible and would be a highly desirable complement to the existing simulations of directed surface growth. The importance of $d = 4$ for the roughening critical point has already been stressed recently [12,20]. For $2 \leq d \leq 4$, this universality class is accessible by renormalized perturbation theory, which produces the exponents $\chi^\star = 0$ and $z^\star = 2$ exactly to all orders. At $d = 4$, however, there are singularities in some observables (for example, in the exponent $y^\star$ defined below).

Via the well-known Hopf-Cole transformation,

$$\exp\left[\frac{\lambda}{2\nu} h(\mathbf{r}_f, t_f)\right] = \int \mathcal{D}\mathbf{r}\, \delta[\mathbf{r}(t_f) - \mathbf{r}_f] \exp[-\beta_0 \mathcal{H}] \tag{3}$$

with the Hamiltonian

$$\mathcal{H} = \int_0^{t_f} dt \left[\frac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - \lambda \eta[\mathbf{r}(t), t]\right], \tag{4}$$

the Kardar-Parisi-Zhang equation can be mapped onto the equilibrium problem of a directed polymer $\mathbf{r}(t)$ living in the quenched random potential $\lambda\eta(\mathbf{r}, t)$ at temperature $\beta_0^{-1} = 2\nu$. The polymer is characterized by its transversal displacement

$$\overline{\langle[\mathbf{r}(t_1) - \mathbf{r}(t_2)]^2\rangle} \sim |t_1 - t_2|^{2\zeta} \tag{5}$$

and by free energy quantities like the "Casimir" term

$$\overline{f}_c(R) \equiv \lim_{T\longrightarrow\infty} \partial_T[\overline{F}(T, R) - \overline{F}(T, \infty)] \sim R^{(\omega-1)/\zeta} \tag{6}$$

in a system of longitudinal size T and transversal size R. (Averages over the disorder are denoted by overbars, thermal averages by brackets $\langle\cdots\rangle$.) The asymptotic scaling in (5) and (6) is related to the growth exponents by $\zeta = 1/z$ and $\omega = \chi/z$; the scaling relation due to Galilei invariance now reads $\omega = 2\zeta - 1$. In the strong-coupling regime for $d < d_>$, the polymer becomes superdiffusive ($\zeta > \frac{1}{2}$), and its free energy acquires an anomalous dimension $-\omega < 0$. These disorder-induced fluctuations persist in the limit $\beta^{-1} \longrightarrow 0$ [21], that is, in the ensemble of minimum energy paths $r_0(t)$. In the weak-coupling (high-temperature) regime for $d > 2$, thermal fluctuations dominate ($\zeta = \frac{1}{2}$) and hyperscaling is preserved ($\omega = 0$). The roughening transition between these two phases takes place at a finite temperature β_c^{-1}. At $d = d_>$ (and probably for $d \geq d_>$), the exponents $\zeta = \frac{1}{2}$ and $\omega = 0$ govern the low-temperature phase as well, albeit with possible logarithmic corrections.

In its mapping to directed polymers, it is a natural extension of the model to study an ensemble of lines $r_i(t)$ that live in the same random potential and are coupled by direct mutual forces [22–24]. The simplest case is that of just two lines with the interaction

$$\mathcal{H}_I = -u \int dt\, \Psi(t) \tag{7}$$

in terms of the local pair field $\Psi(t) \equiv \delta(\mathbf{r}_1(t) - \mathbf{r}_2(t))$. For $d > 2$, this interaction is irrelevant at the Gaussian fixed point governing the weak-coupling regime. However, for $d < d_>$, it turns out to be relevant both at the strong-coupling fixed point [24] and at the transition fixed point. Thus for all temperatures $\beta^{-1} \leq \beta_c^{-1}$, an attractive contact interaction ($u > 0$) leads to a bound state, suppressing the relative fluctuations of the lines on scales $|\mathbf{r}_1 - \mathbf{r}_2| \gtrsim \xi_\perp$. This defines the transversal localization length $\xi_\perp$. A set of similar scales $\xi_\perp^{(m)} (m = 1, 2, \ldots)$ characterizes the moments of the stationary pair distribution, see Eq. (11) below. For $u \longrightarrow 0$ one has the singularities

$$\begin{aligned} &\xi_\perp \sim u^{-\zeta/y} \\ &\xi_\perp^{(m)} \sim \beta^{-1/m} u^{-(\zeta-\omega/m)/y} \end{aligned} \quad \text{with} \quad y = 1 - \omega \tag{8}$$

in the low-temperature phase ($\beta^{-1} < \beta_c^{-1}$), and

$$\xi_\perp \sim \xi_\perp^{(m)} \sim u^{-1/2y^*} \quad \text{with} \quad y^* = (d-2)/2 \tag{9}$$

at the critical temperature β_c^{-1}. According to (8), $\xi_\perp^{(m)}$ increases with temperature for any value of u kept fixed. This is not surprising since the relative fluctuations of the lines are thermally activated. An immediate consequence is the inequalities $y/(\zeta - \omega/m) \geq 2y^*$. These in turn imply an upper bound on the free energy exponent $\omega \leq (4 - d)/d$, yielding our main result $d_> \leq 4$. We now discuss the arguments leading to Eqs. (8) and (9) above.

(a) Strong coupling phase.—At low temperatures and for $\omega > 0$, two noninteracting lines ($u = 0$) in the same random potential have a stationary pair distribution $P(|\mathbf{r}|) \equiv \lim_{t, T-t\longrightarrow\infty} \overline{\langle\delta[\mathbf{r}_1(t) - \mathbf{r}_2(t) - \mathbf{r}]\rangle}$ given by

$$P(r) = \beta^{-1} r^\theta \quad \text{with} \quad \theta = -d - \omega/\zeta \tag{10}$$

in a suitable normalization [24,25]. The nonintegrable short-distance singularity is cut off on scales $r \lesssim \tilde{\xi}_\perp$, where $\tilde{\xi}_\perp \sim \beta^{-\zeta/\omega}$ is the crossover length to the asymptotic strong-coupling behavior. Thus with finite probability, the two lines share a common "tube" of width $\tilde{\xi}_\perp$ along the minimum energy path $r_0(t)$ for a single line. For any fixed $\rho > 0$, the probability of finding the lines at a distance $r > \rho$ approaches zero for $\beta^{-1} \longrightarrow 0$, signaling the *uniqueness* [26] of the ground state. At any finite temperature, however, the lines make thermally activated individual excursions on all length scales from the tube $r_0(t)$. These excursions generate the power law distribution (10), all positive integer moments of which diverge.

Additional interactions between the lines probe the uniqueness of the ground state as well as the statistics of the low energy excursions [27]. For example, a mutual repulsion (7) with $u < 0$ forces one of the lines onto a distant excited path. (In superconductors this is important to stabilize a dilute ensemble of flux lines in the strong-coupling regime against collapse.) For the purpose of this Letter it is more useful to study a weak attractive interaction ($u > 0$), which localizes the two lines to each other. The normalized bound state distribution has the form $P(r, \xi_\perp) = N^{-1}(\xi_\perp)P(r)\mathcal{F}(r/\xi_\perp)$. On scales $r \gtrsim \xi_\perp$, it falls off exponentially as given by the scaling function $\mathcal{F}$, but in the scaling regime $r \ll \xi_\perp$, its dependence on $\xi_\perp$ originates only from the overall normalization $N(\xi_\perp) \equiv \int d^d r P(r)\mathcal{F}(r/\xi_\perp)$. The mth moment ($m = 1, 2, \ldots$) of $P(r, \xi_\perp)$ has the characteristic scale

$$\xi_\perp^{(m)} \equiv \left(\int d^d\mathbf{r}\, r^m P(r, \xi_\perp)\right)^{1/m} \sim \beta^{-1/m}\xi_\perp^{1-\omega/m\zeta} . \tag{11}$$

Pair interactions have been treated in Ref. [24] by renormalized perturbation theory for the Hamiltonian (7), based on the short-distance expansion $\Psi(t)\Psi(t') \sim \beta^{-1}|t - t'|^{-\omega}\Psi(t)$ at the strong-coupling fixed point of noninteracting lines ($u = 0$). For $u > 0$, one finds a bound state with the singularities (8). This is in agreement with results in $d = 1$ from numerical work [22,23] and from the dynamic renormalization of an extended Kardar-Parisi-Zhang equation [28].

It is instructive to cast the field-theoretic derivation of Ref. [24] into the form of a variational scaling argument: A weak bound state has a localization length that minimizes its free energy per unit of t, balancing the free energy gain $\delta\overline{f}_u$ from the overlap of the two lines with the loss $\delta\overline{f}_c$ due to the confinement of their relative fluctuations [29]. The overlap free energy is proportional to the stationary expectation value of the pair field

$$\delta\overline{f}_u = -u\overline{\langle\Psi\rangle} \sim -uN^{-1}(\xi_\perp) \sim -u[1 + O(\beta^{-1}\xi_\perp^{-\omega/\zeta})] . \tag{12}$$

The confinement energy has to vanish in the limit $\beta^{-1} \longrightarrow 0$, where all relative fluctuations of the lines are suppressed even without attractive forces. Therefore, instead of the temperature-independent Casimir energy $\xi_\perp^{(\omega-1)/\zeta}$ analogous to (6), one expects

$$\delta\overline{f}_c \sim \beta^{-1}\xi_\perp^{-1/\zeta} , \tag{13}$$

a term that is analytic in the scaling variable β^{-1} and respects hyperscaling [31]. The variation of $\delta\overline{f}_u + \delta\overline{f}_c$ with respect to $\xi_\perp$ then leads to (8).

(b) Roughening transition.—In contrast to the strong-coupling fixed point, the renormalization group for the critical fixed point in $2 < d < 4$ is well understood. The polymer partition function (3) has been shown to be *one-loop renormalizable* [12,20]. It is this property that produces the exact dimension-independent exponents $\chi^\star = 0$ and $z^\star = 2$; these values agree with previous results from numerical work in $d = 3$ [32], from dynamic renormalization group calculations to one-loop [9] and two-loop [10] order, and from scaling arguments [11]. At the critical point λ_c, small variations of λ are a relevant perturbation of dimension $y^\star = (d - 2)/2$ (with t as the basic scale) [12]. The local field conjugate to λ, $\Phi_\eta(t) \equiv \int d^d\mathbf{r}'\, \eta(\mathbf{r}', t)\delta^d(\mathbf{r}(t) - \mathbf{r}')$, encodes the random potential evaluated along the polymer paths. This field generates the crossover from the critical fixed point to the strong-coupling fixed point ($\lambda \longrightarrow \infty$) and to the Gaussian fixed point ($\lambda = 0$). It is now a purely formal matter to relate its disorder-averaged correlation functions to those of the pair field Ψ for $u = 0$ [24,33]. The simplest example is the stationary one-point function $\overline{\langle\Phi_\eta\rangle}(R)$ obtained from the free energy per unit of t in a system of transversal size R, $\overline{f}(R) = -\beta^{-1}\lim_{T\longrightarrow\infty}\partial_T\overline{\log \mathrm{Tr}\exp(-\beta\mathcal{H})}(T, R)$. With the disorder correlation and the pair interaction regularized on the microscopic scale a, $\overline{\eta(\mathbf{r}', t')\eta(\mathbf{r}'', t'')} = \sigma^2\delta_a(\mathbf{r}' - \mathbf{r}'')\delta(t' - t'')$ and $\Psi_a(t) \equiv \delta_a(\mathbf{r}_1(t) - \mathbf{r}_2(t))$, the universal parts [34] of $\overline{\langle\Phi_\eta(t)\rangle} = \overline{\langle\Phi_\eta\rangle}$ and $\overline{\langle\Psi_a\rangle}$ are seen to be proportional:

$$\begin{aligned}\lambda^{-1}\overline{\langle\Phi_\eta\rangle} &= -\lambda^{-1}\partial_\lambda\overline{f} = -\lambda^{-2}\sigma^2\int d^d\mathbf{r}'d^d\mathbf{r}''\delta_a(\mathbf{r}' - \mathbf{r}'')\overline{\frac{\delta^2 f}{\delta\eta(\mathbf{r}', t)\delta\eta(\mathbf{r}'', t)}} \\ &= \beta\sigma^2\int d^d\mathbf{r}'d^d\mathbf{r}''\delta_a(\mathbf{r}' - \mathbf{r}'')\overline{\langle\delta(\mathbf{r}(t) - \mathbf{r}')\delta(\mathbf{r}(t) - \mathbf{r}'')\rangle^c} \\ &= -\beta\sigma^2\int d^d\mathbf{r}'d^d\mathbf{r}''\delta_a(\mathbf{r}' - \mathbf{r}'')\overline{\langle\delta(\mathbf{r}(t) - \mathbf{r}')\rangle\langle\delta(\mathbf{r}(t) - \mathbf{r}'')\rangle} = -\beta\sigma^2\,\overline{\langle\Psi_a\rangle} .\end{aligned} \tag{14}$$

At $\lambda = \lambda_c$, the correlation functions of Φ_η are scale-invariant with the exponent $x^\star = 1 - y^\star = (4 - d)/2$. By (14), the same holds for the correlation functions of Ψ_a; for example,

$$\overline{\langle\Psi_a\rangle}(R, \lambda_c) \sim \overline{\langle\Phi_\eta\rangle}(R, \lambda_c) \sim R^{-x^\star/\zeta^\star} . \tag{15}$$

Hence the local pair interaction $\Psi_a(t)$ is like $\Phi_\eta(t)$ a relevant scaling field of dimension $x^\star$ at the critical fixed point. This result is consistent with the one-loop dynamic renormalization group discussed in Ref. [28]. It is then straightforward to show that the bound state generated by this interaction has the singularities (9) at β_c^{-1} [35]. Just below that temperature there is a crossover from (9) to (8) at the scale $u \sim \beta_c^{-1} - \beta^{-1}$.

The singularity of $\xi_\perp$ in (9) can again be obtained from a variational scaling argument. At β_c^{-1}, the competing free energy contributions analogous to (12) and (13) are the overlap term

$$\delta \overline{f}_u = -u\overline{\langle \Psi_a \rangle} \sim -u\xi_\perp^{-x^\star/\zeta^\star} = -u\xi_\perp^{d-4} \qquad (16)$$

following from (15) and a Casimir term of the form (6),

$$\delta \overline{f}_c \sim \xi_\perp^{(\omega^\star - 1)/\zeta^\star} = \xi_\perp^{-2} . \qquad (17)$$

It is tempting to speculate about the nature of the strong-coupling regime in high dimensions. Below $d_>$, the pair distribution of noninteracting lines at fixed temperature has the finite limit (10) for $R \longrightarrow \infty$, and this limit distribution collapses to $\delta(\rho)$ for $\beta^{-1} \longrightarrow 0$. For $d \geq d_>$, it is expected that the lines no longer cluster in the vicinity of the minimal path $r_0(t)$ but exploit multiple near-minimal paths at any finite temperature [24]. The pair distribution should then depend on β^{-1} and R in an essential way. Hence its asymptotic behavior will depend on the order in which the zero-temperature limit and the thermodynamic limit $R \longrightarrow \infty$ are taken. Similar properties are familiar from glassy systems.

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