A users guide to K-theory

Spectral sequences

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Bonn-Cologne Intensive Week: Tools of Topology for Quantum Matter, July 2014

The Atiyah-Hirzebruch spectral sequence

In the last lecture we have learnt that there is a wonderful connection between topology and analysis on manifolds: K-theory. This begs the question: how does one calculate the K-groups?

From CW-complexes to K-theory

We saw that a CW-decomposition makes calculating cohomology easy(er). It's natural to wonder whether one can somehow use a CW-decomposition to help calculate K-theory. We begin with a simple observation.

A simple observation

Let X have a CW-decomposition:

$$\varnothing \subseteq \Sigma_0 \subseteq \Sigma_1 \cdots \subseteq \Sigma_n = X.$$

Define

$$K^\bullet_{ZP_k}(X) = \ker i^* : K^\bullet(X) \to K^\bullet(\Sigma_k).$$

This gives a filtration of $K^\bullet(X)$:

$$K^\bullet(X) \supseteq K^\bullet_{ZP_0}(X) \supseteq \cdots \supseteq K^\bullet_{ZP_n}(X) = 0.$$

- One might hope that one can calculate the K-theory of a space inductively, with each step moving up one stage in the filtration.
- This is what the Atiyah-Hirzebruch spectral sequence does: it computes the groups

$$K^\bullet_{ZP_{q/q+1}}(X).$$

- What then remains is to solve the extension problem: given that one knows $K^\bullet_{ZP_{q/q+1}}(X)$ and $K^\bullet_{q+1}(X)$, one must somehow determine $K^\bullet_{ZP_q}(X)$, which fits into

$$1 \to K^\bullet_{ZP_{q/q+1}}(X) \to K^\bullet_{ZP_q}(X) \to K^\bullet_{ZP_{q+1}}(X) \to 1.$$

Spectral sequences, the setup

A *spectral sequence* is made up of a collection of *pages*, each of which is a bi-graded collection of abelian groups. The k'th page, then, looks something like this:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & E_k^{p-1,q-1} & E_k^{p-1,q} & E_k^{p-1,q+1} & \\
\cdots & E_k^{p,q-1} & E_k^{p,q} & E_k^{p,q+1} & \cdots \\
 & E_k^{p+1,q-1} & E_k^{p+1,q} & E_k^{p+1,q+1} &
\end{array}
$$

- Each page in a spectral sequence is a bi-graded complex, and subsequent pages are computed from the cohomology of this complex.
- A spectral sequence is said to converge when there exists some n such that for all $n' > n$, $E_n^{p,q} \cong E_{n'}^{p,q}$. One writes $E_*^{p,q} \Rightarrow E_\infty^{p,q}$.
- The idea is that one finds a spectral sequence that starts somewhere that's easy to compute, and converges to something related to what you want.

The Atiyah-Hirzebruch Spectral sequence

- The Atiyah-Serre spectral sequence is a convergent spectral sequence $E_2^{p,q} = H^p(X, K^q(\mathrm{pt})) \Rightarrow K^{p+q}_{ZP_{p/p+1}}(X) = E_\infty^{p,q}$, and differentials $d_r : E_2^{p,q} \to E_2^{p+r,q-r+1}$.
- One can replace K with any extraordinary cohomology theory.
- For K-theory the first non-zero differential is d_2. Exercise: show this!
- The groups $K^\bullet_{ZP_q}(X)$ may be more invariantly defined: $x \in K^\bullet_{ZP_q}(X) \subseteq K^\bullet(X)$ iff for any CW-complex A with dimension less than q and continuous map $i : A \to X$, $i^*x = 0$. In particular, $K^\bullet_{ZP_1}(X) = \tilde{K}^\bullet(X)$.

- Write down the E^2-page for S^n.
- Show that all the differentials vanish.
- Conclude that one has $K^0(S^{2k+1}) = K^1(S^{2k+1}) = \mathbb{Z}$.
- What about the even case?

- Write down the E^2-page for $\mathbb{CP}^n$.
- Show that all the differentials vanish.
- Conclude that one has $K^1(\mathbb{CP}^n) = 0$,

$$K^0(\mathbb{CP}^n) \xleftarrow{\mathbb{Z}} K^0_{ZP_1}(\mathbb{CP}^n) \xleftarrow{0} K^0_{ZP_2}(\mathbb{CP}^n) \xleftarrow{\mathbb{Z}} \cdots$$

- Argue that $K^0(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}$.

A sketch of the working out of the exercise

- The cohomology of $\mathbb{CP}^n$ is concentrated in even degrees, and is non-zero between degree zero and the dimension of $\mathbb{CP}^n$.
- The two-periodicity of complex means that the E_2-page has "$\mathbb{Z}$"s on points with even p and q (within the support of the cohomology) and zero elsewhere.
- We note that the differential d_2 sends even q to odd q and vice-versa, so must vanish. Thus the $E_2^{p,q} = E_3^{p,q}$.
- Similar reasoning allows us to argue that the d_r vanish for all $r \geq 2$, so that $E_2^{\bullet,\bullet} = E_\infty^{\bullet,\bullet}$.
- Reading of the E_∞ page, we see that $K^0(\mathbb{CP}^n)$ is $\mathbb{Z}$ extended by $\mathbb{Z}$ n-times, and thus $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$.

Surface of genus g

- Write down the E^2-page for Σ_g, the surface of genus g.
- Show that all the differentials vanish.
- Compute the $K^\bullet(\Sigma_g)$.

We have by now seen that often, spectral sequence calculations come down to arguing that the differentials vanish (or are tractable), and doing an extension argument. The next examples are a little trickier.

- Write down the E^2-page for $\mathbb{RP}^2$.
- Show that all the differentials vanish.
- Compute the $K^\bullet(\mathbb{RP}^2)$. Be careful with extensions!

$SO(3)$

- Write down the E^2-page for $SO(3) \cong \mathbb{RP}^3$.
- Show that all the differentials vanish: hint, use the Chern character!
- Compute the $K^\bullet(SO(3))$.

The Künneth Theorem

For our next computations, we need the Künneth theorem in K-theory (Atiyah).

Theorem

Let X be such that $K^\bullet(X)$ is finitely generated, and Y be cellular. Then there is a short exact sequence of $\mathbb{Z}/2\mathbb{Z}$-graded modules

$$0 \to K^*(X) \otimes K^*(Y) \to K^*(X \times Y) \to Tor_1(K^*(X), K^*(Y)) \to 0,$$

where the first map has degree 0, *and the second degree* 1.

The Tor functor

We list some properties of the Tor functor. Here G is an abelian group.

- $\mathrm{Tor}_1(\mathbb{Z}/n, G) = \{g \in G;\ ng = 0\}$,
- $\mathrm{Tor}_1(\mathbb{Z}, G) = 0$,
- $\mathrm{Tor}_1(\bigoplus_i G_i, \bigoplus_j G'_j) \cong \bigoplus_{i,j} \mathrm{Tor}_1(G_i, G'_j)$, for finite sums.

Compute

- $K^\bullet(T^n)$,
- $K^\bullet(SO(4))$, (hint: $SO(4) \cong SO(3) \times S^3$ as spaces),
- $K^\bullet(O(4))$.

Time permitting, use the Chern character to investigate the ring structure of $K^\bullet(T^n)$.

References

Dugger, Daniel, “A geometric introduction to K-theory”,
`http://math.uoregon.edu/ ddugger/kgeom.pdf`
the original papers of Atiyah et al.
the book “K-theory” by Atiyah.
Hatcher: “K-theory”