Spectral flow and the essential spectrum

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Preamble

Motivation for this work comes from the uses of spectral flow in condensed matter theory (where operators can have very complicated spectra).

Fredholm theory is about the discrete spectrum of a self adjoint operator. Scattering theory probes the essential spectrum. In this talk I will mention some results inspired by scattering theory.

Typically a self adjoint Fredholm operator, particularly those arising in quantum theory, will have both discrete and essential spectrum. Most[1] of the mathematical theory is however, devoted to operators with compact resolvent as arises in the study of compact manifolds.

On non-compact manifolds, which is where scattering theory is studied, the essential spectrum is both physically and mathematically important.

[1] except of course for type II theory

On the mathematical side it may contain topological information (a good example is provided by the Novikov-Shubin invariants for covering spaces). Note that the 'type II' spectral flow is relevant to this discussion as well.

Furthermore there are many examples of non-Fredholm operators that arise naturally on non-compact manifolds and it is of interest to know if spectral geometry can be developed for them.

Colleagues contributing to this work: Nurulla Azamov, Peter Dodds, Fritz Gesztesy, Harald Grosse, Jens Kaad, Galina Levitina, Denis Potapov, Fedor Sukochev, Yuri Tomilov, Dima Zanin among many others.

Part 1. Spectral flow and spectral shift

In the 1950s I. Lifshitz introduced the spectral shift function. He was motivated by studying the S-matrix in quantum scattering problems. This spectral shift function was made mathematically precise (in the framework of trace class perturbations) by M. Krein in his famous 1953 paper.

For a pair of self-adjoint bounded operators A_0 and A_1 such that their difference (being the perturbation) is trace class, there exists a unique $\xi \in L^1(\mathbb{R})$ satisfying the trace formula:

$$Tr(\phi(A_1) - \phi(A_0)) = \int \xi(\mu)\phi'(\mu)d\mu$$

whenever ϕ belongs to a class of admissible functions.

NB ξ is not determined pointwise in general but only almost everywhere.

1.1 Spectral flow

At first sight this is an unrelated concept.

Recall that spectral flow was introduced in the index theory papers of Atiyah-Patodi-Singer in the 1970's. They give Lusztig the credit for the idea.

Consider a norm continuous path $F_t; t \in [0,1]$ of bounded self adjoint Fredholms joining F_1 and F_0. J. Phillips in 94-95 introduced an analytic definition of spectral flow that is more useful than the original approach of APS.

We let, for each t, P_t be the spectral projection of F_t corresponding to the non-negative reals. Then we can write $F_t = (2P_t - 1)|F_t|$. Phillips showed that if one subdivides the path into small intervals $[t_j, t_{j+1}]$ such that $P_{t_j}, P_{t_{j+1}}$ are 'close' in the Calkin algebra then they form a Fredholm pair and the spectral flow along $\{F_t\}$ is

$$\sum_j \text{index}(P_{t_j} P_{t_{j+1}} : \text{range } P_{t_{j+1}} \to \text{range } P_{t_j})$$

1.2 Recent interactions between spectral flow and spectral shift

• To my knowledge the first person to understand that spectral shift and spectral flow are the same (when both are defined as occurs in some examples) was Werner Mueller (Bonn) in 1998.

• In 2007, (with Nurulla Azamov, Peter Dodds and Fedor Sukochev), we showed that under very general conditions[2] that guarantee that both are defined, spectral shift and spectral flow are the same.

Questions

• The spectral shift function is defined more generally, even for non-Fredholm operators. Can it be regarded as giving a generalisation of spectral flow in the non-Fredholm case?

• Index/spectral flow theory is built today around Kasparov theory (and this is tied closely to Fredholm operators). What replaces Kasparov theory in the non-Fredholm case if the spectral shift function is giving some generalised notion of spectral flow?

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[2] and also in semi-finite von Neumann algebras

Answers

• We have proved some index/spectral flow formulas when the Fredholm operators in question have some essential spectrum. These formulas extend to the non-Fredholm case providing a partial answer to the first question.

• We have shown in some examples that the replacement for K-theory (or Kasparov theory) in the non-Fredholm case is cyclic homology.

Part 2: The Witten index: a proposal for the non-Fredholm case

Witten was studying supersymmetric quantum field theory and as a toy model introduced supersymmetric quantum mechanics in the early 1980s.

Mathematically we interpret ‘supersymmetry’ as a $\mathbb{Z}_2$ grading.

Witten speculated that there might be a notion of index for non-Fredholm operators and proposed a formula for this, extending an idea due to Callias.

Witten’s idea was taken up by mathematical physicists, Bollé, Gesztesy, Grosse and Simon in 1987 and Borisov, Schrader, Mueller in 1988. They were able to give it a mathematical basis.

Early results

• Bollé et al found simple examples in two dimensions of Dirac operators coupled to connections that had only continuous spectrum for which Witten's formula gave a finite answer.

• When these operators are Fredholm then Witten's formula gives the Fredholm index (even if the operators have some essential spectrum). In non-Fredholm situations the formula can give any real number.

• They proved a stability result that gave invariance of the Witten index for relatively trace class perturbations but not compact perturbations in the non-Fredholm case.

• Callias studied a three dimensional example but unfortunately his methods are mathematically incomplete. All other early examples of the Witten index were one and two dimensional.

2.1 Gesztesy-SImon

The first substantial theoretical advance was due to Gesztesy- Simon (1987) in a subsequent paper.

We are given a Hilbert space equipped with a $\mathbb{Z}_2$ grading γ and a self adjoint unbounded operator $\mathcal{D}$ such that $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$.

For simplicity take the space to be $\mathcal{H}^{(2)} := \mathcal{H} \oplus \mathcal{H}$ for a fixed separable infinite dimensional Hilbert space $\mathcal{H}$. Then we may write:

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

Gesztesy-Simon consider, for z in the intersection of the resolvent sets, the situation where the difference

$$(z + \mathcal{D}_-\mathcal{D}_+)^{-1} - (z + \mathcal{D}_+\mathcal{D}_-)^{-1}$$

is in the trace class but the individual operators are not trace class.

Then they define nonnegative self-adjoint operators

$$H_1 = \mathcal{D}_- \mathcal{D}_+; \quad H_2 = \mathcal{D}_+ \mathcal{D}_-$$

and introduce the spectral shift function $\xi(\mu; H_2; H_1)$ associated with a pair $(H_2; H_1)$ such that $\xi(\mu; H_2; H_1) = 0$, $\mu < 0$ and

$$\mathrm{Tr}[(z + H_1)^{-1} - (z + H_2)^{-1}] = -\int_0^\infty \xi(\mu; H_2; H_1)(\mu + z)^{-2} d\mu$$

and they introduce the 'Witten index' of $\mathcal{D}_+$ as

$$\lim_{z \to 0} z \mathrm{Tr}[(z + H_1)^{-1} - (z + H_2)^{-1}]$$

whenever the limit exists. They prove that in some cases it is given by $\lim_{\mu \downarrow 0} \xi(\mu; H_2; H_1)$.

Heat kernel approach

Gesztesy-Simon also introduced a heat kernel approach. A more extensive treatment of the heat kernel approach linking it to both index theory and scattering theory proper was developed by N. V. Borisov, W. Müller and R. Schrader in 1988.

They did not discuss operators with ‘nasty spectrum’ as occur in condensed matter theory or the non-Fredholm case but did lay out a general set of results for geometric scattering problems in higher dimensions using the heat kernel approach to the Witten index and ‘supersymmetric scattering theory’.

2.2 Relation with spectral flow

Consider spectral flow along a path of self adjoint Fredholm operators $\{A(s)|s \in \mathbb{R}\}$ joining A_0 and A_1 on $\mathcal{K}$.

We convert this into an index problem on a $\mathbb{Z}_2$ graded space by introducing on $L^2(\mathbb{R}, \mathcal{K})$ the operators:

$$\mathcal{D}_+ = \frac{\partial}{\partial s} + A(s), \quad \mathcal{D}_- = \mathcal{D}_+^*$$

and on $L^2(\mathbb{R}, \mathcal{K}^{(2)})$ the operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

Recently it was shown by GLMST that for Fredholm operators $A_\pm$ with essential spectrum there is a situation where the index of $\mathcal{D}_+$ is the spectral flow along the path $A(s)$.

2.3 The GLMST trace formula.

In a long paper in Advances in Mathematics in 2011 GLMST established a trace formula that relates the Fredholm index of $\mathcal{D}_+$ with spectral flow along the path $A(s)$ even when there is essential spectrum.

But a very restrictive hypothesis was needed.

HYPOTHESIS: the relatively trace class perturbation assumption holds namely: $(A_+ - A_-)(1 + A_-^2)^{-1/2}$ is trace class.

This means their result does not apply directly to geometric situations and differential operators

The GLMST trace formula is:

$$\operatorname{tr}_{L^2(\mathbb{R},\mathcal{K}^2)} z\big((H_2 + z)^{-1} - (H_1 + z)^{-1}\big) = \frac{1}{2} \operatorname{tr}_{\mathcal{K}} \left(g_z(A_+) - g_z(A_-)\right), \quad (1)$$

where $g_z(x) = x(x^2 + z)^{-1/2}$. The operator differences on both sides are trace class under the HYPOTHESIS.

They also showed that (in the Fredholm case) if we take $\lim_{z\to 0}$ on both sides then the LHS is the Fredholm index and the RHS is spectral flow.

In recent work we have shown:

(i) The GLMST formula holds for relatively trace class perturbations in the non-Fredholm case. Now when we take $\lim_{z\to 0}$ the LHS is the Witten index and the RHS is given by the spectral shift function at zero for the pair $A_\pm$.

This supports the idea that the spectral shift function for the pair $A_\pm$ is a generalised spectral flow.

(ii) The GLMST formula can be applied to certain pseudodifferential operators in all dimensions.

(iii) The formula can be extended to the situation of relatively Hilbert-Schmidt perturbations and this allows one and two dimensional differential operator examples both Fredholm and non-Fredholm.

We are thinking of these results as showing that when the path does not consist of Fredholm operators, we may think of spectral flow as being replaced by the spectral shift function and the Fredholm index by the Witten index.

BUT the explicit examples for differential operators (not pseudo-differential operators) are greatly restricted. We have generalised spectral flow in one dimension= the Witten index in two dimensions.

For applications we want to know what happens in two and three dimensions.

2.4 Speculation on higher dimensions

For inspiration we turn to some older work on operators with ‘compact’ resolvent.

In BCPRSW, we established a formula for Dirac type operators on covering spaces. The set-up there is that we have a simply connected space $\tilde{M}$ admitting an action by a discrete group Γ such that the quotient $M = \tilde{M}/\Gamma$ is a compact manifold.

Assume $A_{\pm}$ are unitarily equivalent Γ-invariant Dirac type operators on $\tilde{M}$ and τ_{Γ} is the Γ trace of Atiyah.

We were interested in the ‘type II spectral flow’ along a path $A(s), s \in [0, 1]$ of Γ invariant self adjoint operators joining unitarily equivalent $A_{\pm}$.

This is defined by taking Phillips analytic approach to spectral flow and generalising it to the type II setting.

Note that in this type II setting operators can have continuous spectrum including zero but still be ‘Breuer-Fredholm’.

We form $\mathcal{D}_\pm$ as before (that is $\mathcal{D}_+ = \frac{\partial}{\partial s} + A(s)$) and regard these as differential operators on a Hilbert space $\mathcal{H}$ of sections of a bundle on $S^1 \times \tilde{M}$.

Then the following trace formula holds:

$$\tau_{\Gamma,\mathcal{H}}(e^{-\epsilon\mathcal{D}_-\mathcal{D}_+} - e^{-\epsilon\mathcal{D}_+\mathcal{D}_-}) = \sqrt{\frac{\epsilon}{\pi}} \int \tau_\Gamma(\frac{dA(s)}{ds} e^{-\epsilon A(s)^2}) ds$$

The LHS is the (Breuer)-Fredholm index and the RHS is semi-finite spectral flow.

Using a Laplace transform argument we obtain for $r > 0$:

$$\tau_{\mathcal{H},\Gamma}((1 + \mathcal{D}_-\mathcal{D}_+)^{-r} - (1 + \mathcal{D}_+\mathcal{D}_-)^{-r})$$

$$= C_{r+1/2} \int \tau_\Gamma(\frac{dA(s)}{ds}(1 + A(s)^2)^{-(r+1/2)})ds$$

where $C_{r+1/2} = \Gamma(r + 1/2)\pi^{-1/2}\Gamma(r)^{-1}$ provided both sides are trace class.

This is our guess for the higher dimensional GLMST formula. For $r = 1$ and when the operators $A(s)$ have 'compact' resolvents it is exactly the GLMST formula.

2.5 Cyclic homology: the work of Jens Kaad, R.W. Carey and J. Pincus

Consider bounded operators T satisfying $[T, T^*]$ is trace class. Such ‘almost normal operators’ formed the subject of an extensive series of papers by Carey-Pincus in both the Fredholm case and the non-Fredholm case.

Carey-Pincus (1986) wrote $T = X + iY$ where X is the real part and Y is the imaginary part of T. Then we have $[T, T^*] = 2i[X, Y]$ and thus we are dealing with an ‘almost commuting’ pair of self adjoint operators.

They then introduced their ‘principal function’, observed that it can be related to the spectral shift function and created a functional calculus for almost commuting pairs.

They introduced an index for such almost commuting pairs using the limit at zero from above of the spectral shift function (when it exists) just as for the Gesztesy-Simon ‘Witten index’.

The punchline

Witten goes to Carey-Pincus via the map

$$\mathcal{D}_+ \to T = \mathcal{D}_+(1+\mathcal{D}_-\mathcal{D}_+)^{-1/2}$$

because

$$(1+\mathcal{D}_-\mathcal{D}_+)^{-1} - (1+\mathcal{D}_+\mathcal{D}_-)^{-1} = [T, T^*]$$

Form the algebra $\mathcal{A}$ generated by T, T^* and consider the quotient $\mathcal{B} = \mathcal{A}/(\mathcal{A} \cap \text{Trace class})$ with $q : \mathcal{A} \to \mathcal{B}$ the quotient map..

The Witten index is defined via $q(T) \otimes q(T^*) \to \mathrm{Tr}([T, T^*])$ which turns out to be a function on the degree one cyclic homology group of $\mathcal{B}$.

By analogy the Fredholm index would be a function on the K-theory of $\mathcal{A}$/compacts.

Higher dimensions: there is an 'index' constructed from

$$\mathrm{Tr}((1+\mathcal{D}_-\mathcal{D}_+)^{-n} - (1+\mathcal{D}_+\mathcal{D}_-)^{-n}) = \mathrm{Tr}((1-T^*T)^n - (1-TT^*)^n)$$

which is a functional on a more complicated cyclic homology group for the algebra $\mathcal{A}$.