A users guide to $K$-theory

Spectral sequences

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In the last lecture we have learnt that there is a wonderful connection between topology and analysis on manifolds: $K$-theory. This begs the question: how does one calculate the $K$-groups?
From CW-complexes to $K$-theory

We saw that a CW-decomposition makes calculating cohomology easy(er). It’s natural to wonder whether one can somehow use a CW-decomposition to help calculate $K$-theory. We begin with a simple observation.

A simple observation

Let $X$ have a CW-decomposition:

$$\emptyset \subseteq \Sigma_0 \subseteq \Sigma_1 \cdots \subseteq \Sigma_n = X.$$

Define

$$K_{\mathbb{Z}P_k}(X) = \ker i^* : K^\bullet(X) \to K^\bullet(\Sigma_k).$$

This gives a filtration of $K^\bullet(X)$:

$$K^\bullet(X) \supseteq K_{\mathbb{Z}P_0}(X) \supseteq \cdots \supseteq K_{\mathbb{Z}P_n}(X) = 0.$$
One might hope that one can calculate the $K$-theory of a space inductively, with each step moving up one stage in the filtration.

This is what the Atiyah-Hirzebruch spectral sequence does: it computes the groups

$$K_{ZP_{q/q+1}}^\bullet(X).$$

What then remains is to solve the extension problem: given that one knows $K_{ZP_{q/q+1}}^\bullet(X)$ and $K_{q+1}^\bullet(X)$, one must somehow determine $K_{ZP_{q}}^\bullet(X)$, which fits into

$$1 \to K_{ZP_{q/q+1}}^\bullet(X) \to K_{ZP_{q}}^\bullet(X) \to K_{ZP_{q+1}}^\bullet(X) \to 1.$$
A *spectral sequence* is made up of a collection of *pages*, each of which is a bi-graded collection of abelian groups. The $k$'th page, then, looks something like this:

\[
\begin{array}{ccc}
E_{k}^{p-1,q-1} & E_{k}^{p-1,q} & E_{k}^{p-1,q+1} \\
E_{k}^{p,q-1} & E_{k}^{p,q} & E_{k}^{p,q+1} \\
E_{k}^{p+1,q-1} & E_{k}^{p+1,q} & E_{k}^{p+1,q+1} \\
\end{array}
\]
Each page in a spectral sequence is a bi-graded complex, and subsequent pages are computed from the cohomology of this complex.

A spectral sequence is said to converge when there exists some $n$ such that for all $n' > n$, $E_n^{p,q} \cong E_n^{p,q}$. One writes $E_*^{p,q} \Rightarrow E_\infty^{p,q}$.

The idea is that one finds a spectral sequence that starts somewhere that’s easy to compute, and converges to something related to what you want.
The Atiyah-Hirzebruch Spectral sequence

The Atiyah-Serre spectral sequence is a convergent spectral sequence

$$E_2^{p,q} = \left. H^p(X, K^q(\text{pt}) \right) \Rightarrow K_{\mathbb{Z}P_{p/p+1}}^{p+q}(X) = E_\infty^{p,q},$$ and differentials $d_r : E_2^{p,q} \rightarrow E_2^{p+r,q-r+1}$.

One can replace $K$ with any extraordinary cohomology theory.

For $K$-theory the first non-zero differential is $d_2$. Exercise: show this!

The groups $K_{\mathbb{Z}P_q}^\bullet(X)$ may be more invariantly defined:

$x \in K_{\mathbb{Z}P_q}^\bullet(X) \subseteq K^\bullet(X)$ iff for any CW-complex $A$ with dimension less than $q$ and continuous map $i : A \rightarrow X$, $i^*x = 0$. In particular, $K_{\mathbb{Z}P_1}^\bullet(X) = \tilde{K}^\bullet(X)$. 
Write down the $E^2$-page for $S^n$.
Show that all the differentials vanish.
Conclude that one has $K^0(S^{2k+1}) = K^1(S^{2k+1}) = \mathbb{Z}$.
What about the even case?
Write down the $E^2$-page for $\mathbb{C}P^n$.

Show that all the differentials vanish.

Conclude that one has $K^1(\mathbb{C}P^n) = 0$,

$$
\begin{array}{cccc}
K^0(\mathbb{C}P^n) & \xleftarrow{\mathbb{Z}} & K^0_{ZP_1}(\mathbb{C}P^n) & \xleftarrow{0} K^0_{ZP_2}(\mathbb{C}P^n) \\
& \xleftarrow{\mathbb{Z}} & & \xleftarrow{\mathbb{Z}} \\
\end{array}
\ldots
$$

Argue that $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$.
The cohomology of $\mathbb{CP}^n$ is concentrated in even degrees, and is non-zero between degree zero and the dimension of $\mathbb{CP}^n$.

The two-periodicity of complex means that the $E_2$-page has “$\mathbb{Z}$”s on points with even $p$ and $q$ (within the support of the cohomology) and zero elsewhere.

We note that the differential $d_2$ sends even $q$ to odd $q$ and vice-versa, so must vanish. Thus the $E_2^{p,q} = E_3^{p,q}$.

Similar reasoning allows us to argue that the $d_r$ vanish for all $r \geq 2$, so that $E_2^{\bullet,\bullet} = E_\infty^{\bullet,\bullet}$.

Reading of the $E_\infty$ page, we see that $K^0(\mathbb{CP}^n)$ is $\mathbb{Z}$ extended by $\mathbb{Z}$ $n$-times, and thus $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$. 
Write down the $E^2$-page for $\Sigma_g$, the surface of genus $g$.

Show that all the differentials vanish.

Compute the $K^\bullet(\Sigma_g)$. 

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We have by now seen that often, spectral sequence calculations come down to arguing that the differentials vanish (or are tractable), and doing an extension argument. The next examples are a little trickier.

- Write down the $E^2$-page for $\mathbb{RP}^2$.
- Show that all the differentials vanish.
- Compute the $K^\bullet(\mathbb{RP}^2)$. Be careful with extensions!
Write down the $E^2$-page for $SO(3) \cong \mathbb{RP}^3$.

Show that all the differentials vanish: hint, use the Chern character!

Compute the $K^\bullet(SO(3))$. 
For our next computations, we need the Künnett theorem in $K$-theory (Atiyah).

**Theorem**

Let $X$ be such that $K^\bullet(X)$ is finitely generated, and $Y$ be cellular. Then there is a short exact sequence of $\mathbb{Z}/2\mathbb{Z}$-graded modules

$$0 \to K^\ast(X) \otimes K^\ast(Y) \to K^\ast(X \times Y) \to \text{Tor}_1(K^\ast(X), K^\ast(Y)) \to 0,$$

where the first map has degree 0, and the second degree 1.
The Tor functor

We list some properties of the Tor functor. Here $G$ is an abelian group.

- $\text{Tor}_1(\mathbb{Z}/n, G) = \{g \in G; ng = 0\}$,
- $\text{Tor}_1(\mathbb{Z}, G) = 0$,
- $\text{Tor}_1(\bigoplus_i G_i, \bigoplus_j G'_j) \cong \bigoplus_{i,j} \text{Tor}_1(G_i, G'_j)$, for finite sums.
Compute

- $\mathcal{K}\!\mathcal{T}(T^n)$,
- $\mathcal{K}\!\mathcal{T}(SO(4))$, (hint: $SO(4) \cong SO(3) \times S^3$ as spaces),
- $\mathcal{K}\!\mathcal{T}(O(4))$.

Time permitting, use the Chern character to investigate the ring structure of $\mathcal{K}\!\mathcal{T}(T^n)$.
References

the original papers of Atiyah et al.
the book “K-theory” by Atiyah.
Hatcher: “K-theory”