A users guide to K-theory

K-theory

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Outline

1 Vector bundles

2 Clifford algebras

3 K-theory and differential operators

Let X be a (compact, Hausdorff) topological space. Define

$$\mathrm{Vect}_{\mathbb{R},\mathbb{C}}(X) = \{\text{Topological vector bundles on } X\}/\text{isomorphism},$$

the set of isomorphism classes of real/complex vector bundles on X.

- $\oplus$ gives $\mathrm{Vect}_{\mathbb{R},\mathbb{C}}(X)$ the structure of a *commutative monoid* (group without inverses).
- $\otimes$ gives $\mathrm{Vect}_{\mathbb{R},\mathbb{C}}(X)$ the structure of a commutative *rig* (ring without negation), with unit the trivial line bundle.
- The assignment $X \mapsto \mathrm{Vect}_{\mathbb{R},\mathbb{C}}(X)$ is a contravariant functor.

Exercise

Prove these assertions.

Learning to subtract

Let's take a step back, and discuss subtraction.

Question

Suppose we have a commutative monoid (for example, the natural numbers). How do we make it into a *group*?

The Grothendieck K-construction

Definition

Suppose M is an abelian monoid. Define

$$K(M) = M \times M/\Delta.$$

where Δ is the diagonal submonoid, and the quotient is a quotient of monoids.

Exercise: show that $K(\mathbb{N}) \cong \mathbb{Z}$, $K(\mathbb{Z}^\times) = \mathbb{Q}^\times$.

Lemma

$K(M)$ is an abelian group.

The K-theory of a space

Definition

Let X be .... Define $K^0(X) = K(\mathrm{Vect}_{\mathbb{C}}(X))$, the K-theory of X (resp. $KO^0(X) = K(\mathrm{Vect}_{\mathbb{R}}(X))$).

Lemma

The assignment $X \mapsto K^0(X)$ (resp. $X \mapsto KO^0(X)$), is a contravariant functor from (compact, Hausdorff) topological spaces into commutative rings. In fact, the assignment is homotopy invariant.

Exercises

- Compute the rings $K^0(\mathrm{pt})$ and $KO^0(\mathrm{pt})$.
- Show that every complex vector bundle on a circle is trivial. Conclude $K^0(S^1) = \mathbb{Z}$.
- What about $KO^0(S^1)$?

The Chern character

Definition

Let $V \to X$ be a complex hermitian vector bundle, with a compatible connection ∇. Then define the Chern character form $\mathrm{Ch}(\nabla) \in \Omega^{\text{even}}(X, \mathbb{C})$ by

$$\mathrm{Ch}(\nabla) = \mathrm{Tr}\exp\left[-\frac{i}{2\pi}\mathrm{Curv}(\nabla)\right].$$

Properties of the Chern character

- The Chern character form is closed and integral.
- For (V, ∇_V), (W, ∇_W),

$$\mathrm{Ch}(\nabla_{V \oplus W}) = \mathrm{Ch}(\nabla_V) + \mathrm{Ch}(\nabla_W),$$
$$\mathrm{Ch}(\nabla_{V \otimes W}) = \mathrm{Ch}(\nabla_V) \wedge \mathrm{Ch}(\nabla_W).$$

- For connections ∇, ∇', there exists an $\alpha \in \Omega^{\mathrm{odd}}(X)$ so that

$$\mathrm{Ch}(\nabla) - \mathrm{Ch}(\nabla') = \mathrm{d}\alpha.$$

- $\Rightarrow$ the Chern character map descends to a homomorphism of rings

$$\mathrm{Ch} : K^0(X) \to H^{\mathrm{even}}(X; \mathbb{R}).$$

Exercises

Prove the assertions on the previous slide.

K-theory as a cohomology theory?

Question

Can one extend the functors K and KO to define something like cohomology? It turns out the answer is yes, but in order to do so one needs somehow to be able to shift degree.

Definition

We denote by X_* denote a pointed topological space, and X^+ the union of X with a disjoint basepoint. The *smash product* of two pointed topological spaces X_*, Y_*, is defined by

$$X_* \wedge Y_* = (X \times Y)/(X_* \vee Y_*).$$

The *reduced suspension* of a pointed topological space X_* is defined by

$$S(X_*) = S^1 \wedge X_*.$$

One writes $S^n(X_*)$ for the n-fold reduced suspension.

Exercises

Show that:

- $S^n \wedge S^m \cong S^{n+m}$,
- $S^n(X_*) \cong S^n \wedge X$.

Definition

Let X_* be a pointed topological space, and $i : \mathrm{pt} \hookrightarrow X_*$ the canonical inclusion.

- The *reduced K-theory* of the space is defined as

$$\tilde{K}^0(X_*) := \ker i^* : K^0(X) \to K^0(\mathrm{pt}).$$

- for $n \in \mathbb{Z}_{\geq 0}$, define
  - $\tilde{K}^{-n}(X_*) := \tilde{K}^0(S^n(X_*))$,
  - $K^{-n}(X) := \tilde{K}^0(S^n(X^+))$,
  - $K^{-n}(X, Y) := \tilde{K}^0(S^n(X/Y))$.

One can make similar definitions for KO theory.

Exercise

Show that:

- $K^0(X) \cong \tilde{K}^0(X^+)$,
- $K^{-1}(\mathrm{pt}) = 0$.

Theorem (Bott periodicity)

Let $[H] \in K^0(\mathbb{CP}^1)$ denote the class of the canonical line bundle. Then the map $\tilde{K}^0(X_*) \to \tilde{K}^{-2}(X_*) = \tilde{K}^0(S^2(X_*))$ defined by

$$[V] \mapsto ([H] - 1) \boxtimes [V]$$

is an isomorphism. Here $\boxtimes$ is the reduced exterior product.

Definition

Using periodicity, we define $K^n(X)$ for positive n.

Remark

There is a similar statement for KO, but with an *eightfold* periodicity!

K-theory as a cohomology theory

- $K^\bullet$ satisfies all the axioms of an ordinary cohomology theory except the dimension axiom: periodicity shows $K^{2n}(\mathrm{pt}) = \mathbb{Z}$ for all n.
- This makes it a *generalised*, or *extraordinary* cohomology theory.
- KO theory is a second example of such, with the values on a point being eight periodic.
- Tensor products extend to make K-theory into a ringed theory.
- One may show the Chern character gives an isomorphism of graded rings

$$\mathrm{Ch} : K^\bullet(X) \otimes \mathbb{R} \to H(X; \mathbb{R}[u, u^{-1}])^\bullet,$$

where u has degree two.

Clifford algebras: an interlude in algebra.

Clifford algebras

Definition

Let V be a real vector space, with inner product $(\cdot, \cdot)$. Define

$$\mathrm{Cliff}(V) = \bigoplus V^{\otimes n} / \{v \otimes w + w \otimes v \sim -2(v, w)\}.$$

Define $\mathbb{C}\mathrm{liff} = \mathrm{Cliff}(V) \otimes \mathbb{C}$. For convenience, we write $\mathrm{Cliff}_n = \mathrm{Cliff}(\mathbb{R}^n)$.

Remarks

$\mathrm{Cliff}(V)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded, $\mathbb{Z}$-filtered ring. It is not (graded) commutative for $\dim(V) > 0$.

Exercises

Show that:

- $$\mathrm{Cliff}(\mathbb{R}) \cong \mathbb{C},$$
  $$\mathrm{Cliff}(\mathbb{R}^2) \cong \mathbb{H}.$$

  (as ungraded algebras)
- $\mathrm{Cliff}_{n-1} \cong \mathrm{Cliff}_n^{ev}$ (as algebras)
- Show that $\mathrm{Gr}^\bullet(\mathrm{Cliff}(V)) = \bigwedge^\bullet(V)$. Use this to give the dimension of $\mathrm{Cliff}(V)$ in terms of that of V, and give an explicit isomorphism (as vector spaces, not rings) $\mathrm{Cliff}(V) \cong \bigwedge^\bullet(V)$.

Classification of complex Clifford algebras

Theorem

There are isomorphisms

$$\mathbb{C}liff_{2n} \cong M_{2^n}(\mathbb{C}), \quad \mathbb{C}liff_{2n+1} \cong M_{2^n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}).$$

Remark

This two-periodic classification is another manifestation of Bott periodicity. There is an eight-periodic classification of real Clifford algebras.

Representation theory of Clifford algebras

- There is a unique (ungraded) irrep $\mathbb{S}_{2n}$ of $\mathbb{C}\mathrm{liff}_{2n}$, and two inequivalent ungraded irreps $\mathbb{S}^+_{2n+1}$, $\mathbb{S}^-_{2n+1}$.
- The inclusion $\mathbb{C}\mathrm{liff}_{n-1} \hookrightarrow \mathbb{C}\mathrm{liff}^{\mathrm{ev}}_n$ relates the ungraded representations theory of $\mathbb{C}\mathrm{liff}_{n-1}$ with the graded representations of $\mathbb{C}\mathrm{liff}_n$:
    - there is a decomposition $\mathbb{S}_{2n} \cong \mathbb{S}^+_{2n} \oplus \mathbb{S}^-_{2n}$ under the action of $\mathbb{C}\mathrm{liff}^{\mathrm{ev}}_{2n}$, and there are *two* inequivalent irreducible graded $\mathbb{C}\mathrm{liff}_{2n}$ modules, $\mathbb{S}^+_{2n} \oplus \mathbb{S}^-_{2n}$ and $\mathbb{S}^-_{2n} \oplus \mathbb{S}^+_{2n}$,
    - there is a unique irreducible graded $\mathbb{C}\mathrm{liff}_{2n+1}$ module: $\mathbb{S}^+_{2n+1} \oplus \mathbb{S}^-_{2n+1}$.
- There is a similar, eight-periodic discussion of the representation theory of real Clifford algebras.

The spin group

- write $\mathrm{Pin}(V) \subseteq \mathrm{Cliff}(V)$ for the (Lie) group of units (invertible elements) in $\mathrm{Cliff}(V)$.
- The grading on $\mathrm{Cliff}(V)$ divides the group into two components.
- Denote by $\mathrm{Spin}(V)$ the connected component of the identity (the even component).
- The following exact sequence holds:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V) \to SO(V) \to 1.$$

- The basic spinor modules form complex representations of the spin group.
- When $\dim(V) = 2k$, the group $\mathrm{Spin}(V)$ has two inequivalent irreducible complex representations, $\mathbb{S}(V)^{\pm}$.

Spinors and the Clifford bundle

- A Riemannian manifold is *spin* if there exists a principal bundle $\mathrm{Spin}_{\dim X}$
$$\mathrm{Spin}(X) \to SO(X) \to X.$$
- A *spin structure* on X is a choice of such a double cover.
- Assume X is spin with a chosen spin structure, and to simplify things, that $\dim X = 2k$. The bundle of spinors is the associated $\mathbb{Z}_2$-graded bundle
$$\mathbb{S}(X) = \mathbb{S}(X)^+ \oplus \mathbb{S}(X)^- = \mathrm{Spin}(X) \times_{\mathrm{Spin}_{\dim X}} \mathbb{S}_{2k}.$$
The Levi-Civita connection induces a connection on $\mathrm{Spin}(X)$.
- The bundle of spinors form a $\mathbb{Z}/2\mathbb{Z}$ module-bundle for the Clifford bundle $\mathrm{Cliff}(X) \to X$. Any other module bundle $M \to X$ decomposes as $M \cong \mathbb{S}(X) \otimes V$, where V is a vector bundle.

Dirac operators

Let (V, ∇) be a complex hermitian vector bundle with unitary connection. Then the *Dirac operator twisted by V* is the first order differential operator defined by

$$D(\nabla):\ \Gamma(\mathbb{S}(X)^{+}\otimes V)\xrightarrow{\nabla}\Omega^{1}(\mathbb{S}(X)^{+}\otimes V)\xrightarrow{c(\cdot)}\Gamma(\mathbb{S}(X)^{-}\otimes V).$$

In local coordinates, writing $\nabla = \partial + a$, one has

$$D = \Gamma^{i}\partial_{i} + \Gamma^{i}a_{i}.$$

The Dirac operator is elliptic (and hence Fredholm) when X is compact.

K-theory and differential operators

- Fix a graded Hilbert space $H = H_0 \oplus H_1$,
- Choose compatible graded actions of Cliff_n (and hence, $\mathbb{C}\mathrm{liff}_n$) on H, for all n.
- A Fredholm operator $f : H_0 \to H_1$ is a bounded linear operator with finite dimensional kernel and co-kernel $(= H_1/\mathrm{Im}(f))$. Denote $\mathrm{Fred}(H)$ be the space of all such operators, with the operator norm topology.
- Define $\mathrm{Fred}_k(H) \subseteq \mathrm{Fred}(H)$ those operators that graded commute with the action of $\mathbb{C}\mathrm{liff}_k$.

The basic topology of $\mathrm{Fred}_k(H)$

Theorem

- *$\pi_0 Fred_{2k}(H) = \mathbb{Z}$, where the map is given by the index:*

$$index\, f = \dim_{\mathbb{C}liff_{2k}} \ker f - \dim_{\mathbb{C}liff_{2k}} coker\, f,$$

for $f \in Fred_{2k}(H)$.

- *$Fred_{2k+1}(H)$ has three connected components: $Fred^{\pm}_{2k+1}(H)$, the essentially positive (negative) components, which are contractible; and $Fred^{*}_{2k+1}(H)$.*

Fredholm operators and K-theory

The following is the basic theorem relating K-theory and Fredholm operators.

Theorem

Let X be as before. Then there is an isomorphism of groups $K^0(X) \cong [X, Fred(H)]$.

Theorem

- $Fred_{2k}(H) \simeq \Omega Fred^*_{2k-1}(H)$, *for* $k > 0$,
- $Fred^*_{2k+1}(H) \simeq \Omega Fred_{2k}(H)$, *for* $k \geq 0$.

As a result, $K^{-k}(X) \cong [X, Fred^{(*)}_k(H)]$.

A remark on KO

There is a similar statement for KO-theory, relating the KO-groups to homotopy classes of maps to spaces of Fredholm operators commuting with the *real* Clifford algebras. In particular, one has that $KO^{-1}(X) = [X, \mathrm{Fred}_{1,\mathbb{R}}(H)]$, and the appropriate index in this case is the dimension of the kernel mod 2.

Pushforwards

- Suppose X is spin, and Riemannian, with a chosen spin structure, and that $\dim X = 2k$.
- Let $(V, \nabla) \to X$ be a vector bundle with connection. The class $[V] \in K^0(X)$ is independent of the connection.
- The Dirac Operator $D(V)$ is elliptic and Fredholm, so induces a class $[D(V)] \in K^0(\mathrm{pt}) = \mathbb{Z}$. This is also independent of the connection.
- The assignment $[V] \in K^0(X) \mapsto [D(V)] \in K^0(\mathrm{pt})$, induces a pushforward map $\pi_! : K^0(X) \to K^0(\mathrm{pt}) \cong K^{-2k}(\mathrm{pt})$.

The Atiyah-Singer Index theorem

- This construction can be extended to families: for a "Riemannian" submersion $f : X \to Y$ with compact, spin fibres of relative dimension n, one obtains a map $f_! : K^\bullet(X) \to K^{\bullet-n}(Y)$.
- Suppose $f : X \to Y$ is a Riemannian family with compact, spin fibres of relative dimension n.
- The Atiyah-Singer index theorem states that the diagramme below commutes:

$$\begin{array}{ccc} K^\bullet(X) & \xrightarrow{\hat{A}(X/Y)\wedge \mathrm{Ch}} & H(X;\mathbb{R}[u,u^{-1}])^\bullet \\ \downarrow f_! & & \downarrow f_* \\ K^{\bullet-n}(Y) & \xrightarrow{\mathrm{Ch}} & H(X;\mathbb{R}[u,u^{-1}])^{\bullet-n} \end{array}$$

- Said differently, the Chern character does not commute with push-forward:
$$\mathrm{Ch} f_! = f_*(\hat{A}(X/Y) \wedge \mathrm{Ch}).$$
- In particular, when $f : X \to \mathrm{pt}$, one obtains the famous Atiyah-Singer theorem: for (V, ∇) a hermitian complex vector bundle with connection,
$$\operatorname{index} D(\nabla) = \int_X \hat{A}(X) \wedge \mathrm{Ch}(\nabla).$$