NCG APPROACH TO TOPOLOGICAL INVARIANTS IN CONDENSED MATTER PHYSICS: LECTURE I

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1. Noncommutative Geometry: an Apology

1.1. **Why Do We Need Noncommutative Geometry.** Why do we need such a complicate mathematical theory such as *Noncommutative Geometry* to describe the properties of electrons and phonons in a solid ? Because

$$\text{No Translation Symmetry} \Rightarrow \text{no Bloch Theory}$$

Here is a discussion of the various aperiodic problems Physicists have dealt with in the past, which eventually lead to the need of a better mathematical approach.

1.1.1. *Periodic Media.* In Condensed Matter Physics, the basic tool to describe models and to perform calculations is Bloch's Theory [13]. The main ingredient is to use the invariance of the Hamiltonian under the action of the translation group and to diagonalize simultaneously the Hamiltonian and the unitary group representing the translations. Since the translation group (either $\mathbb{R}^d$ or $\mathbb{Z}^d$, with $d = 1, 2, 3$ in practice) is both Abelian and locally compact, the diagonalization of the unitaries representing it can be done through its group of characters (Pontryagin dual) known as the *Brillouin zone* [16] in Condensed Matter theory and it will be denoted here by $\mathbb{B}$. If the focus of attention is put on the one-electron motion, then for each *quasi-momentum* $k \in \mathbb{B}$, there is a self-adjoint Hamiltonian $H(k) = H(k)^\dagger$ which is mostly a matrix, either finite dimensional, when the energy is restricted to a neighborhood of the Fermi level (in the so called *tight-binding representation*), or infinite dimensional, whenever the continuum version of the Schrödinger operator is considered in full. In the latter case though, $H(k)$ is unbounded, but its spectrum is discrete with finite multiplicity, so that it has a *compact resolvent*. The eigenvalues of $H(k)$ are usually labeled by some index $\{E_b(k)\,;\, b \in \mathcal{B}\}$ representing the band index, or, equivalently, the label of the internal degrees of freedom, such as *spin* or *orbital* in practice. The maps

$$k \in \mathbb{B} \mapsto E_b(k) \in \mathbb{R}\,,$$

are called *band functions* and it can be shown that the energy spectrum is given by the union of the possible values of $E_b(k)$ as both k and b vary. In most cases these functions are extremely smooth, actually they are analytic, but they may be isolated points in the Brillouin zone, at which two or more of these functions coincide, in which case they may exhibit some singularity. An important example is the so-called *Dirac cone*, like in graphene or like in the dynamic of edge

Work supported in part by NSF grant DMS 1160962.

Three Lectures given at the Erwin Schrödinger Institute, Vienna, within the Workshop *Topological Phases of Condensed Matter*, August 4-8, 2014.

states of $3D$-topological insulators. These local singularities have been the focus of attention from the part of Condensed Matter Physicists since they are usually related to unusual behavior of the conduction electrons and such effects can be observed experimentally.

If the focus is put on the many-body problem, in order to take interactions into account, then $H(k)$ is an operator built out of some Fock space representation defined only for finite volume, with a kinetic and an interacting terms. To define rigorously its infinite volume limit it has to be understood as acting on a local observable algebra as a *derivation*, namely the infinitesimal generator of a one-parameter automorphism group [15] (see [14] for the fundamental mathematical tools used in the study of the N-body problem).

1.1.2. *Disorder and Aperiodicity.* However, if the system investigated has disorder or some aperiodic structure encoded in the distribution of atomic sites, the Hamiltonian describing the electronic motion is no longer translation invariant. In addition, if the origin of the coordinates is shifted, namely if the Hamiltonian is translated, while the new Hamiltonian describes exactly the same physics as the old one, **it does not even commute with the old one either !!**

The main idea behind the Noncommutative formalism is to use a C^*-algebra the one generated by the Hamiltonian and its translated.

Why a C^*-algebra ?

Because

C^*-algebras are nothing but Fourier transform without symmetries !

By comparing the algebraic operations required in the Bloch theory, such as *integration* over the Brillouin zone or *differentiation w.r.t.* the quasi-momentum, with what happens in the aperiodic case, it will be realized that the same *Calculus Rules* can be defined in real space, leading to a natural way to write the formulae found in textbooks. It comes though with a twist: the analysis of the behavior of physical quantities as functions of the physical parameters (such as temperature, magnetic or electric fields) are strikingly different in the aperiodic case from the periodic one.

1.1.3. *Electrons in a Uniform Magnetic Field.* The earliest example of such a situation is probably the problem of understanding the electronic motion in a perfectly periodic crystal, submitted to a uniform magnetic field. After the original works of Landau [31] and Peierls [34], it has been realized that the solution of the electronic motion in such a situation was notoriously difficult. This problem has been studied by every serious Solid State Physicists since then, in particular because the occurrence of magnetic oscillations has been a mystery for decades, until Onsager [33] realized that the magnetic field was playing, for the various components of the quasi-momentum, a role similar to the Planck constant in Quantum Physics, namely a semiclassical parameter that makes the theory amenable to effective calculations. In particular, one of the outcome of Onsager's seminal paper is the measurement of the shape and the size of the Fermi surface. It is, nowadays, a standard experimental method to get this kind of informations on whatever sample of some unknown material. It explains why several laboratories have been built throughout the world to create very high magnetic fields, such as in Grenoble, Tallahasse (permanent field) or Toulouse (oscillating field). The famous contribution of Lifshitz and Kosevitch [32] on this topic is considered as the fundamental result by expert of magnetic oscillations. The Peierls problem has led to a series of mathematical results since the early eighties, in particular on the so-called

Harper model. These highly non-trivial results explain why this problem was so difficult since Bloch Theory simply fails and additional techniques of study are required.

1.1.4. *Semiconductor at Very Low Temperature.* Another important class of material exhibiting disorder are semiconductors at very low temperature. After the discovery of transistors by Bardeen and Brattain in 1948, semiconductors became the cornerstone of the expansion of electronic, phone, radio-transmission, television, and eventually computers. The gigantic development of this industry and the enormity of the financing coming out of it has led to remarkable achievements. For example, semiconductors produced today are nearly perfect crystals: the industry is able to offer 2m long and 25cm wide rods of silicon with a density of defects less than one in 10^{10} atoms !

The need to develop further the performances of semiconductors led the Solid State Physics community, helped by funds coming from the industry, to study them in extreme conditions, in particular at very low temperature. Typically if $T < 10K$, the probability for the electrons in the impurity band to jump into the conduction band is vanishingly small. Since the Fermi level is located in the impurity band, it follows that the conduction electrons only see the impurity sites in the semi-conductor. Consequently, these electrons are evolving in a perfectly random environment, since the distribution of impurities is nearly Poissonian, as a consequence of the thermodynamical equilibrium. If the concentration of impurities is the standard one for semiconductors used in electronic and computer devices, namely one implanted impurity for every 10^9-atomic sites, the distance between impurities is too large so that the electrons are confined in the impurity sites. This is the so-called *Anderson's strong localization* phase. If however the density of impurities is 10^{-6}, like in solar cells, for instance, the impurity electrons can jump by tunneling effect form impurity to impurity leading to a kind of weak metallic behavior, in which the electron motion is an interference enhanced diffusion (*Anderson's weak localization regime*).

The immediate lesson drawn from these predictions and observations is that the electron behavior in such materials is qualitatively different from electrons in a perfect crystals. So, not only the failure of Bloch Theory creates a problem for theoreticians, but the physics of electrons in such a situation is completely different!

1.1.5. *Amorphous Materials.* The most important amorphous material for the purpose of the electronic industry and technology is probably the silicon. Without the treatment used in the industry to make it such a perfect crystal, the silicon is spontaneously obtained in an amorphous phase. So many study have been made of the amorphous phase of silicon in order to figure out whether it could be used efficiently.

On the other hand, during the last two decades, some substantial progress have lead material scientists to find a procedure to cast some alloy in a glassy phase [21, 22]. The most preferred class of such materials is called *vitalloy* and is made mostly of copper and zirconium with a variety of possible other atomic species. These materials are becoming the focus of attention of a large crowd of people in material science, because their mechanical properties are so amazing ! They are used nowadays in various devices for surgery, sport or weapon parts. Their main characteristic structure, at the atomic scale, is to exhibit an icosahedral symmetry in the atomic arrangement at short distance, and a liquid like distribution of atoms at large distance. The liquid-solid transition is mostly dynamical, rather than being an equilibrium discontinuity, in that time scales for viscosity, suddenly become very large. This problem has become recently amenable to a rigorous mathematical treatment and the author of these notes is actively involved in this effort, hoping to publish a first paper soon.

1.1.6. *Quasicrystals.* In 1984, Schechtman *et al.* published the famous paper announcing the first observation of a quasicrystalline order [43]. Since then a long list of metallic alloys have been found to share the same properties. Most of them, like $Al_{72.5}Cu_{25}Fe_{12.5}$ or $AlPdMn$, are ternary alloys. However, more recently some binary alloys were found to be nearly perfect quasicrystal with an icosahedral symmetry like $YbCd_{5.7}$ [45]. The main property of these materials is to exhibit a pointlike diffraction spectrum with an icosahedral symmetry that is incompatible with translation invariance. Since the first discovery, these materials have attracted a lot of attention in the Physics community. First because the structure came as a confirmation of a model proposed by Penrose to built a five-fold symmetry tiling of the plane, hence incompatible with translation invariance. Then, because the mechanical and electrical properties of these material are very unusual. For instance, $AlCuFe$ is an alloy made of the three most conducting metals. **In its icosahedral phase, this alloy becomes an insulator !!** None of the standard theory could account for the variety of strange properties found in those materials. At last, because of their exceptional hardness, applied physicists working for aluminium companies were expecting to use those quasicrystals to produce coating, like in motor of an automobile, with a light material. So far this latter expectation has lead to disappointments. But for the purpose of this lecture, these materials are striking media requiring a more serious approach from the mathematical standpoint, in order to overcome the difficulty of not having the Bloch theory to help.

1.1.7. *A Summary of Achievements.* The author of these notes initiated a program of research in the Fall of 1979, centered around using the theory developed at that time by Alain Connes, which became later known as *Noncommutative Geometry* [18]. Step by step, a coherent formalism was developed to describe the properties of the electronic motion in aperiodic materials. Amazingly, the frame offered by Alain Connes allowed to include the description of topological invariants. In particular three important results were found

- **Gap Labeling Theorem:** The so-called *Gap Labeling Theorem* [5, 6, 10, 23, 11, 12] that explains how K-theory is entering into the game for any material in any dimension. It might be considered as a substitute for diffraction spectra whenever it is blurred by the absence of translation invariance.
- **IQHE:** A first-principle derivation of a completely rigorous theory describing the Integer Quantum Hall Effect [7], including the existence and the quantization of the plateaux for the transverse conductivity, due to Anderson's strong localization regime.
- **Bulk-Edge Correspondence:** It has been realized that the bulk-edge correspondence could be expressed through the crossed-product analog of the Bott maps, namely the Pimsner-Voiculescu exact sequence [39, 26, 27, 28].

Besides, several other results contributed for a better understanding of the electronic properties of aperiodic solid, such as

- **Kubo's Formula:** A noncommutative version of Kubo's formula in the so-called *Relaxation Time Approximation* (RTA) [38, 8]. A consequence was the prediction of an *anomalous Drude formula* relating the conductivity to the relaxation time, explaining the difference between various types of conductivity. In particular it gives a solution to the strange scaling laws observed in the conductivity of quasicrystal [8]. More recently, a description of the variable range hopping transport in semiconductors at very low temperature, a regime in which the RTA fails, was provided in [4]. It involves a many-body formalism to take into account the Fermi statistics of electrons and their effective interaction through phonon emission.

Structure of Aperiodic Solids: Starting with the work of Kellendonk [24, 25], a relation has been established between the previous formulation of the theory of aperiodic media and the theory of tilings (in any dimension). The seminal paper by Anderson and Putnam [3] has allowed to represent an aperiodic ordered system with *Finite Local Complexity* in terms of a family of CW-complexes, leading to an explicit representation of its *Hull*, a compact space on which the translation group acts by homeomorphisms, which plays the role of the set of configurations of disorder. This was supplemented by an important work of Lagarias [29, 30] who introduced the concept of *Delone set* [19] to describe the distribution of the atomic nuclei positions in a solid.

Numerics: Emil Prodan is responsible for having developed an accurate and stable numerical algorithm, based on the C^*-algebra-approach, allowing to compute the electric conductivity in the presence of disorder and a uniform magnetic field [36, 47, 42].

Topological Insulators: The calculation of the magneto-electric response in presence of disorder [40]. A proof of the stability of the topological part of the magneto-electric response in $3D$ topological insulators [37]. Persistence of spin-edge current in presence of disorder [41].

1.2. **An Example of Aperiodic System.** Let us consider a lattice system in dimension d representing the electronic motion in a disordered medium. In the one-particle space a typical tight-binding model can be described by

$$(1)\quad H\psi_\alpha(x) = \sum_{x';\langle x'\, x\rangle} t(x')_\alpha^\beta\Big(\psi_\beta(x') - \psi_\beta(x)\Big) + V(x)_\alpha^\beta\psi_\beta(x)\,, \qquad \psi_\alpha \in \ell^2(\mathbb{Z}^d)\,,\;\; \alpha,\beta \in \mathcal{B}\,.$$

Here, $\langle x'\, x\rangle$ means that x' is nearest neighbor to x. Moreover, the indices $\alpha, \beta \in \mathcal{B}$ is a labeling of possible internal degrees of freedom on site, such as spin or orbitals. In such a model either the hopping terms coefficients $t(x)$ or the potential one $V(x)$ or both will be identically distributed independent random variables ($i.i.d$). However, for the purpose of this argument, the randomness is not essential. It is essential though that both $t(x)$ and $V(x)$ be bounded. Hence the family $(t(x), V(x))_{x\in\mathbb{Z}^d}$ belongs to a space Ω which has the property of being a *topological space* which is *compact*. Compactness is a profound mathematical property leading to many outstanding results. However, this concept might not be well understood by non mathematicians. Roughly speaking:

A space is compact if it can be approximate to any accuracy by a finite subset.

Mathematically this can be expressed as follows: from any cover of the space by open sets, one can extract a finite sub-cover.

As a consequence of the previous description of our model, the Hamiltonian H depends effectively on a choice ω of a point in Ω. For this reason H will rather be written as H_ω to make this dependence more explicit. Hence instead of one Hamiltonian, the system is represented by a *family* $H = (H_\omega)_{\omega\in\Omega}$ of Hamiltonians, *all of which are equally representing the same physical system.* This family is submitted to two important conditions

Covariance: If $U(a)$ denotes the Unitary operator translating the wave function by $a \in \mathbb{Z}^d$, then

$$U(a) H_\omega U(a)^\dagger = H_{\mathrm{T}^a \omega} \,,$$

where T^a represents the configuration of the disorder after being translated by a. In particular $\mathbb{Z}^d$ acts on Ω. In order to preserve the topology $\mathrm{T}^a : \Omega \to \Omega$ must be continuous. Moreover, translating by b first then by a should be equivalent to translating by $a+b$ so that $\mathrm{T}^a \circ \mathrm{T}^b = \mathrm{T}^{a+b}$. In particular T^0 should not change the configuration, namely $\mathrm{T}^0 = id$ while $\mathrm{T}^{-a} \circ \mathrm{T}^a = \mathrm{T}^0 = id$, which shows that T^a is invertible, with inverse map T^{-a} and therefore this inverse is continuous. Namely $\mathbb{Z}^d$ acts by *homeomorphisms*.

Strong Continuity: The second property is more mathematical it says that the map $\omega \in \Omega \mapsto H_\omega$ must be continuous in some sense. It means that the set of operators must be endowed with a topology that will allow to make this statement precise. There are two topologies that could be considered. The first is the *norm topology*. The norm of an operator is defined by

$$\|H\| = \sup\{\|H\psi\| \,;\, \psi \in \mathcal{H} \,,\, ||\psi\| \leq 1\} \,,$$

where $\mathcal{H}$ denotes the Hilbert space of quantum states. However an intensive study may show that such a continuity occurs only if the Hamiltonian H describes an almost periodic system [5] ! The other topology is called the *strong topology*. It is defined by demanding that, for any state ψ in the Hilbert space the following map

$$\omega \in \Omega \mapsto H_\omega \psi \in \mathcal{H} \,,$$

be continuous. Since any state $\psi \in \mathcal{H}$ is *localized* in some sense, this strong continuity expresses the fact that the Hamiltonian is only locally continuous.

Before going forward, it is important to notice that any aperiodic medium admits a space Ω of the previous type, namely compact, with an action of the translation group through homeomorphisms. Even if the system is not random, Ω can be constructed by considering all possible translations of the system, looking at them locally through a fixed window (of any possible size), and adding all possible limits as seen in all possible such windows. The result is called the *Hull*. The Hull for aperiodic solids has been introduced in [5], but over time, more and more details came out in order to describe the Hull and even to compute it. For instance the Hull of an icosahedral quasicrystal looks like a torus in dimension 6, with a topology defined by an irrational 3-space and with a Cantor like topology transversally to this 3-space [9].

1.3. **Building the C^*-algebra.** The set $\mathcal{A}$ of families $A = (A_\omega)_{\omega\in\Omega}$ of operator satisfying to the properties of covariance and strong continuity, makes up a C^*-algebra ! For indeed the following algebraic operations can be defined

Sum: If A and B are two such families then $A + B$ is defined as the family $A + B = (A_\omega + B_\omega)_{\omega\in\Omega}$.
Scalar Multiplication: If $\lambda \in \mathbb{C}$ is a scalar then $\lambda A = (\lambda A_\omega)_{\omega\in\Omega}$.
Product: If A, B are two such families, then $AB = (A_\omega B_\omega)_{\omega\in\Omega}$ defines a product.
Adjoint: Similarly $A^\dagger = (A^\dagger_\omega)_{\omega\in\Omega}$.

As can be seen these algebraic operation are exactly what it takes for $\mathcal{A}$ to be a $*$-algebra. Now, there is a need of a norm, in order to be able to check whether an infinite series is convergent or not. For instance, the usual perturbation theory for, say the Green's functions or for the *resolvent operator*, requires indeed such an infinite series. The most obvious way to define such a norm consists in choosing

$$\|A\| = \sup_{\omega\in\Omega} \|A_\omega\|\,.$$

This leads to the question addressed by the next Section.

2. What is a C^*-algebra ?

A mathematician would define dryly a C^*-algebra in those terms

Definition 1. *(i) A $*$-algebra $\mathcal{A}$ is a complex vector space equipped with a product $(a,b)\in\mathcal{A}\times\mathcal{A}\mapsto ab\in\mathcal{A}$ which is associative and distributive w.r.t. the addition and scalar multiplication. It is also equipped with an adjoint map $a\in\mathcal{A}\mapsto a^\dagger\in\mathcal{A}$ which is antilinear and satisfies $(ab)^\dagger = b^\dagger a^\dagger$.*
(ii) A Banach $*$-algebra is a $*$-algebra equipped with a norm $a\in\mathcal{A}\mapsto\|a\|\in[0,\infty)$ satisfying $\|ab\|\le\|a\|\|b\|$, $\|a^\dagger\|=\|a\|$ and for which it is complete.
(iii) A C^*-algebra is a Banach $*$-algebra for which $\|a^\dagger a\| = \|a\|^2$.

A more intuitive answer to the question is more involved. Defining this concept has been a challenge which took almost thirty years to be established. However a look at History might help understanding why mathematicians ended up with this concept. It has been already advocated earlier in this Lecture that *C^*-algebras are Fourier transform without symmetry.* We intend to develop this idea more fully right now.

2.1. **Fourier Transform.** The story starts in 1807, with a seminal paper written by Joseph Fourier [20], then the Préfet of the "Département de l'Isère", the representative of the Napoleonic state in Grenoble. He was a very busy man, an official in charge of distributing the money provided by the nation taxes to the representatives of local governments. As such, he had administrative duties, occupying most of his time during the day, and he had also mundane duties, offering frequent dinners in the evening, during which deals were made between various local power representatives. Nevertheless, Fourier managed to have several seminal contributions to science, helping Champollion, a fellow living in Grenoble, in translating the "Rosetta Stone" message, a key to translate the ancient hieroglyphs Egyptian writing. In his 1807 paper, Fourier investigates primarily the propagation of heat in solids. He defines the concept of heat capacity (unknown before) and derived the *Heat Equation.* It is worth mentioning in passing that the heat equation is a cornerstone in several area today, such as the theory of Brownian motion in modern Probability, and also its importance in analysis and the study of partial differential equations (PDE). For example, one proof of the Atiyah-Singer Index Theorem was using the heat equation as well. Then Fourier managed to solve this equation in some elementary situation, such as an homogeneous rod with square section. In order to solve the heat equation, Fourier claims that

Fourier (1807): all "functions" can be expanded in trigonometric series

In other words, using a modern version of this claim, Fourier assumes that if $f:[0,1]\to\mathbb{C}$ is a complex valued function, then it can be expanded as

$$f(x) = \sum_{n=-\infty}^{+\infty} \widehat{f}_n \, e^{2\imath\pi nx} ,$$

and he gives a formula to compute the *Fourier* coefficients

$$\widehat{f}_n = \int_0^1 f(x) \, e^{-2\imath\pi nx} \, dx .$$

Of course this presentation is anachronistic. For indeed Fourier used sine and cosine instead. In addition his parameter x was measuring the coordinates of points in the rod so it was not rescaled. But my modern statement is equivalent to his. The only question he did not address, in the true spirit of the Euler 18th-century point of view on computation, was the problem of convergence of these series. And there are good reasons to explain why he did not consider this point. Because he wanted to get a quick result: why bother, since the result can be summed up easily ? But also because the convergence problem is so complex that it led to the development of an entire new discipline in modern mathematics, called *Analysis*.

This first step led to many more steps afterwards. It is worth mentioning some of them. The *Bessel* inequality was a first step towards understanding the convergence problem. Much later came the *Parseval identity* stating the

$$\sum_{n\in\mathbb{Z}} |\widehat{f}|^2 = \int_0^1 |f(x)|^2 \, dx .$$

In the second half of the 19th century, mathematicians discovered many examples of other ways to expand a function. The discovery of the Legendre, Laguerre, Hermite, Jacobi polynomials, for instance, led to strong analogies with the Fourier transform, especially because the Parseval identity seemed to be universal. This led to the definition of the concept of Hilbert space, a rigorous algebraic framework giving a geometrical meaning to these various expansions.

2.2. **Pontryagin Duality and Brillouin Zone.** The story could have stopped there. But in 1934, Pontryagin published a seminal paper that gave the story a completely different course. Namely Pontryagin looked at Fourier transform from the point of view of *symmetries*. In Fourier's analysis there is an extra properties that had been overlooked, in particular in the theory of orthogonal polynomials: translation invariance. Coming back to the previous example, the functions investigated by Fourier have the property of being *periodic*. That is to say

$$f(x+1) = f(x) , \qquad \text{namely} \qquad x \sim x+1 \;\Leftrightarrow\; x \in \mathbb{R}/\mathbb{Z} .$$

In particular f is actually a function over the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The space $\mathbb{R}/\mathbb{Z}$ is the set of equivalence classes of real number under the identification of x and $x+n$ whenever n is an integer. Both $\mathbb{R}$ and $\mathbb{Z}$ are groups, with the law of addition. In addition $\mathbb{Z}$ is an *invariant subgroup* of $\mathbb{R}$. Consequently, their quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is also a group. Moreover these three groups are *Abelian* and *topological*. Thanks to the De Moivre-Euler formula, the trigonometric map

$$x \in \mathbb{T} \mapsto e^{2\imath\pi x} = \cos 2\pi x + \imath \sin 2\pi x \in \mathbb{S}^1 = U(1)$$

is a group isomorphism between the torus $\mathbb{T}$ (which measures angles) and the *unit circle*, denoted by $\mathbb{S}^1$ or by $U(1)$ depending upon which mathematical sub-culture is preferred. Pontryagin realized that the unit circle play the role of a universal reference in the theory of Abelian groups, analogous to the role of $\mathbb{R}$ or of $\mathbb{C}$ in the theory of vector spaces. Then he realized that if G is an Abelian topological group, it is possible to define the concept of *character*, namely maps $\chi : G \to U(1)$, that preserve group operations. Hence

$$\chi(0) = 1\,, \qquad \chi(g+g') = \chi(g)\chi(g')\,, \qquad g, g' \in G\,.$$

The set G^* of characters is itself a topological Abelian group. The group law is just the pointwise multiplication $(\chi\chi')(g) = \chi(g)\chi'(g)$. The neutral element is the constant character $\chi(g) = 1$ for all g's. The topology is called *weak** is the pointwise convergence, namely a sequence $(\chi_n)_{n\in\mathbb{N}}$ of characters converges to χ if and only if $\lim_{n\to\infty}\chi_n(g) = \chi(g)$ for all $g \in G$. Then the limit is automatically a character. Two important results are

(1) if G is locally compact, then so is G^*,
(2) if G is locally compact, the bi-dual $(G^*)^*$ coincides with G itself

In the case of $\mathbb{T}$ any character is given by an integer $n \in \mathbb{Z}$ by $\chi_n : x \mapsto e^{2\imath\pi nx}$. Hence $\mathbb{Z} = (\mathbb{T})^*$. This gives the Fourier theory a group theory basis. Analogously any character of $\mathbb{R}^d$ is defined by a vector $p \in \mathbb{R}^d$ as $\chi_p : x \in \mathbb{R}^d \mapsto e^{\imath px} \in U(1)$. Hence $\mathbb{R}^d$ is isomorphic to its dual. In Physics, a character of $\mathbb{R}^d$ can be identified with the representation of the translation group and therefore p represents the generator of these translations. Consequently, thanks to Noether's Theorem, p can be identified with the *momentum*.

In Solid State Physics, the symmetry problem comes as follows. Let $\mathcal{L}$ denotes the set of the positions in $\mathbb{R}^d$ of the atoms of a crystal. Thanks to the crystalline property, there is a translation subgroup $G \subset \mathbb{R}^d$ which has the property of leaving $\mathcal{L}$ invariant. Then the quotient set $\mathcal{L}/G$ is called the *Bravais zone*, and $\mathcal{L}$ is a Bravais lattice whenever the Bravais zone is reduce to one point. To make sure that translations in G are in sufficient number, it is required that the quotient $\mathbb{R}^d/G$ (which is a group) be compact: then G is said to be *co-compact*.
The dual group G^* is nothing but the quasi-momentun space, called the *Brillouin zone*. Actually, since $G \subset \mathbb{R}^d$, any character on $\mathbb{R}^d$, namely any momentum, restricts to G as a character of G. Now two momenta p and p' admits the same restriction to G if and only if $e^{\imath(p-p')\cdot x} = 1$ for all $x \in G$. That means $p - p'$ belong to the *reciprocal lattice* which, in group theory is called the *orthogonal to* G and is denoted by $G^\perp$. Then $G^* = \mathbb{R}^d/G^\perp$. This explains the name *quasi-momentum* given to a point in the Brillouin zone $\mathbb{B} = G^*$. The original Brillouin zones, as defined by Brillouin himself, are only the *Voronoi cells* of the dual lattice. But using periodic boundary condition permits to identify them with the dual group G^*.

Like in Fourier's theory, Pontryagin generalized the concept of Fourier transform as follows: let $f : G \to \mathbb{C}$ be a continuous function, vanishing at infinity fast enough (say with compact support), then its Fourier transform is defined by

$$\widehat{f}(\chi) = \int_G f(g)\,\chi(g)\,dg\,, \qquad \chi \in G^*\,,$$

where dg denotes the *Haar measure* on G, namely the unique positive measure (modulo normalization) that is invariant by left translations on G. Then he manages to check that among the Haar measures on G^* there is one $d\chi$ such that the inverse Fourier formula and the Parseval identity hold, namely

$$f(g) = \int_{G^*} \widehat{f}(\chi)\,\overline{\chi(g)}\,d\chi\,, \qquad \int_G |f(g)|^2\,dg = \int_{G^*} |\widehat{f}(\chi)|^2\,d\chi\,.$$

One important outcome of this theory is the Fourier transform on finite groups, with a seminal application to computer science, through the algorithm called *Fast Fourier Transform* (FFT) using the group $\mathbb{Z}/2^n\mathbb{Z}$.

2.3. **The work of Gelfand.** In a series of papers published between 1940 and 1950, Gelfand, with the help of several colleagues, like Neumark, Raĭkov, Šilov, Naĭmark, investigated systematically the various classes of normed rings and their embedding into the space of operators on a Hilbert space. Gelfand, a former PhD student of Kolmogorov, was part of the Moscow community of Mathematics and was well aware of the work of Pontryagin. Ironically, Pontryagin was an influential antisemite member of the Academy of Science, while Gelfand was of Jewish and Ukrainian origin. Nevertheless, Gelfand found his way through the difficulties of the Stalinist era and became one of the most prolific and profound mathematicians of the 20th century. He died at the eage of 96 while he was still an active professor at the University of Rutgers in New-Jersey.

One idea that came immediately was to extend the concept of character, defined in group theory, to the case of algebras. In such a case, the unit circle $U(1)$ is naturally replaced by the simplest possible algebra containing it, namely the field $\mathbb{C}$ of complex numbers. Hence, if $\mathcal{A}$ is a complex algebra, a character will be an algebraic homomorphism $\chi : \mathcal{A} \to \mathbb{C}$.

As its stand, such a study is too broad and some restriction are required in order to go further. In view of the possibility for $\mathcal{A}$ to be infinite dimensional, the first restriction is to endow it with a normed, namely a map $a \in \mathcal{A} \mapsto \|a\| \in [0, \infty)$ such that

(i) $\|a\| > 0$ for $a \neq 0$,
(ii) $\|\lambda a\| = |\lambda|\|a\|$ if $\lambda \in \mathbb{C}$ is a scalar,
(iii) $\|a + b\| \leq \|a\| + \|b\|$ (triangle inequality), for $a, b \in \mathcal{A}$,
(iv) $\|ab\| \leq \|a\|\,\|b\|$ for $a, b \in \mathcal{A}$,
(v) endowed with this norm, $\mathcal{A}$ is complete.

Then $\mathcal{A}$ is called a *Banach algebra*.

Motivated by the study of compact groups and their unitary representations, Gelfand added to these axioms, following the ideas developed earlier by von Neumann, the existence of an adjoint, and made $\mathcal{A}$ into a $*$-algebra. Namely any $a \in \mathcal{A}$ admits an adjoint $a^\dagger$ satisfying the following property: if a, b are elements of $\mathcal{A}$ and λ is a scalar,

(a) $(a + b)^\dagger = a^\dagger + b^\dagger$,
(b) $(\lambda a)^\dagger = \overline{\lambda} a^\dagger$,
(c) $(ab)^\dagger = b^\dagger a^\dagger$,
(d) the norm satisfies $\|a^\dagger\| = \|a\|$.

In such a situation, whenever $\mathcal{A}$ is Abelian, namely if $ab = ba$ for all $a, b \in \mathcal{A}$, the set of *continuous* characters $X(\mathcal{A})$ such that $\chi(a^\dagger) = \overline{\chi(a)}$, can be endowed with a weak* topology like in the case of Abelian group studied by Pontryagin. Then using general results on Banach space, it follows immediately that $X(\mathcal{A})$ is locally compact. Moreover, if $a \in \mathcal{A}$, the map $\mathcal{G}_a : \chi \in X(\mathcal{A}) \mapsto \chi(a) \in \mathbb{C}$ is, by definition, a continuous function on $X(\mathcal{A})$. In addition, it not too hard to show that if $X(\mathcal{A})$ is not compact, this function vanishes at infinity. The set of such functions is usually denoted by $\mathcal{C}_0(X(\mathcal{A}))$. The Gelfand Theorem asserts that if, in addition the norm of $\mathcal{A}$ satisfies

$$\|a^\dagger a\| = \|a\|^2 \,, \tag{2}$$

then the map $a \in \mathcal{A} \mapsto \mathcal{G}_a \in \mathcal{C}_0(X(\mathcal{A}))$, called the *Gelfand Transform*, is a $*$-algebraic isomorphism provided the norm on $\mathcal{C}_0(X(\mathcal{A}))$ be the sup-norm, namely

$$\|f\| = \sup_{X(\mathcal{A})} |f(\chi)| .$$

To summarize

Theorem 1 (Gelfand Theorem)**.** *Any Abelian C^*-algebra is isomorphic to the space $\mathcal{C}_0(X)$ of continuous function on a locally compact space X, vanishing at infinity. The space X is unique modulo homeomorphisms and is called the spectrum of the algebra. If, in addition, the algebra admits a unit, then X is compact.*

In other word,

Abelian C^*-algebras are locally compact spaces

Gelfand's transform extends Fourier's Transform

No symmetry Group but an Algebra

2.4. **Non-Abelian C^*-algebras.** It remains to understand the case of non-Abelian algebras. From 1943 on, the axiom given in eq. (2) is postulated in Gelfand's work. However there was some type of insecurity expressed in these early papers, since an additional condition with also required, namely $(1 + a^\dagger a)$ should be invertible. It took a long time to get rid of this extra condition. For indeed, the set of elements of a C^*-algebra of the form $a = b^\dagger b$, is a convex cone which is closed under the norm topology. This means that a norm limit of such elements is still of this form, a non trivial fact. As a consequence, $(1 + a^\dagger a)$ is automatically invertible. An element of this cone is called *positive* and it is denoted by $a \geq 0$.

What is so special about axiom (2) ? The answer was given to me by Alain Connes in the early eighties:

The norm on a C^*-algebra has purely algebraic origin

Why ? There are two ways to see that. The first is to appeal to the concept of *spectral radius*. Namely, assuming that $\mathcal{A}$ has a unit, the spectrum of an element $a \in \mathcal{A}$ is the set of complex numbers $z \in \mathbb{C}$ such that $(z - a)$ does not admit an inverse in $\mathcal{A}$. Such a set is automatically contained in the disk centered at the origin with radius $\|a\|$. However, in general, there might be a smaller such disk containing the spectrum. The spectral radius is the minimum radius of such disks. Then it is possible to check that whenever $a \geq 0$, the spectrum is contained in the interval $[0, \|a\|]$ and $\|a\|$ is exactly the spectral radius. Since the spectrum is about invertibility, the spectral radius of an element is only an algebraic property, there is no topology in such a concept. Hence for any $a \in \mathcal{A}$ $a^\dagger a$ is positive, by definition so that its norm coincide with its spectral radius, namely

The C^*-norm $\|a\|$ is the square root of the spectral radius of $a^\dagger a$.

There is another way to understand Connes claim. Namely if $\mathcal{A}, \mathcal{B}$ are two C^*-algebras, and if $\phi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism (which means that ϕ respect all the algebraic operations including the adjoint) then automatically $\|\phi(a)\| \leq \|a\|$. In particular *all algebraic homomorphism*, because they respect the adjoint, *are continuous*. In addition, if ϕ is one-to-one, then $\|\phi(a)\| = \|a\|$, which means that it also preserves the norm ! This shows that transporting the algebraic structure also transports the norm.

An important example of C^*-algebras is the set of all bounded operators in a Hilbert space. For indeed if $\mathcal{H}$ is a Hilbert space and if $T : \mathcal{H} \to \mathcal{H}$ is linear and continuous, then $\|T\|$ is defined as

$$\|T\| = \sup\{\|T\psi\| \,;\, \psi \in \mathcal{H}\,,\, \|\psi\| = 1\}$$

It is a standard exercise to show that $\|T^\dagger T\| = \|T\|^2$ in this case. The set of such operators is usually denoted by $\mathcal{B}(\mathcal{H})$. As a consequence the best way to define a C^*-norm on an algebra is to *represent* it as a set of bounded operators on some Hilbert space. As a converse, the Gelfand-Naĭmark-Segal (GNS) construction, shows that any C^*-algebra can be seen as a closed sub-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

2.5. **Non-Abelian C^*-algebras and Fourier Analysis.** Why should one consider non Abelian C^*-algebras as a substitute for Fourier Theory when symmetries are absent ? Here are two arguments in favor of seeing a C^*-algebra as the generalization of Fourier's transform.

First, in view of the apology given previously for C^*-algebras, it is only natural to think of a (non-commutative) C^*-algebra as the analog of the space of continuous function on some locally compact space, vanishing at infinity. Such a locally compact space does not necessarily exists as a set. But the usual operations made on topological space can be translated into algebraic properties of continuous functions, that may be extended to non-Abelian C^*-algebras.

Another argument concerns the case of the C^*-algebra associated with a locally compact group. If G is a locally compact group, then it supports a left-invariant (*resp.* a right-invariant) positive Borel measure $\lambda(dx)$ (*resp.* $\rho(dx)$), which is unique modulo a normalization factor. This measure is called the *Haar measure*, where the normalization and the invariant type are kept implicit. It is not true, in general, that the left-invariant and the right invariant Haar measure coincide. However, it is known that they differ by a multiplicative factor Δ, called the *modular function*

$$\rho(dx) = \Delta(x)\,\lambda(dx)\,, \qquad \Delta(x) > 0\,, \qquad \Delta(xy) = \Delta(x)\Delta(y)\,, \;\; x, y \in G\,.$$

In addition, the change of variable $x \in G \mapsto x^{-1} \in G$ leads to the exchange of the left and right Haar measures. A group is called *unimodular* whenever its modular function is identically equal to 1. Examples are Abelian groups, or compact groups. However the group generated by the 2×2 matrices of the form

$$x = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \qquad a, b \in \mathbb{R}\,,\; a \neq 0\,,$$

is not unimodular: its left-invariant Haar measure is $dadb/a^2$ while its right-invariant one is $dadb/|a|$ as can be checked immediately.

In any case, it becomes possible to define the Hilbert space $L^2(G)$, namely the space of functions defined on G, with complex values, which are square integrable *w.r.t.* the left Haar measure (the same construction can be made for the right-invariant Haar measure). The inner product is defined by

$$\langle\psi|\phi\rangle = \int_G \overline{\psi}(x)\,\phi(x)\,\lambda(dx)\,.$$

In addition, the space $\mathcal{C}_c(G)$ of complex valued continuous functions on G with *compact support* becomes a $*$-algebra if endowed with the left convolution product and the adjoint defined by

$$fg(x) = \int_G f(y)g(y^{-1}x)\, dy\,, \qquad\qquad f^\dagger(x) = \Delta(x)\,\overline{f(x^{-1})}\,.$$

This algebra is not Abelian whenever G is not Abelian. In addition this algebra is represented in $L^2(G)$ by bounded operator as follows

$$\pi_L(f)\psi(x) = \int_G f(y)\,\psi(y^{-1}x)\,\lambda(dx)\,.$$

π_L is called the *left-regular representation*. An elementary exercise shows that indeed π_L preserves the algebraic operations. Then a C^*-norm can be defined by setting

$$f \in \mathcal{C}_c(G) \qquad \Rightarrow \qquad \|f\|_{red} = \|\pi_L(f)\|\,.$$

The completion of $\mathcal{C}_c(G)$ *w.r.t.* this norm is called the *reduced* group-C^*-algebra of G and is denoted by $C^*_{red}(G)$. This name comes from the fact that there might be other C^*-norms and that the reduced norm is the smallest among them. This phenomenon happens whenever the group is not *amenable*, which is the case for *free groups* or *hyperbolic groups.*

If the group G is abelian then $C^*(G)$ is isomorphic to the space $\mathcal{C}_0(G^*)$ of continuous functions vanishing at infinity on the Pontryagin dual of G. Namely the spectrum of $C^*(G)$ is G^*.

In addition, whenever G is compact, such as the finite dimensional *orthogonal, unitary or symplectic groups*, then $\mathcal{C}_c(G)$ coincides with the space $\mathcal{C}(G)$ of all complex valued continuous function on G and there is a unique C^*-norm on $\mathcal{C}(G)$. In addition, the Peter-Weyl Theorem permits to prove that

(i) any irreducible representation of $C^*(G)$ is finite dimensional and is given by the corresponding irreducible representation of G itself,

(ii) the left-regular representation can be decomposed into a direct sum of all irreducible representations and each such representation occurs with a multiplicity equal to its dimension,

(iii) each function in $C^*(G)$ can be decomposed into a Fourier series *in which the Fourier coefficients are matrices in one of the irreducible representation space,*

(iv) an inverse Fourier formula and a Parseval identity can be established as well.

This gives the connection with the Fourier transform: $C^*(G)$ contains exactly all the ingredients necessary to define and investigate the Fourier transform of functions on groups. Hence passing from the a group-C^*-algebra to a general C^*-algebra allows to thing of a noncommutative C^*-algebra as the "Fourier transform" of the virtual noncommutative locally compact space it represents.

2.6. **Noncommutative Geometry: Philosophy.** In view of the apology given previously for C^*-algebras, it is only natural to think of a (non-commutative) C^*-algebra as the analog of the space of continuous function, vanishing at infinity, on some locally compact space. Such a locally compact space does not necessarily exists as a set. But the philosophy of Noncommutative Geometry is to work on a C^*-algebra as if it were coming from some locally compact space.

As far as the algebra is concerned, only the topology of the space is encoded. However it is possible to encode both parts of Calculus: (i) Integration (ii) Differential Calculus

Measurability: A *state* on a C^*-algebra is a linear bounded map $\phi : \mathcal{A} \to \mathbb{C}$ such that

Positivity: $\phi(a^\dagger a) \geq 0$ for all $a \in \mathcal{A}$,

Normalisation: $\|\phi\| = \sup\{|\phi(a)|\,;\, \|a\| \leq 1\} = 1$

A state is the noncommutative analog of a *probability distribution*. A *weight* is the unbounded version of a state and correspond to positive unbounded measures. Give a state or a weight, the GNS-construction produces a Hilbert space which is the analog of the L^2-space defined by the state. The C^*-algebra acts on this Hilbert space through the so-called *GNS-representation*. The weak-closure of the C^*-algebra in this space is a *von Neumann algebra*, namely the noncommutative analog of the L^∞-space. It is also possible to generalize the concept of L^p-space.

Differential Structure: While noncommutative measurability is well understood today, the situation with differentiability is still under scrutiny. A classical manifold is usually defined through local coordinates charts and changes of coordinates. This has no analog so far in noncommutative geometry. Actually localizing is impossible in general, because of the mathematical analog of "tunneling" which prevent quantum phase space to be splits into pieces. In addition smooth manifolds have a "tangent space", again a concept that has no equivalent in non commutative geometry in general. However, vectors fields can be seen as acting on functions and this can be defined. To summarize

- **$*$-Derivations:** The analog of *vector fields* are $*$-derivations. A differential structure is defined by a family of $*$-derivations.
- **Vector Bundles:** A noncommutative vector bundle can be defined either through projection or through a projective module. The latter coincides with the analog of sections of a bundle. The concept of *connection* can also be defined noncommutatively. *Complex Hermitian vector bundles* can be described through the concept of *Hilbert-C^*-module.*
- **Riemannian Geometry:** The main tool to study noncommutative Riemannian Geometry is the concept of *Spectral Triple*. However the last decade of study have shown that it describes rather metric spaces instead, including ultrametric Cantors sets, fractals. Nevertheless, this concept is helping proposing various ways to built the analog of Laplace-Beltrami operators, a subject of extreme importance in Metric Geometry today. The main difficulty with this formalism is that there is no obvious candidate for the concept of *curvature*. Metric Geometry has found such a concept for the class of *Alexander spaces* which may not be manifolds but retain several aspects of Riemannian Geometry like the concept of *geodesic* of angle between geodesics and even scalar curvature. So far several ways are investigated by mathematician to extend the concept of curvature to noncommutative spaces, but none of these proposal have reached the status of a consensus nor efficiency.
- **Differential Forms:** *Cyclic Cohomology*, initiated by Connes and by Quillen, in different context has been a successful landmark to generalize usual cohomology theory to the noncommutative world. In particular all topological invariants proposed by condensed matter physicists, can, in principle be calculated through using cyclic cocycles. This is the case for Quantum Hall Effect or for the magneto-electric response in topological insulators.

2.7. **Why C^*-algebras ? Because...** ... because as explained above, C^*-algebras are the latest development in Fourier Theory. Because C^*-algebras were already in the seminal 1925 paper by Heisenberg that gave the foundation of Quantum Mechanics. Because the Schrödinger point of view using wave functions to describe quantum states instead of observables, a point of view much preferred in the Physics Community, has a lot of problems when used in Solid

State Physics. For indeed, while the algebraic point of view can handle the infinite volume limit pretty well, this is not always the case with the wave function theory. Famous results like the *orthogonality catastrophy* have no equivalent in the algebraic theory.

On the other hand the lack of Bloch Theory for aperiodic system is a permanent problem for physicists while the tools of Noncommutative Geometry can handle this problem carefully, realistically and now also numerically, including the question of topological invariants.

3. Observable Algebra

After having indulged ourselves in a flight high in the sky to see the big picture behind Noncommutative Geometry, it is time to return to the ground and to learn how to use this tool. As will be seen the algebra of covariant and strongly continuous observable defined in Section 1.2 is not yet in a state that makes calculations practical. In this section Calculus rules will be described making the tool amenable to practical use.

3.1. **Matrix Elements Formulation.** Going back to the example described in Section 1.2, the system under study lives on the lattice $\mathbb{Z}^d$. The Hilbert space of quantum states is $\mathcal{H} = \ell^2(\mathbb{Z})\otimes\mathbb{C}^{\mathcal{B}}$ where the the finite dimensional space $\mathbb{C}^{\mathcal{B}}$ accounts for the discrete indices describing bands, orbitals and spin. In addition, the disorder configurations are encoded in the compact space Ω, which is endowed with a $\mathbb{Z}^d$-action through homeomorphism, denoted by T. Let then $A = (A_\omega)_{\omega\in\Omega}$ be an observable. Thanks to the *covariance condition*, its matrix elements satisfy

$$\langle x, \alpha| A_\omega |y, \beta\rangle = \langle 0, \alpha| A_{\mathrm{T}^{-x}\omega} |y-x, \beta\rangle = A_{\alpha,\beta}(\mathrm{T}^{-x}\omega, y-x)\,.$$

In other words, in addition to the band indices, the matrix elements depend only upon two variables, namely $\omega\in\Omega$ and $y-x$ which describes the labeling of the diagonals. For $x=y$ represents the main diagonal, but as $y-x$ grows larger it labels diagonal farther form the main one.

In addition, thanks to the *strong-continuity* of the family of operators A *w.r.t.* ω, the *function* $A_{\alpha,\beta}(\omega, x) = \langle 0, \alpha| A_\omega |x, \beta\rangle$ is also continuous *w.r.t.* $(\omega, x)\in\Omega\times\mathbb{Z}^d$. Consequently, to describe the family of operators $A = (A_\omega)_{\omega\in\Omega}$ it is sufficient to give the matrix valued map $A(\omega, x)$. What follows are the rules describing the algebraic operations on the observable algebra.

Function Space: Like for the Hamiltonian (1), it will be assumed that $A(\omega, x) = 0$ for $|x|$ large enough. Namely $A\in\mathcal{C}_c(\Omega\times\mathbb{Z}^d)\otimes M_{\mathcal{A}}(\mathbb{C}) = \mathcal{A}_0$ is a *matrix valued continuous function with compact support.*

Vector Space: The addition and scalar multiplication in $\mathcal{A}_0$ are defined pointwise.

Product: The matrix elements formula for the product of two observables leads to the following definition

$$AB(\omega, x) = \sum_{y\in\mathbb{Z}^d} A(\omega, y)\, B(\mathrm{T}^{-y}\omega, x-y)\,. \tag{3}$$

If a magnetic field is present, a term $e^{\imath Bx\wedge y}$ should be added inside the sum. As can be seen this is a convolution product, but twisted by disorder and by the magnetic field.

Adjoint: The formula for the matrix elements of the adjoint is

$$A^\dagger(\omega, x) = A(\mathrm{T}^{-x}\omega, -x)^\dagger\,.$$

$\mathcal{A}_0$ is $*$-algebra: Endowed with the previous operations, $\mathcal{A}_0$ becomes a $*$-algebra.

Representation: Let $\pi_\omega(A)$ denotes the operator previously called A_ω, namely $\pi_\omega(A)$ acts on a vector $\psi \in \mathcal{H}$ as follows

$$(\pi_\omega(A)\psi)_\alpha(x) = \sum_{\beta\in\mathcal{A}} \sum_{y\in\mathbb{Z}^d} A_{\alpha,\beta}(\mathrm{T}^{-x}\omega, y-x)\,\psi_\beta(y)\,.$$

Consequently the map $\pi_\omega : A \in \mathcal{A}_0 \mapsto \pi_\omega(A) \in \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism.

Norm: As suggested in Section 1.2, the C^*-norm is described by

$$\|A\| = \sup_{\omega\in\Omega} \|\pi_\omega(A)\|\,.$$

Then $\mathcal{A}_0$ becomes a normed $*$-algebra. However $\mathcal{A}_0$ is *not complete* ! To understand this, let us consider the following *power series*, corresponding to approximating the *resolvent* $(z-H)^{-1}$,

$$R_N(z) = \frac{1}{z} + \frac{H}{z^2} + \cdots + \frac{H^N}{z^{N+1}}\,.$$

If $|z| > \|H\|$, a short calculation shows that, for any $M > N$

$$\|R_N(z) - R_M(z)\| \le \left(\frac{\|H\|}{|z|}\right)^N \frac{1}{|z| - \|H\|}\,.$$

In particular the sequence $(R_N(z))_{N\in\mathbb{N}}$ is Cauchy. However, the matrix elements $R_N(z;\omega,x)$ gets more and more non zero diagonal as $N \to \infty$, so that it is not true that the limit, which exists, has a finite number of non vanishing diagonals. Consequently this limit is not in $\mathcal{A}_0$. Therefore there is a need to *complete* the algebra $\mathcal{A}_0$ in order to include also the limit points. The *completion* $\mathcal{A}$ is usually denoted by

$$\mathcal{A} \stackrel{\text{def}}{=} \overline{\mathcal{A}_0}^{\|\cdot\|} \stackrel{\text{def}}{=} C^*(\Omega \rtimes_{\mathrm{T}} \mathbb{Z}^d) \otimes M_{\mathcal{B}}(\mathbb{C})\,.$$

Here the definition of $\mathcal{A}$ corresponds to the reduced algebra, but since $\mathbb{Z}^d$ is amenable, in the present case all C^*-norms coincide.

3.2. **The Noncommutative Brillouin Zone.** What is the meaning of the C^*-algebra $\mathcal{A}$? As advocated in Section 2.4, $\mathcal{A}$ should be seen a the set of continuous function on a non commutative topological space. Actually, by analogy with the Gelfand Theorem, this topological space is locally compact. In the present case, in addition, because the set of lattice sites is *discrete*, $\mathcal{A}$ admits a unit given by

$$1_{\alpha,\beta}(\omega, x) = \delta_{\alpha,\beta}\,\delta_{x,0}\,.$$

Consequently this locally compact space is actually compact. But because of the noncommutativity of the algebra, such a space is not really existing as an object that can be manipulated. Nevertheless, by comparing with the periodic case, one can easily understand the physical meaning of it. For indeed, if there is no disorder, then Ω is just trivial, namely it is reduce to a point, on which the $\mathbb{Z}^d$ action is also trivial. Hence in the eq. (3), ω can be forgotten, leading to

$$AB(x) = \sum_{y\in\mathbb{Z}^d} A(y)\,B(x-y)\,,$$

which is nothing but the *convolution product* over the Abelian group $\mathbb{Z}^d$!! In order to simplify the formula, it becomes convenient to use the usual Fourier transform:

$$\widehat{A}(k) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} A(x)\, e^{2\imath\pi k\cdot x}\,.$$

This gives a continuous function defined on the Brillouin zone $k \in \mathbb{B}$. Then, the product formula becomes

$$\widehat{AB}(k) = \widehat{A}(k)\widehat{B}(k)\,,$$

namely this is the *pointwise product* over the Brillouin zone. Similarly

$$\widehat{A^\dagger}(k) = (\widehat{A}(k))^\dagger\,.$$

In view of the Gelfand Theorem, this is nothing but the algebra of matrix valued continuous functions over the Brillouin zone !! Consequently, in general, even if the system becomes aperiodic, the C^*-algebra $\mathcal{A}$ will be interpreted as the set of continuous functions over the Brillouin zone. However, now, because of the magnetic field and/or of the aperiodicity, this *Brillouin zone becomes noncommutative.*

3.3. **Dual Action.** The dual group to $\mathbb{Z}^d$, which has been interpreted as the Brillouin zone before, actually acts on the C^*-algebra $\mathcal{A}$: this *dual action* was pointed out by Connes in his PhD Thesis [17] and by Takesaki for von Neumann algebras [46]. The definition of this dual action for C^*-algebras is due to Takai [44].

Given a quasi-momentum $k \in \mathbb{B}$, let η_k be the map defined on $\mathcal{A}_0$ by

$$\eta_k(A)(\omega, x) = e^{\imath k\cdot x}\, A(\omega, x)\,.$$

This operation is nothing but the *translation by k in the Brillouin zone.* It is elementary to check that (i) η_k is linear, (ii) $\eta_k(AB) = \eta_k(A)\,\eta_k(B)$, (iii) $\eta_k(A)^\dagger = \eta_k(A^\dagger)$, hence defining a $*$-homomorphism from $\mathcal{A}_0$ into itself. In addition, $\eta_{k+k'} = \eta_k \circ \eta_{k'}$ and $\eta_0 = id$, as can be checked immediately. In particular η_k is actually an automorphism. In view of the properties of the C^*-norm reported in Section 2.4, this implies that η_k is preserving the norm and as such can be extended to the whole algebra. At last it can be checked that, given any $A \in \mathcal{A}$, the map $k \in \mathbb{B} \mapsto \eta_k(A) \in \mathcal{A}$ is *norm continuous.*

3.4. **Calculus: Integration over the Brillouin Zone.** As explained previously, since $\mathcal{A}$ is supposed to represent the space of continuous function on the Noncommutative Brillouin zone, it is legitimate to address the question of knowing whether such a function can be integrated. The answer is yes ! To see this let us consider the average over the quasi-momentum

$$\int_{\mathbb{B}} \eta_k(A)(\omega, x)\; dk = A(\omega, 0)\; \delta_{x,0}\,.$$

The result is not a scalar, but a matrix valued function over the space Ω describing the configurations of disorder. In order to get a scalar, the main idea is to *average* also over the disorder and to take the *trace* over the band indices. How can we do that ! By choosing on Ω a *probability measure* with the property of being translation invariant. Let $\mathbb{P}$ be such a probability. Then

$$\mathcal{T}_{\mathbb{P}}(A) \stackrel{\text{def}}{=} \int_\Omega \mathrm{Tr}\,\left(A(\omega, 0)\right)\; d\mathbb{P}(\omega)\,.$$

The remarkable fact is that $\mathcal{T}_{\mathbb{P}}$ is a *trace.* Namely

(i) $\mathcal{T}_{\mathbb{P}} : \mathcal{A} \mapsto \mathbb{C}$ is linear,

(ii) for $A \in \mathcal{A}$, then $\mathcal{T}_{\mathbb{P}}(A^\dagger A) \geq 0$ (Check it !),
(iii) $\mathcal{T}_{\mathbb{P}}(1) = 1$ (Normalization)
(iii) for $A, B \in \mathcal{A}$, then $\mathcal{T}_{\mathbb{P}}(AB) = \mathcal{T}_{\mathbb{P}}(BA)$ (Trace property) ! This last property comes from the translation invariance of $\mathbb{P}$.

In general, there might be several possible choices for $\mathbb{P}$. However, the space of such invariant probabilities over Ω is a convex set, which generalizes the concept of simplex (segment, triangle, tetrahedron,...) to infinite dimensional spaces. Any simplex has extremal points and it has been shown in the past that extremal invariant probability measures are nothing but the ones which are *ergodic*. Then, using the Birkhoff ergodic Theorem, it becomes possible to prove the following

$$\mathcal{T}_{\mathbb{P}}(A) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \operatorname{Tr}\left(\pi_\omega(A) \restriction_\Lambda\right), \qquad \text{for } \mathbb{P} \text{ almost every } \omega\,.$$

In other words, $\mathcal{T}_{\mathbb{P}}$ is nothing but a *trace per unit volume* provided $\mathbb{P}$ is $\mathbb{Z}^d$-invariant and ergodic.
Question: is there a *natural* choice for $\mathbb{P}$?
Answer: mathematically every ergodic invariant measure will do. However physics gives a criterion to choose $\mathbb{P}$. Indeed the lattice underlying our system is coming from the distribution of atoms and the electrons under consideration are all coming from the bands crossing the Fermi level. These atoms are crystallizing because they are in thermal equilibrium. If some disorder occurs, it is also because of the thermal equilibrium, hence the space Ω represents the possible *atomic configurations at equilibrium*. Therefore it is only natural to choose for $\mathbb{P}$ the Gibbs measure representing mathematically such an equilibrium.

3.5. **Calculus: Differentiating in the Brillouin zone.** It was argued in Section 3.3 that the dual action represents translating the observable in the quasi-momentum space. Hence differentiating with respect to the quasi-momentum is just an infinitesimal translation, namely, if $\{e_1, \cdots, e_d\}$ represents the standard basis in $\mathbb{R}^d$ (seen as the tangent space to $\mathbb{B}$), then

$$\partial_i A = \lim_{\epsilon \downarrow 0} \frac{\eta_{\epsilon e_i}(A) - A}{\epsilon}\,.$$

Written in matrix form gives, with $x_i = \langle e_i | x \rangle$

$$(\partial_i A)(\omega, x) = \imath x_i\, A(\omega, x)\,, \qquad \nabla = (\partial_1, \cdots, \partial_d)\,.$$

The remarkable fact is that, the representation of ∇A is given by the *commutator by the position operator* (check it !)

$$\pi_\omega(\nabla A) = -\imath[X, \pi_\omega(A)]\,, \qquad X = (X_1, \cdots, X_d) \text{ is the position operator}\,.$$

By construction, ∇ is linear, is invariant by the adjoint and it satisfies the Leibniz formula

$$\nabla(A^\dagger) = (\nabla A)^\dagger\,, \qquad \nabla(AB) = (\nabla A)B + A(\nabla B)\,.$$

In addition, it generates an automorphism group, namely, by construction it is the infinitesimal generator of $(\eta_k)_{k \in \mathbb{B}}$

$$e^{k \cdot \nabla}(A) = \eta_k(A)\,.$$

This leads to the definition of differential one-forms as

$$dA = \sum_{i=1}^{d} \partial_i A \, dk_i$$

Since $\partial_i \partial_j = \partial_j \partial_i$ it follows that $d^2 A = 0$. One example of use consists in taking the Fermi projection $A = P_F$ of the Hamiltonian. If the Fermi level is in a gap, P_F belongs to $\mathcal{A}$. Then

$$P_F dP_F \, dP_F = \sum_{i<j} P_F [\partial_i P_F, \partial_j P_F] \, dk_i \wedge dk_j \, .$$

3.6. **Universality of the Hull Ω.** In Section 1.2 the space Ω was introduced as a representative of the disordered configurations. However, as soon as a system loose periodicity, there is a compact metrizable space Ω with an action of the translation group (namely $\mathbb{R}^d$ in most cases, but sometime $\mathbb{Z}^d$ if there is an underlying crystal) by homeomorphisms. This space is usually called the *Hull*. Relevant examples of aperiodic situations are, in increasing order of complexity (i) isolated impurities, (ii) modulated crystals, (iii) quasicrystal, (iv) tiling with *finite local complexity*, (v) amorphous and glassy materials, (vi) disordered materials.

In order to define the Hull, the procedure is always the same. One usually starts from a reference configuration of atoms or from the coefficients of the Hamiltonian, depending of the problem investigated. Then one considers the *orbit* of this reference object under the translation group. The third step is to define a topology, which is *"local"*, namely the translated objects are all considered in finite *windows* (for instance a bounded open set when looking at an atomic configuration in $\mathbb{R}^d$). Then a sequence of such translated is said to converges if in each window the trace of these objects do converge. The Hull is the closure of the orbit under this topology. In almost all cases known so far, the condition on the object are such that

Compactness: The orbit closure is compact. Practically, this means that, *given any window, and any error ϵ, the number of possible configurations seen in that window modulo the error ϵ is finite.*

Metrizability: The construction always leads to a metrizable space, namely the topology can be defined by a metric, even if such a metric is neither unique nor canonical.

Translation Group: The translation group acts on the Hull by homeomorphisms.

The Hull has been computed in many situations. For a local impurity on a fixed crystal, the Hull is a copy of $\mathbb{Z}^d$ compactified by one point at infinity. For tiling having tiles touching face-to-face, with a finite number of tiles modulo translation (finite local complexity), which is the case for quasicrystals, the Hull is a compact space foliated by the translation group $\mathbb{R}^d$ with transversal given by the Cantor set. For glassy materials, the Hull is the space of all Delone sets with a given set of parameters. Conversely, given any compact metrizable set with an $\mathbb{R}^d$ action, admitting at least one dense orbit (topological transitivity) it is always possible to built an atomic configuration admitting this space as a Hull.

3.7. **Textbook Formulae in Solid State Physics.** Using the dictionary provided by the Table 1 it becomes possible to transform every formula found in textbooks using the Bloch theory into formulae in the Noncommutative formalism. The first example concerns the electron density n_{el} [7]

$$n_{el} = \mathcal{T}_{\mathbb{P}}\left(f_{\beta,\mu}\right) , \qquad\qquad f_{\beta,\mu} = \frac{1}{1 + e^{\beta(H-\mu)}} \, .$$

Periodic Media	Aperiodic Media
Integration over Brillouin zone	Trace per unit Volume on the NCBZ
Derivative w.r.t. quasimomentum	$\nabla A(\omega, x) = \imath x\, A(\omega, x)$

Table 1. Analogy between Bloch Theory and NCG

Here, $f_{\beta,\mu}$ is the Fermi-Dirac *density matrix*, which belongs to $\mathcal{A}$ and

$$\beta = \frac{1}{k_B T} \;\; (\text{inverse temperature})\,, \qquad\qquad \mu = \text{chemical potential}$$

Another example is the *Kubo formula* that can be written as follows in the *relaxation time approximation* (RTA) [38, 8]. Namely if one assumes that all dissipative phenomena affecting the electron are described by one time scale τ_{rel} then the electric conductivity tensor is given by

$$\sigma_{ij} = \frac{e^2}{\hbar}\, \mathcal{T}_{\mathbb{P}} \left(\partial_j f_{\beta,\mu} \frac{1}{1/\tau_{rel} + \mathcal{L}_H}\, \partial_i H \right)\,, \qquad\qquad \mathcal{L}_H = \frac{\imath}{\hbar}[H, \cdot]\,,$$

Using the Noncommutative Geometry formalism, Prodan and his collaborators [36, 47, 42] have been able to design a code allowing for the numerical computation of these quantities. The result is numerically very stable and much more performant than previous numerical schemes.

3.8. **Non Continuous Smooth Functions and Localization.** One of the most fascinating result of using Noncommutative Geometry concerns the localization theory [7]. Before explaining this result, it is worth remembering that if a function $f : [0,1] \to \mathbb{C}$ defined on the unit interval of $\mathbb{R}$ admits almost everywhere a derivative which is square integrable, then a use of the Cauchy-Schwarz inequality implies that

$$|f(t) - f(s)| = \left| \int_s^t f'(u)\, du \right| \leq |t-s|^{1/2} \, \left(\int_s^t |f'(u)^2\, du \right)^{1/2}\,, \qquad\qquad 0 \leq s \leq t \leq 1\,.$$

In particular f is automatically continuous. A suitable generalization of this argument in higher dimension shows that if a function $f : U \subset \mathbb{R}^d \to \mathbb{C}$, defined on a bounded open set U in $\mathbb{R}^d$, admits enough square integrable derivatives, then it must be continuous.

Such a result needs not be true on noncommutative spaces. The Anderson localization is one example of such a strange phenomenon. The first ingredient is the concept of *density of states* (DoS), defined as the derivative of the *integrated density of states* (IDS). Namely the IDS is a nondecreasing function of the energy E, counting the number of energy eigenstates per unit volume available through the Hamiltonian, with energy less than E. Not surprisingly this function can be expressed in the noncommutative formalism by using the trace per unit volume [5]

$$\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}\left(P(E)\right) , \qquad\qquad \textbf{(Shubin's Formula)} ,$$

where $P(E)$ is the projection onto the states of the Hamiltonian with energy at most E. This formula, initially due to Shubin, in the context of differential operators with almost periodic coefficients, can be re-expressed as

$$\int_{\mathbb{R}} f(E)\, d\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}\left(f(H)\right) , \qquad\qquad f \in \mathcal{C}_c(\mathbb{R}) .$$

One of the important result in the non-commutative theory of localization is that if $\Delta \subset \mathbb{R}$ is an interval of energy in which all states are localized, then the *localization length* can be defined in Δ as a function $E \in \Delta \mapsto \ell(E)$ which is square integrable *w.r.t.* the DoS and such that

$$\mathcal{T}_{\mathbb{P}}\left(|\nabla P(E_1) - \nabla P(E_0)|^2\right) \leq \int_{E_0}^{E_1} \ell(E)^2\, d\mathcal{N}(E) , \qquad\qquad E_0, E_1 \in \Delta .$$

The definition of the localization length can be done in several ways, either through the dynamics or through the Green's functions, using the *current-current correlation function* (see [7] for a long discussion of this definition). As it turns out both definitions coincide. Using various approach proposed between 1979 and the end of the nineties, in particular the techniques developed by Aizenman and Mochanov [2] in the version of [1], it is possible to prove that a random perturbation of the Landau Hamiltonian admits such intervals Δ between two consecutive Landau levels, and using these results, it is possible to prove that the localization length is actually well defined and square integrable *w.r.t.* the DoS. As a consequence, the projections $P(E)$ are differentiable in the sense of Sobolev. Actually, using those techniques, it is even possible to prove that for any power n

$$\mathcal{T}_{\mathbb{P}}\left(|\nabla^n P(E)|^2\right) < \infty ,$$

In other words, $P(E)$ is infinitely differentiable in the sense that all of its derivatives are square integrable over the Brillouin zone. However, the same technique show that if the disorder is large enough, there is no gap in the energy spectrum. One of the consequence is that $P(E)$ does not belong to the observable algebra $\mathcal{A}$. In other words

In the localization regime, the Fermi projection is infinitely smooth without being continuous !

This is a property of noncommutative spaces: they can support infinitely differentiable functions which are not continuous. And the physicists equivalent is precisely the Anderson strong localization regime !

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