A users guide to K-theory

Homology and Cohomology

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Outline

CW complexes

a short reminder

- A (finite) *CW*-complex is a collection of topological spaces Σ_k, $k \in \mathbb{Z}$, called the k-skeleta, with $\Sigma_{k-1} \subseteq \Sigma_k$ for all k. Furthermore,
  - $\Sigma_k = \varnothing$, for $k < 0$,
  - the n-skeleton Σ_n is obtained by attaching a (finite) collection of n-disks via attaching maps
  $$\phi_n^i : \partial D^n \to \Sigma_{n-1},$$
  so that
  $$\Sigma_n = \Sigma_{n-1} \cup_{\phi_n^0} D^n \cup_{\phi_n^1} D^n \cdots \cup_{\phi_n^k} D^n.$$
- A space X is said to have a CW-decomposition if we can write $X = \bigcup_n \Sigma_n$ for some CW-complex $\{\Sigma_k\}$. We assume there is always some n_0 such that for all $k > n_0$, $\Sigma_k = \Sigma_{n_0}$.

Exercise: (a short reminder)

- Give a CW-decomposition of a surface of genus g.
- Give a CW-decomposition of real, complex and quaternionic projective spaces.

Definition

Let X_*, Y_+ be pointed topological spaces. The *wedge product* of X and Y, denoted $X \vee Y$, is the quotient space

$$X \vee Y = X \coprod Y/(* \sim +).$$

Exercise

Let X be a connected topological space with a CW-decomposition. Show that for all n, Σ_n/Σ_{n-1} is a wedge product of n-spheres, one for each n-cell.

We write

$$\Sigma_n/\Sigma_{n-1} = \bigvee_i e_i^{(n)},$$

where $e_i^{(n)}$ is the sphere corresponding to the n-cell $\phi_n^i : D^n \to \Sigma_{n-1}$.

The boundary map

We now introduce some maps:

- $i_n : S^n \to D^{n+1}$ is the inclusion of the boundary,
- $p_k : \Sigma_n \to e_k^{(n)} = S^n$ be the projection induced by the quotient

$$\Sigma_n \to (\Sigma_n/\Sigma_{n-1}) \Big/ \bigcup_{\{i \neq k\}} e_i^n,$$

  in other words, the projection onto the k'th sphere in the decomposition of Σ_n/Σ_{n-1} into a wedge of spheres.
- define $\delta_{kl}^{(n)} : S^n \to S^n$ by

$$\delta_{kl}^{(n)} = p_l \circ \phi_{n+1}^k \circ i_n.$$

  This should be interpreted as a decomposition of the boundary of the k'th $n+1$ cell into its boundary cells.

Maps between spheres

To any continuous map between oriented spheres of the same dimension, one can attach an integer called the *degree*. When the map is smooth, this can be computed in a particularly easy way: for a smooth map $f : S^n \to S^n$,

$$\deg(f) = \int_{S^n} f^* \mathrm{vol},$$

where vol is the unit volume form on S^n. The degree is an integer, and a homotopy invariant.

A second interpretation of the degree

- We would like to interpret the degree as counting something. Consider $f : S^n \to S^n$.
- Generically $f^{-1}(p)$ is finite, for $p \in S^n$.
- One can show that for generic $p \in S^n$

$$\deg(f) = \sum_{x \in f^{-1}(p)} \sigma(x),$$

where

$$\sigma(x) = \begin{cases} 1, & \text{if } x \text{ has the same orientation as that induced by } f, \\ -1, & \text{otherwise.} \end{cases}$$

Exercises:

- Let $f : S^1 \subset \mathbb{C} \to S^1 \subset \mathbb{C}$ be the map $z \mapsto z^n$. Show $\deg(f) = n$.
- Let $p : S^n \to S^n$ be the constant map $p(x) = x_p$, for some chosen $x_p \in S^n$. Show $\deg(p) = 0$.
- Let $a : S^n \to S^n$ be the antipodal map. Show $\deg(a) = (-1)^n$.

Chains

Definition

Let X be a finite CW-complex. The *degree k chains*, denoted by $C_k(X)$, is the abelian group freely generated by the k-cells: a typical element is thus of the form

$$\sum_i n_i \phi_k^i,$$

with the $n_i \in \mathbb{Z}$.

One can think of the k-chains as being generated by the "k-dimensional" part of X.

The boundary map

There is a canonical homomorphism $\delta_{(k)} : C_k(X) \to C_{k-1}(X)$, called the boundary map, defined on generators by

$$\delta_{(k)}\phi_i^k = \sum_l \deg(\delta_{il}^k)\phi_l^{k-1}.$$

Furthermore, one can show that $\delta_{(k)} \circ \delta_{(k+1)} = 0$ for all k (where we define $\delta_{(0)} = 0$). In other words, $\ker \delta_{(k)} \subseteq \operatorname{im} \delta_{(k+1)}$.

Chain complexes

A $\mathbb{Z}$-indexed collection of abelian groups (henceforth referred to as a graded abelian group) C_k along with homomorphisms $\delta_{(k)} : C_k(X) \to C_{k-1}(X)$ satisfying $\delta_{(k)} \circ \delta_{(k+1)} = 0$ is called a chain complex.

Homology

Definition

We call k-chains in the kernel of δ_k *closed*, and those in the image *exact*. The *degree-k homology of X*, written $H_k(X)$, is defined as

$$H_k(X) = \frac{\ker \delta_{(k)}}{\operatorname{im} \delta_{(k+1)}}.$$

Intuitively, one can think of closed chains as representing closed subspaces of X, exact chains as being boundaries of subspaces of X, and degree k homology classes as representing "k-dimensional holes".

Exercises

Compute the homology of

- a point,
- S^n, $n > 1$,
- S^1,
- a "bouquet of circles", $S^1 \vee S^1 \vee S^1 \vee \dots S^1$,
- $\mathbb{CP}^n$,
- a surface of genus g,
- $\mathbb{RP}^n$.

Properties of homology

Homology satisfies a set of axioms due to Eilenberg and Steenrod. We won't list them here, but the most important are that it is a covariant functor from suitable topological spaces to abelian groups, that is homotopy invariant.

Co-chains

We will now do something a little odd, and consider "functions on chains".

Definition

Let X be a CW complex. The degree k co-chains on X, denoted by $C^k(X)$, is the group $\mathrm{Hom}(C_k(X), \mathbb{Z})$.

Remark

$C^k(X)$ is generated by the "δ-functions" $f_i^{(k)}$:

$$f_i^{(k)}\left(\sum_j n_j \phi^j_{(k)}\right) = n_i.$$

A general co-chain is thus of the form

$$g^{(k)} = \sum_i m^i f_i^{(k)}.$$

The co-boundary map

There is a canonical map $\delta^{(k)} : C^k(X) \to C^{(k+1)}(X)$, defined as follows. Suppose $g \in C^k(X)$. Then, for a $c_{k+1} \in C_{k+1}(X)$

$$(\delta^{(k)} g)(c_{k+1}) = g\left(\delta_{(k+1)} c_{k+1}\right).$$

The co-boundary map also squares to zero: this follows from the analogous statement for the boundary map. We call co-chains in the kernel of δ *closed*, and those in the image, *exact*.

Cohomology

Definition

The cohomology of a CW-complex X is the $\mathbb{Z}$-graded collection of abelian groups

$$H^k(X) = \frac{\ker \delta^{(k)}}{\operatorname{im} \delta^{(k-1)}}.$$

Remark

A collection of abelian groups A^k indexed by $\mathbb{Z}$, along with maps $\delta : A^k \to A^{(k+1)}$ which square to zero is called a co-chain complex, and for any such one can define the cohomology completely analogously.

Exercises

Compute the cohomology of

- a point,
- S^n, $n > 1$,
- S^1,
- a "bouquet of circles", $S^1 \vee S^1 \vee S^1 \vee \dots S^1$,
- $\mathbb{CP}^n$,
- a surface of genus g,
- $\mathbb{RP}^n$.

de Rham cohomology

Let X, for now, be a smooth manifold. We recall that one can always associated the differential forms $\Omega^\bullet(X)$. These, in fact, form a *co-chain complex*, the co-boundary map being given by the de Rham $\mathrm{d} : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$:

$$\cdots \xrightarrow{\mathrm{d}} \Omega^{k-1}(X) \xrightarrow{\mathrm{d}} \Omega^k(X) \xrightarrow{\mathrm{d}} \Omega^{k+1}(X) \xrightarrow{\mathrm{d}} \cdots$$

One may thus take the cohomology of this complex! This is the so-called de Rham cohomology:

$$H^k_{\mathrm{dR}}(X) = \frac{\ker \mathrm{d}^{(k)}}{\operatorname{im} \mathrm{d}^{(k-1)}}.$$

Exercises

- Compute the de Rham cohomology of
  - a point,
  - S^1,
  - $S^1 \times \ldots \times S^1$.
- Show that the wedge product "descends to cohomology", giving $H^\bullet_{\mathrm{dR}}(X)$ the structure of a graded commutative ring (hint: the Leibnitz rule is key).
- Let $p : X \to Y$ be an oriented submersion with compact fibres of dimension n. Show that integration descends to cohomology, and defines a map $p_* : H^k_{\mathrm{dR}}(X) \to H^{k-n}_{\mathrm{dR}}(Y)$.
- Suppose X is a manifold. Show that a class $[\omega] \in H^k_{\mathrm{dR}}(X)$ defines a function on oriented closed k-dimensional submanifolds of X, with the property that if Y, $Y' \subseteq M$ are closed k-dimensional submanifolds, with a $k+1$-dimensional submanifold Z such that $\delta Z = Y \coprod Y'$, then $\int_Y [\omega] = \int_{Y'} [\omega]$.

The Steenrod Axioms

We now have two different "cohomologies" defined on smooth manifolds. Worse, the one "cohomology" is defined in terms of a choice of CW-complex. This begs the question: how are they related? It turns out they (and essentially any other version of cohomology one might dream up) are, in a strong sense, the same. In order to explain this, we need a little algebra, and a notion from topology.

Abelian groups as $\mathbb{Z}$ modules, and tensor products

Any abelian group G is canonically a $\mathbb{Z}$-module: for $n \in \mathbb{Z}$ and $g \in G$, one defines $n \cdot g = \sum_{i=1}^{n} g$. Now suppose M, N are $\mathbb{Z}$ modules. One can form their tensor product:

$$M \otimes_{\mathbb{Z}} N = M \times N / \{(km, n) \sim (n, km) \text{ for all } k \in \mathbb{Z}\}.$$

This is, of course, also a $\mathbb{Z}$-module.

Exercises

- Compute: $\mathbb{Z} \otimes G$ (for any G), $\mathbb{Z}_2 \otimes \mathbb{Z}_3$, $\mathbb{Z}_2 \otimes \mathbb{R}$, $\mathbb{R}^k \otimes \mathbb{R}^l$.
- Prove that the action of $\mathbb{Z}$ on G really does make G into a $\mathbb{Z}$-module. Why is it necessary for G to be abelian?
- An element $g \in G$ is called *torsion* if there is a non-zero $n \in \mathbb{Z}$ with $n \cdot g = 0$. Show that if F is a field of characteristic zero, then for any abelian G, $F \otimes G$ has no torsion elements, and is, in fact, canonically an F-vector space. We define the *rank* of a $\mathbb{Z}$-module G to be

$$\mathrm{Ran}\, G = \dim_F F \otimes G.$$

Interlude : cohomology with coefficients

In defining cellular cohomology, co-chains were $\mathbb{Z}$-valued functions on chains. There is, of course, nothing sacred about this choice! One can define cellular cohomology with values in any abelian group: for an abelian group M, define the M-valued co-chains $C^\bullet(X; M) \cong \mathrm{Hom}(C_k(X), M)$, and the cellular cohomology with values in M, $H^\bullet(X; M)$ analogously. It is not hard to see that

$$H^\bullet(X; M) = H^\bullet(X) \otimes M.$$

The Euler characteristic can be recovered from the cohomology with values in $\mathbb{R}$:

$$\chi(X) = \sum_i (-1)^i \dim H^i(X; \mathbb{R}).$$

Exact sequences

A sequence of abelian groups (equivalently, $\mathbb{Z}$-modules) M_i indexed by $\mathbb{Z}$ with homomorphisms

$$\dots M_{i-1} \xrightarrow{\phi_{i-1}} M_i \xrightarrow{\phi_i} M_{i+1} \cdots$$

is called *exact* if $\operatorname{im} \phi_i = \ker \phi_{i+1}$ for all i. An exact sequence of the form

$$0 \to F \to G \to H \to 0$$

is called a *short*. An exact sequence with (countably) infinite terms is called *long*.

Exercises

- Suppose the sequence

$$\dots F \xrightarrow{\phi} G \longrightarrow 0$$

  is exact. Show that ϕ is surjective.
- Suppose the sequence

$$0 \longrightarrow G \xrightarrow{\phi} \dots$$

  is exact. Show that ϕ is injective.
- A short exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\phi} G \longrightarrow 0$$

  is called *split* if there exists $s : G \to F$ such that $\phi \circ s = \mathrm{id}_G$. Show that this gives a canonical isomorphism $F \to E \oplus G$.

The category of pairs

In the setting of topology, a *pair* is an inclusion of topological spaces $i : A \hookrightarrow X$, denoted by (X, A). These form a category, where a morphism between (X, A) and (X', A') is given by two maps $f : X \to X'$, $g : A \to A'$ that commute with the inclusions. There is a functor from spaces to pairs, induces by the inclusion $X \mapsto (X, \varnothing)$.

The cellular cohomology of a pair

A continuous map between CW-complexes is cellular if it takes skeletons to skeletons (keeping degree). One defines a pair in the CW setting to be a topological pair, where the inclusion is cellular. This is not a drastic restriction: the CW-approximation theorem states that any continuous map between CW-complexes is homotopic to a cellular map.

Definition

The cellular cohomology of the CW pair (X, A) is the cohomology of $C^\bullet(X, A) \subseteq C^\bullet(X)$ is defined by

$$C^\bullet(X, A) = \{f \in C^\bullet(X),\ \text{such that } f(c) = 0 \text{ if } c \in C_\bullet(A)\}.$$

The de Rham cohomology of a pair

Definition

Let (X, A) be a pair of smooth manifolds (in particular, the inclusion is smooth, and A is a closed sub manifold). Define the relative complex $(\Omega^\bullet(X, A), \mathrm{d})$, with

$$\Omega^\bullet(X, A) = \{\omega \in \Omega^\bullet(X),\ \omega|_A = 0\}.$$

Then the de Rham cohomology of X relative to A is the cohomology of the complex $(\Omega^\bullet(X, A), \mathrm{d})$.

The Eilenberg-Steenrod axioms

An *ordinary cohomology theory* is a collection $\{H^i\}_{i\in\mathbb{Z}}$ such that:

- for each $n \in \mathbb{Z}$, H^i is a contravariant functor from pairs to abelian groups,
- (homotopy invariance) if $f \simeq g$ are homotopic maps of pairs, then the induced maps on cohomology are the same,
- (LES) for each pair (X, A), there exists a long exact sequence

  $$\cdots \longrightarrow H^i(X,A) \longrightarrow H^i(X) \longrightarrow H^i(A) \xrightarrow{\delta} H^{i+1}(X,A) \longrightarrow \cdots$$

  such that the "boundary map" δ is natural,
- (preserves products) $H^k(\coprod X_\alpha) = \prod H^k(X^\alpha)$.
- (excision) If $Z \subseteq A \subseteq X$ and $\bar{Z} \subseteq A^\circ$, then the induced map

  $$H^i(X,A) \to H^i(X - Z, A - Z)$$

  is an isomorphism for each i,
- (dimension) $H^i(\mathrm{pt}) = 0$ for $i \neq 0$.

Both cellular and de Rham cohomology are ordinary cohomology theories. It turns out that all ordinary cohomology theories with the same value on a point are equivalent, so that one may talk about *the* cohomology of a space, choosing the model as needed. One may ask whether the additional structure we saw for de Rham cohomology: the ring structure and the existence of push-forwards exists for cohomology with other coefficients. It does: ordinary cohomology can be given a (graded commutative) ring structure, and for nice enough maps push forwards (or wrong-way or Umkehr maps) can be defined.

References

The books

- Hatcher, Alan: *Algebraic Topology*
- Bott, Raoul, and Tu, Loring: *Differential Forms in Algebraic Topology*

are a wonderful introduction and reference for everything in this talk.