– BCGS Intensive Week –
Tools from Quantum Information Theory
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The first set of exercises is supposed to be solved during the lecture. They are relatively easy and should not take much time. The second set of exercises is for the separate exercise session. Simply pick out the exercises you are most interested in. Exercises marked by a star $\star$ are more involved and give a perspective going beyond the immediate scope of the lecture.

In-Lecture Exercises

1. Majorana modes

1. Let $c_k^\dagger$ and c_k be the usual fermionic creation and annihilation operators. Derive the commutation relations for the Majorana modes a_k and b_k which are defined as

$$c_k \;=\; \tfrac{1}{2}(a_k + ib_k) \qquad \text{and} \qquad c_k^\dagger \;=\; \tfrac{1}{2}(a_k - ib_k) \quad . \tag{1}$$

2. The AKLT model

1. Verify that the projector onto $2 \subset \frac{1}{2} \otimes \frac{1}{2}$ is given by

$$P_2 \;=\; \tfrac{1}{2}\,\vec{S}_1 \cdot \vec{S}_2 + \tfrac{1}{6}\,(\vec{S}_1 \cdot \vec{S}_2)^2 + \tfrac{1}{3} \quad . \tag{2}$$

2. Consider an arbitrary Hamiltonian $H = \sum_k P_k$ which is a sum over hermitean projectors P_k (not necessarily orthogonal). Show that H is non-negative and that a zero-energy eigenstate needs to be annihilated by each P_k separately. This argument can be used for the discussion of the AKLT model.

3. Group cohomology

1. Derive the cocycle condition for the functions $\omega : G \times G \to U(1)$ that follows from the associativity of a projective representation $\rho : G \to U(1)$.

2. Verify that every function of the form

$$\alpha(g_1, g_2) \;=\; \frac{f(g_1)f(g_2)}{f(g_1 g_2)} \quad , \tag{3}$$

   where $f : G \to U(1)$, is a solution to the cocycle condition.

3. Let ρ be a projective representation with cocycle ω. Show that $\rho'(g) = f(g)\rho(g)$ defines a projective representation and derive its cocycle ω'.

4. Verify that the sets of cocycles and coboundaries have the structure of an abelian group.

Separate Exercises

4. The Majorana chain I (entanglement spectrum)

1. Verify the validity of the rewriting (if you don't believe it, otherwise do something more interesting)

$$H = \sum_k \Big\{ -w(c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k) - \mu(c_k^\dagger c_k - \tfrac{1}{2}) + (\Delta c_k c_{k+1} + \Delta c_{k+1}^\dagger c_k^\dagger) \Big\} \tag{4}$$

$$= \frac{i}{2} \sum_k \Big\{ -\mu a_k b_k + (\Delta + w) b_k a_{k+1} + (\Delta - w) a_k b_{k+1} \Big\} \quad . \tag{5}$$

Note that we assumed Δ to be real. A potential phase could be absorbed in a redefinition of the operators c and $c^\dagger$.

2. Let us consider the non-trivial phase associated with $\mu = 0$ and $\Delta = w$. In this exercise we are going to derive the entanglement spectrum of the groundstate.

   (a) Verify that the state

   $$|\psi\rangle = \frac{1}{\sqrt{2^{L-1}}}(1 + c_1^\dagger)(1 + c_2^\dagger) \cdots (1 + c_L^\dagger)|0\rangle \tag{6}$$

   is a groundstate of the open chain. [Hint: Check that the projectors $\tilde{P}_k$ have the desired action. Rewrite them in terms of the fermions $c, c^\dagger$ for this purpose.]

   (b) The state $|\psi\rangle$ is an equal weight superposition of all states available. We split it according to

   $$|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle \quad , \tag{7}$$

   where $|\psi_\pm\rangle$ are the projection onto even and odd fermion numbers. Show that $|\psi_-\rangle$ is the groundstate of the closed chain. [Hint: See previous hint.]

   (c) Calculate the reduced density matrix of the state $|\psi_-\rangle$ for an arbitrary segment by expressing it in terms of the states $|\psi_\pm^{A/B}\rangle$, the analogue of the states $|\psi_\pm\rangle$ but now for the two subsystems A and B.

   (d) Write down the Schmidt decomposition and show that the entanglement spectrum is two-fold degenerate.

   (e) How do groundstate, reduced density matrix and entanglement spectrum look like for the trivial chain with $\Delta = w = 0$ and $\mu < 0$? Again we can consider an arbitrary segment for the entanglement cut.

5. The Majorana chain II (Pfaffian)

1. The Hamiltonian of a general non-interacting superconducting system can be written in terms of Majorana fermions γ_k (with $k = 1, \ldots, 2L$) as

$$H = \frac{i}{4} \sum_{k,l} A_{kl} \gamma_k \gamma_l \qquad (\text{with } \bar{A}_{kl} = A_{kl} = -A_{lk}) \quad . \tag{8}$$

Assuming the uniqueness of the associated groundstate, its parity Π can be expressed as the sign of the Pfaffian [1],

$$\Pi \;=\; \operatorname{sign}\operatorname{Pf}(A) \quad , \tag{9}$$

where the Pfaffian is defined as

$$\operatorname{Pf}(A) \;=\; \frac{1}{2^L L!}\sum_{\sigma\in S_{2L}} \operatorname{sign}(\sigma) A_{\sigma(1)\sigma(2)}\cdots A_{\sigma(2L-1)\sigma(2L)} \quad . \tag{10}$$

(a) Determine the Pfaffian of a general anti-symmetric 2×2 and 4×4 matrix.

(b) Prove the relation

$$\operatorname{Pf}(WAW^T) \;=\; \det(W)\;\operatorname{Pf}(A) \quad . \tag{11}$$

(c) Using a transformation $A\mapsto WAW^T$, a general anti-symmetric $2L\times 2L$ matrix can be brought to the standard form

$$\Lambda_1\oplus\Lambda_2\oplus\cdots\oplus\Lambda_L \qquad \text{with} \qquad \Lambda \;=\; \begin{pmatrix} 0 & \lambda \\ -\lambda & 0\end{pmatrix} \quad . \tag{12}$$

Determine the Pfaffian of such a matrix.

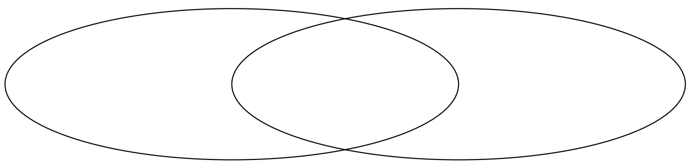
(d) Determine the Pfaffian of the two special Hamiltonians discussed in the lecture and verify the validity of the relation (for these two cases)

$$\Gamma \;=\; \frac{\Pi(L_1+L_2)}{\Pi(L_1)\Pi(L_2)} \quad . \tag{13}$$

6. The Majorana chain III (time reversal invariance)

1. Time reversal is an *anti-linear* operation which acts as $Tc_kT^{-1}=c_k$ and $Tc_k^\dagger T^{-1}$. Derive the transformation properties of the Majorana modes a_k and b_k.

2. Write down all possible bilinears in a_k and b_k (i.e. complex multiples of a_ka_l, b_kb_l and a_kb_l) and classify them according to whether they are hermitean or time-reversal invariant (or both).

Hermitean  **T-invariant**

3. In the lecture we have argued that a 1D superconductor with symmetry $\mathbb{Z}_2$ (fermionic parity) has two phases which are classified by $\mathbb{Z}_2$. A crucial question for the non-triviality of a phase is whether the edge modes can be gapped out or not (i.e. whether one can write down a term in the Hamiltonian which gives them a mass). Try to stagger Majorana chains and investigate whether enforcing time-reversal symmetry changes the topological classification.

4. [$\star$] [This exercise requires to know the projective representations of $\mathbb{Z}_2^T$, see one of the other exercises.] Consider an arbitrary 1D fermionic model with the following symmetries: Fermionic parity $\mathbb{Z}_2$ and time-reversal invariance $\mathbb{Z}_2^T$. Denote the generators by Q and T. Using the method of symmetry fractionalization, work out all possible topological phases. [Hint: There are eight distinct phases.]

7. Projective representations and central extensions

1. Work out the possible classes of projective representations for the following groups

   (a) $\mathbb{Z}_N \times \mathbb{Z}_N$

   (b) $\mathbb{Z}_M \times \mathbb{Z}_N$

2. Work out the possible classes of projective representations of the group $\mathbb{Z}_2^T$ generated by time-reversal T. Bear in mind that T is represented by an anti-unitary (hence anti-linear) operator. Show that there are only two classes corresponding to the freedom of choosing $\rho(T)^2 = \pm 1$.

3. Consider the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(3)$ of π-rotations R_x, R_y, R_z around the axes x, y, z. Show that the quantity

$$\rho(R_x)\rho(R_y)\rho(R_x)^{-1}\rho(R_y)^{-1} \in \mathbb{Z}_2 \tag{14}$$

   determines the class of the projective representation $\rho : \mathbb{Z}_2 \times \mathbb{Z}_2 \to U(1)$. Write down an arbitrary 2D and an arbitrary 3D representation (both non-trivial) and calculate the previous quantity. Briefly discuss the physical implications.

4. Consider a cocycle ω associated with a projective representation of a group G. Use ω to define a non-trivial multiplication on the set $U(1) \times G$, i.e. a non-trivial central extension H of G. Verify the group axioms. Argue that two central extensions obtained from equivalent cocycles (same class) are isomorphic.

5. [$\star$] Work out the cocycle conditions for a general group G involving unitary and anti-unitary elements. (Be careful about the order of factors!) Use that G comes with a homomorphism $\phi : G \to \mathbb{Z}_2$. Elements which map to 1 are unitary, the others anti-unitary. How should the associated coboundaries look like?

8. [$\star$] Majorana chain versus transverse field Ising model

1. Consider the non-local transformations

$$S_k^x = (c_k + c_k^\dagger)\prod_{l<k}(1 - 2c_k^\dagger c_k) \tag{15}$$

$$S_k^y = -i(c_k - c_k^\dagger)\prod_{l<k}(1 - 2c_k^\dagger c_k) \tag{16}$$

$$S_k^z = 1 - 2c_k^\dagger c_k \tag{17}$$

   and derive what the Hamiltonian of Kitaev's Majorana chain is mapped to. Start with the two special choices discussed in the lecture. Discuss the fate of the topologically trivial and the topologically non-trivial phase under this map.

Further reading

Unfortunately, there is no single reference which will cover all the topics of this lecture course. Some of the relevant original articles are

- Majorana chains
  - The discussion of Majorana edge modes in the non-interacting chain [1]
  - Classification using symmetry fractionalization [2, 3]
- AKLT model
  - Original construction and discussion [4]
  - Discussion of the entanglement spectrum [5]
- Matrix product states [6]
- For the classification of topological phases using group cohomology [8, 7, 9, 10, 11]
- Tools related to entanglement
  - Entanglement entropy scaling in 1D critical systems [12]
  - Topological entanglement entropy in 2D gapped systems [13]
  - Entanglement spectra in 2D fractional quantum Hall systems [14]

References

[1] A. Y. Kitaev, "Unpaired Majorana fermions in quantum wires," *Physics Uspekhi 44* (2001) 131, arXiv:cond-mat/0010440.

[2] L. Fidkowski and A. Kitaev, "Topological phases of fermions in one dimension," *Phys. Rev. B83* (2011) 075103, arXiv:1008.4138.

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[13] A. Kitaev and J. Preskill, "Topological Entanglement Entropy," *Physical Review Letters 96* (2006) 110404, hep-th/0510092.

[14] H. Li and F. D. M. Haldane, "Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States," *Physical Review Letters 101* (2008) 010504, arXiv:0805.0332.

Solutions

1. The Majorana chain

1. The relations for c_k and $c_k^\dagger$ may easily be inverted to read

$$a_k \;=\; c_k + c_k^\dagger \qquad \text{and} \qquad b_k \;=\; -i(c_k - c_k^\dagger) \quad . \tag{18}$$

   One then easily finds

$$\{a_k, a_l\} \;=\; \{b_k, b_l\} \;=\; 2\delta_{kl} \qquad \text{and} \qquad \{a_k, b_l\} \;=\; 0 \quad . \tag{19}$$

   These relations are known to define a Clifford algebra.

2. The AKLT model

1. We have the decomposition $\frac{1}{2} \otimes \frac{1}{2} \;=\; 0 \oplus 1 \oplus 2$. The individual components can be distinguished by the eigenvalues $C_j = j(j+1)$ of the quadratic Casimir operator

$$C \;=\; \vec{S}^2 \;=\; (\vec{S}_1 + \vec{S}_2)^2 \;=\; \vec{S}_1^2 + \vec{S}_2^2 + 2\,\vec{S}_1 \cdot \vec{S}_2 \;=\; 4 + 2\,\vec{S}_1 \cdot \vec{S}_2 \quad . \tag{20}$$

   Here we used $\vec{S}_1^2 = \vec{S}_2^2 = 2$ (the physical sites have $j = 1$). By evaluation on the individual components of the tensor product one can verify that the projector is given by

$$P_2 \;=\; \frac{(C - C_0)(C - C_1)}{(C_2 - C_0)(C_2 - C_1)} \quad . \tag{21}$$

   The result follows immediately by simplification of this expression.

2. An operator H is non-negative if all its expectation values $\langle\psi|H|\psi\rangle$ are non-negative. The projectors satify $P_k^2 = P_k = P_k^\dagger$. In the present case one therefor has

$$\langle\psi|H|\psi\rangle \;=\; \sum_k \langle\psi|P_k|\psi\rangle \;=\; \sum_k \langle\psi|P_k^2|\psi\rangle \;=\; \sum_k \langle\psi|P_k^\dagger P_k|\psi\rangle \;=\; \sum_k ||\,P_k|\psi\rangle\,||^2 \quad . \tag{22}$$

   This proves both assertions.

3. Group cohomology

1. Using the properties of a projective representation one has

$$\big[\rho(g_1)\rho(g_2)\big]\rho(g_3) \;=\; \rho(g_1)\big[\rho(g_2)\rho(g_3)\big] \tag{23}$$

$$\omega(g_1, g_2)\,\rho(g_1g_2)\rho(g_3) \;=\; \omega(g_2, g_3)\,\rho(g_1)\,\rho(g_2g_3) \tag{24}$$

$$\omega(g_1, g_2)\omega(g_1g_2, g_3)\,\rho(g_1g_2g_3) \;=\; \omega(g_2, g_3)\omega(g_1, g_2g_3)\,\rho(g_1g_2g_3) \quad . \tag{25}$$

   The comparison of both sides yields

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) \;=\; \omega(g_2, g_3)\omega(g_1, g_2g_3) \quad . \tag{26}$$

2. Plugging the given expression into the cocycle condition one finds

$$\text{lhs} = \frac{f(g_1)f(g_2)}{f(g_1g_2)}\frac{f(g_1g_2)f(g_3)}{f(g_1g_2g_3)} \tag{27}$$

$$\text{rhs} = \frac{f(g_2)f(g_3)}{f(g_2g_3)}\frac{f(g_1)f(g_2g_3)}{f(g_1g_2g_3)} \quad . \tag{28}$$

   Obviously both sides agree.

3. We calculate

$$\rho'(g_1)\rho'(g_2) = f(g_1)f(g_2)\,\rho(g_1)\rho(g_2) = f(g_1)f(g_2)\,\omega(g_1,g_2)\,\rho(g_1g_2) \tag{29}$$

$$= \frac{f(g_1)f(g_2)}{f(g_1g_2)}\,\omega(g_1,g_2)\,\rho'(g_1g_2) \quad . \tag{30}$$

   The cocycle ω' is therefor given by

$$\omega'(g_1,g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)}\,\omega(g_1,g_2) \quad . \tag{31}$$

   In other words: The two cocycles are related by a coboundary.

4. As functions with values in $U(1)$ the respective group structure is inherited from that of $U(1)$. All one needs to check is whether the corresponding sets close under multiplication. But this is rather obvious. Remark: Both cases can be related to tensor products $\rho_1 \otimes \rho_2$ of projective representations.

4. The Majorana chain I (entanglement spectrum)

1. This is straightforward and will not be stated explicitly.

2. Entanglement spectrum of the groundstate.

   (a) We have to verify that all operators

$$\tilde{P}_k = -ib_ka_{k+1} = -(c_k - c_k^\dagger)(c_{k+1} + c_{k+1}^\dagger) \tag{32}$$

   act as the identity element. For this we only need to check

$$\tilde{P}_k(1+c_k^\dagger)(1+c_{k+1}^\dagger)|0\rangle = -(c_k - c_k^\dagger)(c_{k+1} + c_{k+1}^\dagger)(1+c_k^\dagger)(1+c_{k+1}^\dagger)|0\rangle \tag{33}$$

$$= -(c_k - c_k^\dagger)(1-c_k^\dagger)(c_{k+1} + c_{k+1}^\dagger)(1+c_{k+1}^\dagger)|0\rangle \tag{34}$$

$$= -(c_k - c_kc_k^\dagger - c_k^\dagger)(c_{k+1} + c_{k+1}c_{k+1}^\dagger + c_{k+1}^\dagger)|0\rangle \tag{35}$$

$$= (1+c_k^\dagger)(1+c_{k+1}^\dagger)|0\rangle \quad . \tag{36}$$

   Note indeed that $\tilde{P}_k$ is bosonic and can be moved through all operators c_l and $c_l^\dagger$ with $l < k$ without problems. Also, additional operators with $l > k+1$ in front of $|0\rangle$ do not alter the conclusion.

(b) In addition to the previous relations we now also have to demand that $\tilde{P}_L = -ib_L a_1$ acts trivially. But since the total fermionic parity is given by

$$Q \;=\; (-i)^L a_1 b_1 \cdots a_L b_L \;=\; -\tilde{P}_1 \cdots \tilde{P}_L \tag{37}$$

this implies that Q acts as minus the identity.

(c) First we note that $|\psi_-\rangle$ has unit norm due to the special convention used in Eq. (6). Since the state $|\psi_-\rangle$ has odd fermion parity one has

$$|\psi_-\rangle \;=\; \tfrac{1}{\sqrt{2}}\Big[|\psi_-^A\rangle \otimes |\psi_+^B\rangle + |\psi_+^A\rangle \otimes |\psi_-^B\rangle\Big] \quad . \tag{38}$$

By writing out the density matrix one has

$$\rho \;=\; \tfrac{1}{2}\Big[|\psi_-^A\rangle|\psi_+^B\rangle\langle\psi_-^A|\langle\psi_+^B| + |\psi_+^A\rangle|\psi_-^B\rangle\langle\psi_-^A|\langle\psi_+^B| \tag{39}$$

$$+ |\psi_-^A\rangle|\psi_+^B\rangle\langle\psi_+^A|\langle\psi_-^B| + |\psi_+^A\rangle|\psi_-^B\rangle\langle\psi_+^A|\langle\psi_-^B|\Big] \quad . \tag{40}$$

Since states with different fermion number are orthogonal one immediately finds the reduced density matrix

$$\rho_A \;=\; \tfrac{1}{2}\,|\psi_-^A\rangle\langle\psi_-^A| + \tfrac{1}{2}\,|\psi_+^A\rangle\langle\psi_+^A| \;=\; e^{-\ln 2}(|\psi_-^A\rangle\langle\psi_-^A| + |\psi_+^A\rangle\langle\psi_+^A|) \tag{41}$$

Since both states appearing here are orthogonal, this provides a Schmidt decomposition with equal weight (the normalization can be checked by taking the trace which needs to equal 1). We have thus proven the degeneracy of the entanglement spectrum $\epsilon_1 = \epsilon_2 = \ln 2$.

(d) See previous statement.

(e) For the trivial chain the groundstate is $|0\rangle$ which is a product state. The reduced density matrix is $\rho_A = |0^A\rangle\langle 0^A|$ where $|0^A\rangle$ is the vacuum in the reduced chain. For this reason there is a single entanglement eigenvalue $\epsilon_1 = 0$.

5. The Majorana chain II (Pfaffian)

1. Pfaffians of small rank anti-symmetric matrices are given by

$$\mathrm{Pf}\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \;=\; \lambda \;, \qquad \mathrm{Pf}\begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & 0 & \lambda_4 & \lambda_5 \\ -\lambda_2 & -\lambda_4 & 0 & \lambda_6 \\ -\lambda_3 & -\lambda_5 & -\lambda_6 & 0 \end{pmatrix} \;=\; \lambda_1\lambda_6 - \lambda_2\lambda_5 + \lambda_3\lambda_4 \quad . \tag{42}$$

2. We start with the product

$$(WAW^T)_{kl} \;=\; \sum_{m,n} W_{km} A_{mn} W_{ln} \quad . \tag{43}$$

Plugging this into the definition of the Pfaffian we find

$$\mathrm{Pf}(WAW^T) = \frac{1}{2^L L!}\sum_{\sigma\in S_{2L}}\mathrm{sign}(\sigma)\sum_{\{i_k\}}W_{\sigma(1)i_1}W_{\sigma(2)i_2}A_{i_1i_2}\cdots W_{\sigma(2L-1)i_{2L-1}}W_{\sigma(2L)i_{2L}}A_{i_{2L-1}i_{2L}} \tag{44}$$

$$= \frac{1}{2^L L!}\sum_{\{i_k\}}\sum_{\sigma\in S_{2L}}\mathrm{sign}(\sigma)\Big[W_{\sigma(1)i_1}W_{\sigma(2)i_2}\dots W_{\sigma(2L)i_{2L}}\Big]\Big[A_{i_1i_2}\cdots A_{i_{2L-1}i_{2L}}\Big]\ . \tag{45}$$

We note that the summand vanishes if any two of the i_k concide. This is due to summing over the transposition which exchanges the two but gives a different sign factor. For this reason, all i_k can be assumed to be distinct. This means that the sum over all $\{i_k\}$ can be replaced by a sum over all permutations π with $i_k = \pi(k)$. We then find

$$\mathrm{Pf}(WAW^T) = \frac{1}{2^L L!}\sum_{\pi\in S_{2L}}\sum_{\sigma\in S_{2L}}\mathrm{sign}(\sigma)\Big[W_{\sigma(1)\pi(1)}W_{\sigma(2)\pi(2)}\dots W_{\sigma(2L)\pi(2L)}\Big] \tag{46}$$

$$\Big[A_{\pi(1)\pi(2)}\cdots A_{\pi(2L-1)\pi(2L)}\Big]\ . \tag{47}$$

Next we note that the sum over the permutations σ and the product of W just gives the determinant $\det(W)$ if $\pi(k) = k$. In all the other cases one obtains $\mathrm{sign}(\pi)\det(W)$ since the factors first need to be brought into the correct order (the permutation in the arguments of π can be mapped into the corresponding permutation of the arguments of σ). This immediately implies

$$\mathrm{Pf}(WAW^T) = \frac{1}{2^L L!}\det(W)\sum_{\pi\in S_{2L}}\mathrm{sign}(\pi)\Big[A_{\pi(1)\pi(2)}\cdots A_{\pi(2L-1)\pi(2L)}\Big] \tag{48}$$

$$= \det(W)\ \mathrm{Pf}(A)\ . \tag{49}$$

3. The Pfaffian is just the product of the corresponding λ_i,

$$\mathrm{Pf}(\Lambda_1\oplus\cdots\oplus\Lambda_L) = \lambda_1\cdots\lambda_L\ . \tag{50}$$

4. We set $(\gamma_1,\gamma_2,\gamma_3,\dots) = (a_1,b_1,a_2,\dots)$. In this basis, the matrix A for Case i) has the form

$$A = \begin{pmatrix}0 & -\mu\\ \mu & 0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0 & -\mu\\ \mu & 0\end{pmatrix}\ . \tag{51}$$

Here one has $\mathrm{Pf}(A) = (-\mu)^L$ and $\Pi(L) = (-\operatorname{sign}\mu)^L$ and hence $\Gamma = 1$. Similarly, the matrix A for Case ii) has the form

$$A = \begin{pmatrix}0 & & & \cdots & -2w\\ & 0 & 2w & & \\ & -2w & 0 & & \\ \vdots & & & & \vdots\\ 2w & & & \cdots & 0\end{pmatrix} \tag{52}$$

To bring this matrix to standard form one needs to apply two cyclic permutations, one bringing the last column in front of the second column and one bringing the last row in front of the second row. Both transformations are implemented in terms of a W with $\det(W) = 1$. We note that all blocks have the same sign except for the first. We thus find $\mathrm{Pf}(A) = -(2w)^L$. This then implies $\Pi(L) = -(\operatorname{sign} w)^L$, and hence $\Gamma = -1$.

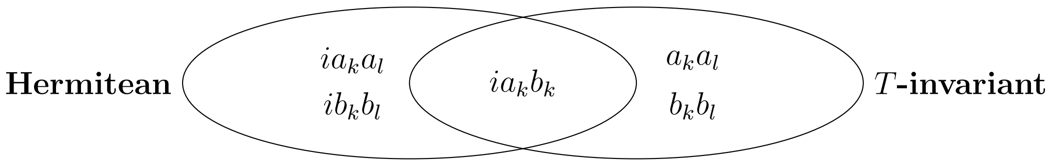
6. The Majorana chain III (effect of time reversal invariance)

1. A simple calculation yields that the Majorana modes of type a and b need to be considered on a different footing:

$$Ta_kT^{-1} = T(c_k + c_k^\dagger)T^{-1} = c_k + c_k^\dagger = a_k \tag{53}$$

$$Tb_kT^{-1} = T[-i(c_k - c_k^\dagger)]T^{-1} = +i(c_k - c_k^\dagger) = -b_k \quad . \tag{54}$$

2. One easily finds



3. By staggering systems with edge modes it looks as if there was a topological invariant in $\mathbb{Z}$ which is just counting the number of edge modes (on one side of the chain). However, not all configurations are stable since pairs of Majorana fermions can be gapped out using interactions of the form ia_ka_l, ib_kb_l or ia_kb_l. However, we note that the first two are forbidden by time-reversal invariance. Hence imposing time-reversal invariance suggests a $\mathbb{Z}$-classification of topological phases (in 1D). However, as will be shown in the next part of this exercise, this classification is actually reduced to $\mathbb{Z}_8$ if one allows for interactions.

4. See [2, 3].

7. Projective representations and central extensions

1. Projective representations of $\mathbb{Z}_M \times \mathbb{Z}_N$

   (a) The final result is $H^2\big(\mathbb{Z}_N \times \mathbb{Z}_N\big) = \mathbb{Z}_N$.

   (b) The result is $H^2\big(\mathbb{Z}_M \times \mathbb{Z}_N\big) = \mathbb{Z}_{\gcd(M,N)}$, where gcd denotes the greatest common divisor.

   For many more explicit results, see [10].

2. By the replacement $\rho(g) \mapsto \rho(g)/\rho(1)$ one can always assume $\rho(1) = \mathbb{1}$. Let us then investigate

$$\rho(T)\rho(T) = \omega(T,T)\,\rho(T^2) = \omega(T,T)\,\rho(1) = \omega(T,T) \quad . \tag{55}$$

In contrast to the case of a unitary symmetry, the right hand side cannot be absorbed into a redefinition of $\rho(T)$. Indeed, for any $\xi \in U(1)$ one finds

$$\big[\xi\rho(T)\big]\big[\xi\rho(T)\big] = \xi\bar{\xi}\,\rho(T)\rho(T) = \omega(T,T)\,\mathbb{1} \quad . \tag{56}$$

It remains to find the allowed values of $\omega = \omega(T,T)$. These follow from the associativity condition

$$\text{lhs} = \big[\rho(T)\rho(T)\big]\rho(T) = \omega\rho(T)\rho(T) = \omega^2\,\mathbb{1} \tag{57}$$

$$\text{rhs} = \rho(T)\big[\rho(T)\rho(T)\big] = \rho(T)\omega\rho(T) = \bar{\omega}\,\rho(T)\rho(T) = \bar{\omega}\omega\,\mathbb{1}\ . \tag{58}$$

Hence we conclude $\omega = \bar{\omega}$. This only leaves the two possibilities $\omega = \pm 1$.

3. The corresponding matrices can be obtained by investigating which matrices are rotating vectors and Pauli matrices in the desired way. One obtains

$$\rho(R_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \rho(R_y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \rho(R_z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{59}$$

and

$$\rho(R_x) = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho(R_y) = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \rho(R_z) = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{60}$$

One can indeed verify that these square to the identity (a potentially arising phase in a projective representation could be absorbed by multiplying the matrices with a constant). One then clearly sees that

$$2D: \qquad \rho(R_x)\rho(R_y)\rho(R_x)^{-1}\rho(R_y)^{-1} = -\mathbb{1} \tag{61}$$

$$3D: \qquad \rho(R_x)\rho(R_y)\rho(R_x)^{-1}\rho(R_y)^{-1} = \mathbb{1} \tag{62}$$

The sign in the first equation signals the presence of a projective representation.

4. The group multiplication should read

$$(\alpha, g)\cdot(\beta, h) = (\alpha\beta\omega(g,h)\omega(1,1)^{-1}, gh)\ . \tag{63}$$

The rest is more or less calculation.

5. Some complex conjugation symbols need to be placed at various places depending on the type and the position of certain factors. Where they appear should actually be rather clear when going through the associativity condition and considering the replacement $\rho(g) \to f(g)\rho(g)$.

8. [$\star$] Majorana chain versus transverse field Ising model

1. The result should be a transverse field Ising model of the form

$$H = -f\sum_k S_k^z - J\sum_k S_k^y S_{k+1}^y\ . \tag{64}$$

Specifically, the images of the Hamiltonians discussed in the lecture are

$$H_{\text{trivial}} = -\frac{1}{2}\sum_k S_k^z \qquad \text{and} \qquad H_{\text{non-trivial}} = -\frac{1}{2}\sum_k S_k^y S_{k+1}^y\ . \tag{65}$$

The first Hamiltonian has a disordered (unique) groundstate with $S_k^z = 1$ on every site. The second Hamiltonian has two groundstates, corresponding to a spontaneous breaking of the $\mathbb{Z}_2$ symmetry $S^z \to -S^z$. Remark: Non-local mappings of this type have been used in various cases in order to classify phases of 1D systems.