Reducing and increasing dimensionality of topological insulators

Anton Akhmerov
with Bernard van Heck, Cosma Fulga, Fabian Hassler, and Jonathan Edge

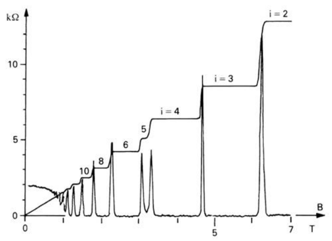
PRB **85**, 165409 (2012), PRB **89**, 155424 (2014).

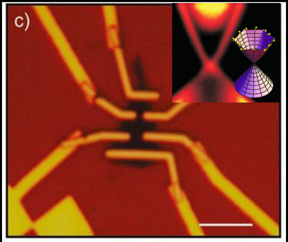
ESI, 1 September 2014

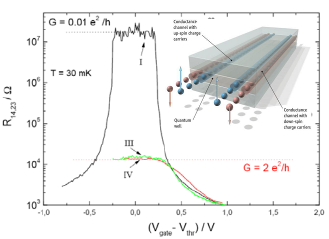
Plan

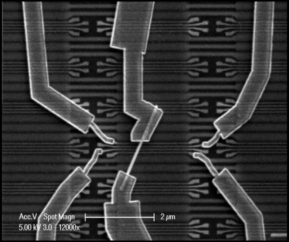
1. Intro: Topological insulators
2. Reducing dimension: Understanding a TI through scattering
3. Increasing dimension: TIs protected by a statistical symmetry

Topological insulators and superconductors









Topological insulators and superconductors

Topological insulator is

- A material with a band gap in the bulk
  (and a certain discrete symmetry)
- It has protected zero energy states at the edge
- Number of these states is a *topological invariant* $\mathcal{Q}[H(\mathbf{k})]$, an integer which does not change under small perturbations.
- $\mathcal{Q}$ is a macroscopic quantity, defined for any insulator with proper symmetry.

Classification

Three discrete symmetries (Altland&Zirnbauer):
$\mathcal{T}: H(k) = U_\mathcal{T} H^*(-k) U_\mathcal{T}^\dagger$, $\mathcal{P}: H(k) = -U_\mathcal{P} H^*(-k) U_\mathcal{P}^\dagger$,
$\mathcal{C}: H(k) = -U_\mathcal{C} H(k) U_\mathcal{C}^\dagger$,

Classification

Three discrete symmetries (Altland&Zirnbauer):
$\mathcal{T} : H(k) = U_{\mathcal{T}} H^*(-k) U_{\mathcal{T}}^\dagger$, $\mathcal{P} : H(k) = -U_{\mathcal{P}} H^*(-k) U_{\mathcal{P}}^\dagger$,
$\mathcal{C} : H(k) = -U_{\mathcal{C}} H(k) U_{\mathcal{C}}^\dagger$,
give 10 symmetry classes and

Classification

Three discrete symmetries (Altland&Zirnbauer):
$\mathcal{T} : H(k) = U_{\mathcal{T}} H^*(-k) U_{\mathcal{T}}^\dagger$, $\mathcal{P} : H(k) = -U_{\mathcal{P}} H^*(-k) U_{\mathcal{P}}^\dagger$,
$\mathcal{C} : H(k) = -U_{\mathcal{C}} H(k) U_{\mathcal{C}}^\dagger$,
give 10 symmetry classes and
a lot of topological insulators (Kitaev, Schnyder *et al.*):

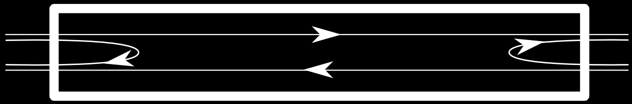
Symmetry class	d = 1	2	3	4	5	6	7	8
A		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AIII	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AI				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
AII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$
CII	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
C		$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
CI			$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Many descriptions

- surface Hamiltonian avoids fermion doubling
- K-theory (Kitaev)
- top.-term in σ-model (Schnyder, Ryu, Ludwig)
- ... in field theory (Qi, Hughes, Zhang & Ryu, Moore, Ludwig)
- string theory (Ryu, Takayanagi)
- Green's functions (Gurarie, Essin)
- c^*-algebra (Hastings, Loring)
- ...

Part I

Scattering matrix



$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}_{\text{out}} = S \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}_{\text{in}}$$

- Describes scattering of free particles from the system at the Fermi level.
- Is also constrained by symmetry.
- Easy to tell an insulator from a conductor.

What about $\mathcal{Q}(S)$?

Simple case: Majorana fermions (1D superconductor)

Reflection matrix r has
Current conservation:

$$rr^\dagger = 1 \Rightarrow |\det r| = 1$$

Particle-hole symmetry:

$$r = \begin{pmatrix} r_{ee} & r_{he} \\ r_{eh} & r_{hh} \end{pmatrix} = \begin{pmatrix} r_{ee} & r_{he} \\ r^*_{he} & r^*_{ee} \end{pmatrix} \Rightarrow \mathrm{Im}\det r = 0$$

Together:

$$\det r = \pm 1$$

Simple case: Majorana fermions (1D superconductor)

$\det r = -1 \Rightarrow \det(r-1) = 0 \Leftrightarrow$ bound state at zero energy.
$\Rightarrow$ Superconductor is in topologically nontrivial phase.

Scattering invariant

$$\mathcal{Q} = \operatorname{sign}\det r$$

Scattering invariant

$$\mathcal{Q} = \operatorname{sign} \det r$$

Phase transition is accompanied by a single fully transmitted mode.

Other TI's in 1D

Idea:

1. Find all disconnected groups of fully reflecting r's.
2. Find what distinguishes them.
3. Check that this quantity is indeed $\mathcal{Q}(r)$.

Other TI's in 1D

Idea:

1. Find all disconnected groups of fully reflecting r's.
2. Find what distinguishes them.
3. Check that this quantity is indeed $\mathcal{Q}(r)$.

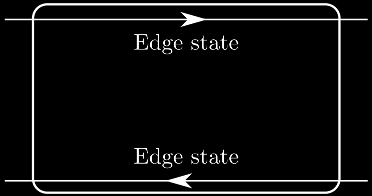
It works!

Symmetry	D	DIII	AIII	BDI	CII
$\mathcal{Q}(r)$	$\operatorname{sign}\det r$	$\operatorname{sign}\operatorname{Pf} r$	$\nu(r)$	$\nu(r)$	$\nu(r)$

Question

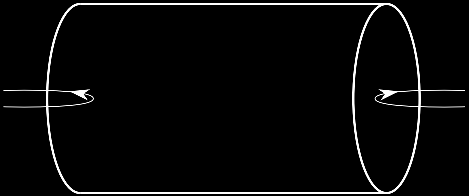
What about higher dimensions?

Higher dimensions: QHE



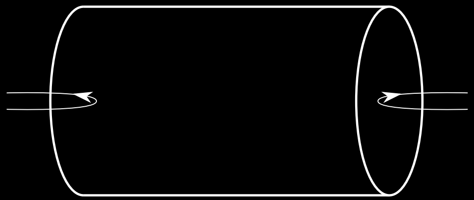
- Not insulating due to edge states?

Higher dimensions: QHE



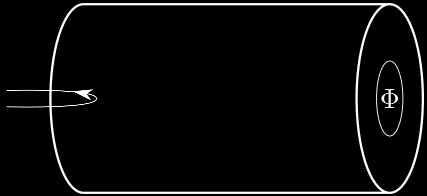
- Not insulating due to edge states?
  Solution: roll it up.

Higher dimensions: QHE



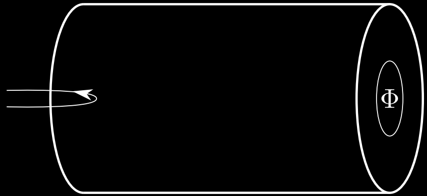
- Not insulating due to edge states?
  Solution: roll it up.
- No difference from 1D?

Higher dimensions: QHE



- Not insulating due to edge states?
  Solution: roll it up.
- No difference from 1D?
  Solution: thread flux, quantized charge pumping appears.

Higher dimensions: QHE



- Not insulating due to edge states?
  Solution: roll it up.
- No difference from 1D?
  Solution: thread flux, quantized charge pumping appears.
- Charge pumping is a winding number of $\det r(\Phi)$:
  $$\mathcal{Q}(r) = \int_0^{2\pi} d\Phi \frac{d}{d\Phi} \operatorname{Im} \log \det r(\phi)$$

Dimensional reduction

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.

Dimensional reduction

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d-1$ dimensions with twisted boundary conditions.

Dimensional reduction

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d - 1$ dimensions with twisted boundary conditions.
3. Calculate $r(\mathbf{k}_{d-1})$.

Dimensional reduction

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d-1$ dimensions with twisted boundary conditions.
3. Calculate $r(\mathbf{k}_{d-1})$.
4. Classify topologically disconnected families of $r(\mathbf{k})$.

Dimensional reduction

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d-1$ dimensions with twisted boundary conditions.
3. Calculate $r(\mathbf{k}_{d-1})$.
4. Classify topologically disconnected families of $r(\mathbf{k})$.

Q: Isn't that a lot of work?

Dimensional reduction II

Idea: reduce problem to a known one.

Dimensional reduction II

Idea: reduce problem to a known one.
With chiral symmetry $\mathcal{C}$, $r(\mathbf{k}) = r^\dagger(\mathbf{k})$, so define

$$H_{d-1}(\mathbf{k}) = r(\mathbf{k})$$

Dimensional reduction II

Idea: reduce problem to a known one.
With chiral symmetry $\mathcal{C}$, $r(\mathbf{k}) = r^\dagger(\mathbf{k})$, so define

$$H_{d-1}(\mathbf{k}) = r(\mathbf{k})$$

Without chiral symmetry define

$$H_{d-1}(\mathbf{k}) = \begin{pmatrix} 0 & r(\mathbf{k}) \\ r^\dagger(\mathbf{k}) & 0 \end{pmatrix}$$

Dimensional reduction II

Idea: reduce problem to a known one.
With chiral symmetry $\mathcal{C}$, $r(\mathbf{k}) = r^\dagger(\mathbf{k})$, so define

$$H_{d-1}(\mathbf{k}) = r(\mathbf{k})$$

Without chiral symmetry define

$$H_{d-1}(\mathbf{k}) = \begin{pmatrix} 0 & r(\mathbf{k}) \\ r^\dagger(\mathbf{k}) & 0 \end{pmatrix}$$

This $H_{d-1}(\mathbf{k})$ has the same topology as $r(\mathbf{k})$,
(Symmetry of H_{d-1} is shifted according to the Kitaev's periodic table.)

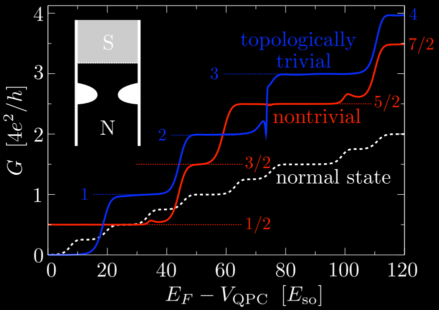
Algorithm to calculate $\mathcal{Q}(S)$

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d-1$ dimensions with twisted boundary conditions.
3. Calculate $r(\mathbf{k}_{d-1})$ and $H_{d-1}(\mathbf{k})$.
4. Look up in the table $\mathcal{Q}(H_{d-1})$.

Symmetry class	d 1	2	3	4	5	6	7	8
A	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$
AIII	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-
AI	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-
AII	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$
CII	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-
C	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-
CI	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-

Summary I

1. Topology of $S(k)$ coincides with that of $H(k)$
2. This provides a highly efficient method to calculate $\mathcal{Q}$
   systems of 2000×2000 vs 60×60 in 2D
   and of $50 \times 50 \times 50$ vs $12 \times 12 \times 12$ in 3D
3. Any observable consequence of topology in transport must be connected to $\mathcal{Q}(S)$



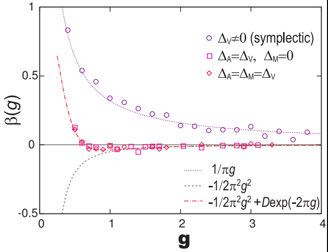
Part II

Non-TI protected surface states

Systems with conducting surface, but without bulk top. invariant:

- 3D topological insulator with random magnetic field (broken time-reversal):
  surface at the critical point of QHE transition, finite conductivity $\sigma \approx 0.5$

  (Nomura, Ryu, Koshito, Mudry, Furusaki)



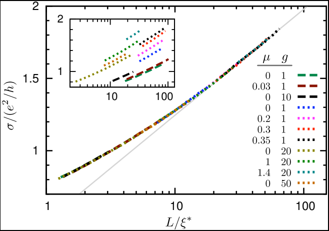
Non-TI protected surface states

Systems with conducting surface, but without bulk top. invariant:

- 3D topological insulator with random magnetic field (broken time-reversal):
  surface at the critical point of QHE transition, finite conductivity
  $\sigma \approx 0.5$

  (Nomura, Ryu, Koshito, Mudry, Furusaki)
- weak topological insulator
  two randomly coupled Dirac cones, always metallic

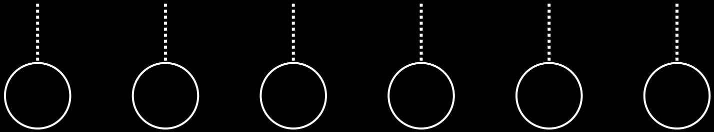
  (Ringel, Kraus, Stern & Mong, Bardarson, Moore)



The general idea: undefined surface topological invariant

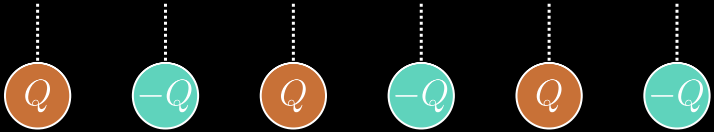
Statistical topological insulators:
Surface of a system *can* be a topological insulator,
but it cannot choose which kind of a TI.
Hence it cannot become an insulator.

Constructing a toy model



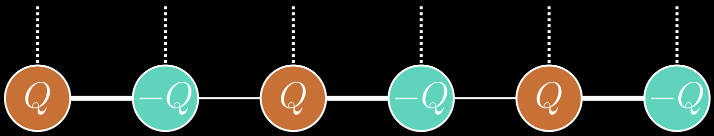
1. Make a layered system

Constructing a toy model



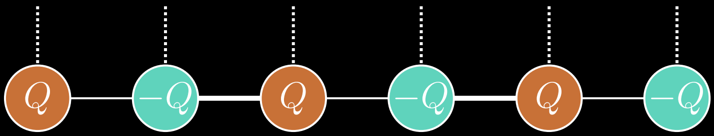
1. Make a layered system
2. Layers carry a staggered topological invariant.

Constructing a toy model



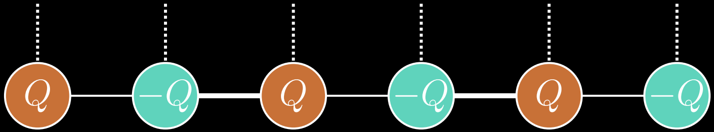
1. Make a layered system
2. Layers carry a staggered topological invariant.

Constructing a toy model



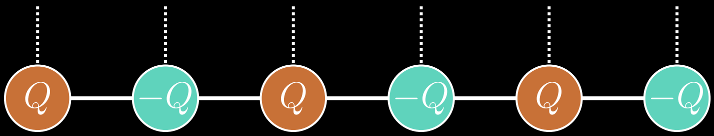
1. Make a layered system
2. Layers carry a staggered topological invariant.

Constructing a toy model



1. Make a layered system
2. Layers carry a staggered topological invariant.
3. Different staggering of couplings changes # of edge states (of the surface).

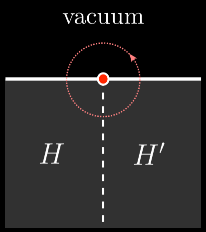
Constructing a toy model



1. Make a layered system
2. Layers carry a staggered topological invariant.
3. Different staggering of couplings changes # of edge states (of the surface).
4. If the ensemble is reflection symmetric, surface cannot be localized.

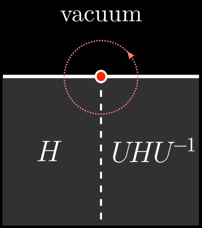
STIs and their bulk invariant ($\mathbb{Z}_2$ symmetry)

- Assume the surface is gapped and has definite topology.



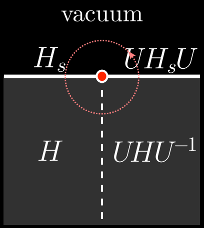
STIs and their bulk invariant ($\mathbb{Z}_2$ symmetry)

- Assume the surface is gapped and has definite topology.
- Consider a 'domain wall' between H and UHU^{-1}:
  it has to carry no protected edge states.



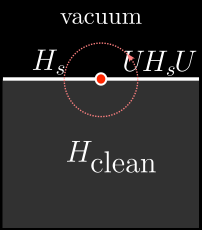
STIs and their bulk invariant ($\mathbb{Z}_2$ symmetry)

- Assume the surface is gapped and has definite topology.
- Consider a 'domain wall' between H and UHU^{-1}:
  it has to carry no protected edge states.
- Add H_s and UH_sU^{-1} on the surface, such that the surface will stay gapped.



STIs and their bulk invariant ($\mathbb{Z}_2$ symmetry)

- Assume the surface is gapped and has definite topology.
- Consider a 'domain wall' between H and UHU^{-1}: it has to carry no protected edge states.
- Add H_s and UH_sU^{-1} on the surface, such that the surface will stay gapped.
- Remove disorder (without closing the bulk gap)

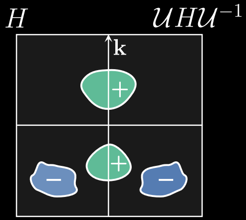


STIs and their bulk invariant (continued)

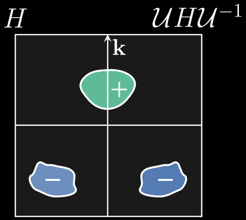
- Number of states at the domain wall can be counted by counting gap closings ΔQ in the path $H_\text{clean} + H_s \to H_\text{clean} + U H_s U^{-1}$.

STIs and their bulk invariant (continued)

- Number of states at the domain wall can be counted by counting gap closings ΔQ in the path $H_{\text{clean}} + H_s \to H_{\text{clean}} + UH_sU^{-1}$.
- $\Delta Q \neq 0$ if H_{clean} has odd # of Fermi surfaces at the surface (odd mirror Chern number, one Majorana per unit cell, etc.)



$\mathcal{Q} \equiv (-1)^{\Delta Q} = +1$, trivial STI.



$\mathcal{Q} \equiv (-1)^{\Delta Q} = -1$, nontrival STI.

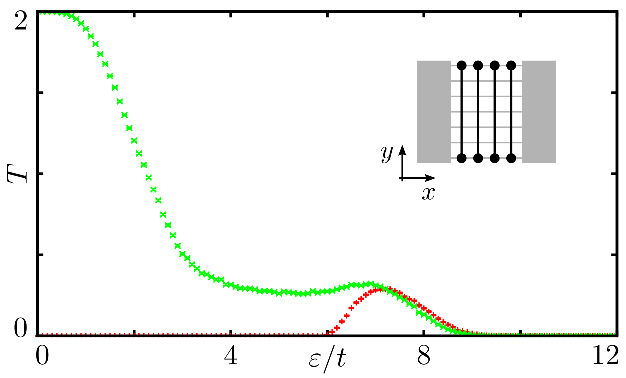
Really good things are good more than once

The construction can be repeated *ad infinitum* by adding extra symmetries and dimensions.

$$\mathcal{Q} = (-1)^{\Delta \mathcal{Q}} = (-1)^{\Delta (-1)^{\Delta Q}}$$

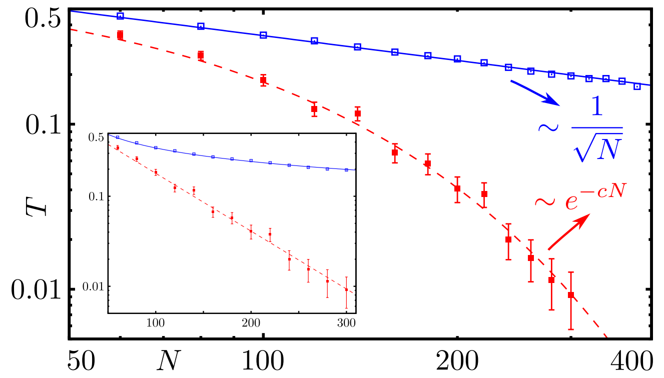
Applications I

Comparison an array of Kitaev chains (p_x-wave superconductor) and a stack of $Q = \pm 2$ BDI wires.



Applications I

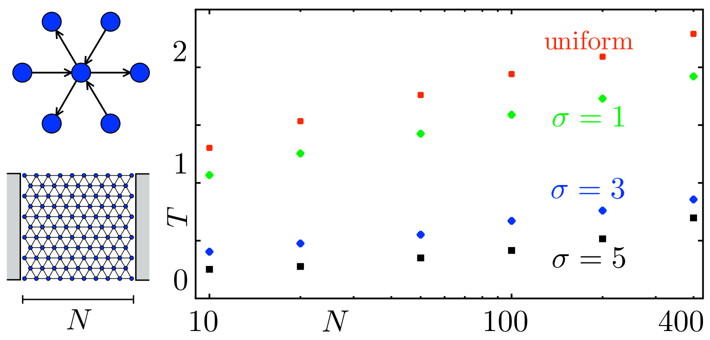
Comparison an array of Kitaev chains (p_x-wave superconductor) and a stack of $Q = \pm 2$ BDI wires.



For Kitaev chains $G \sim L^{-1/2}$; for BDI wires $G \sim e^{-cL}$

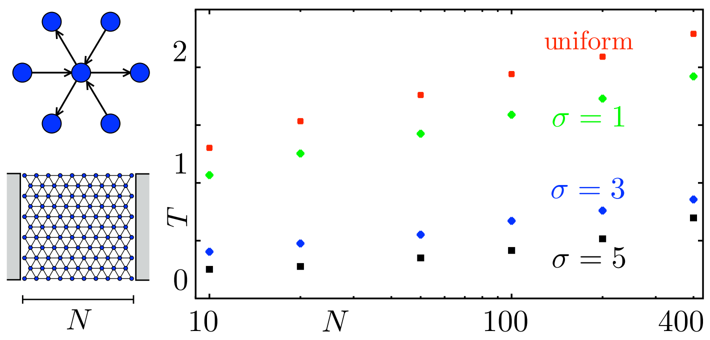
Applications II

Triangular Majorana lattice (Laumann, Ludwig, Huse, Trebst & Kraus, Stern)



Applications II

Triangular Majorana lattice (Laumann, Ludwig, Huse, Trebst & Kraus, Stern)



Always metallic if two statistical reflection symmetries are present:

$$dG/d\log L > 0$$

Summary II

Before:

Symmetry				$d =$							
AZ	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{C}$	1	2	3	4	5	6	7	8
A	0	0	0		✓		✓		✓		✓
AIII	0	0	1	✓		✓		✓		✓	
AI	1	0	0				✓		✓	✓	✓
BDI	1	1	1	✓				✓		✓	✓
D	0	1	0	✓	✓				✓		✓
DIII	-1	1	1	✓	✓	✓				✓	
AII	-1	0	0		✓	✓	✓				✓
CII	-1	-1	1	✓		✓	✓	✓			
C	0	-1	0		✓		✓	✓	✓		
CI	1	-1	1			✓		✓	✓	✓	

Summary II

After:

Symmetry				$d=$							
AZ	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{C}$	1	2	3	4	5	6	7	8
A	0	0	0		✓	✓	✓	✓	✓	✓	✓
AIII	0	0	1	✓	✓	✓	✓	✓	✓	✓	✓
AI	1	0	0				✓	✓	✓	✓	✓
BDI	1	1	1	✓	✓	✓	✓	✓	✓	✓	✓
D	0	1	0	✓	✓	✓	✓	✓	✓	✓	✓
DIII	-1	1	1	✓	✓	✓	✓	✓	✓	✓	✓
AII	-1	0	0		✓	✓	✓	✓	✓	✓	✓
CII	-1	-1	1	✓	✓	✓	✓	✓	✓	✓	✓
C	0	-1	0		✓	✓	✓	✓	✓	✓	✓
CI	1	-1	1			✓	✓	✓	✓	✓	✓

Summary II

1. There are many topological phases protected by a statistical symmetry.
2. These have bulk-edge correspondence and a bulk invariant protecting their surface from localization.
3. Statistical symmetry can be applied to prove the absence of an insulating phase.

Conclusions

- Topology of a topological insulator manifests in one dimension lower through its scattering matrix.
- Adding an ensemble symmetry allows to make a new topological insulator in dimensions higher than the original one.

Conclusions

Thank you all.
The end.