Platonic QHE

Chern numbers

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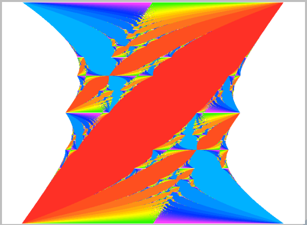


Outline

1. Prelims
   - Physics
   - Math
2. Platonic QHE
3. Virtual work & Hall conductance
4. Chern=Kubo

Outline

1 Prelims
- Physics
- Math

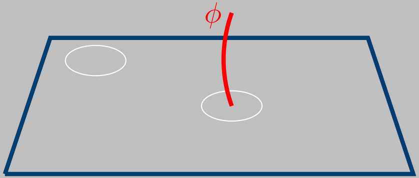
2 Platonic QHE

3 Virtual work & Hall conductance

4 Chern=Kubo

Aharonov-Bohm flux tubes

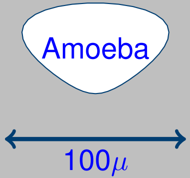
Quantum flux



- $$\oint A\cdot dx = \begin{cases} \phi & \textit{winds origin} \\ 0 & \textit{otherwise} \end{cases}$$

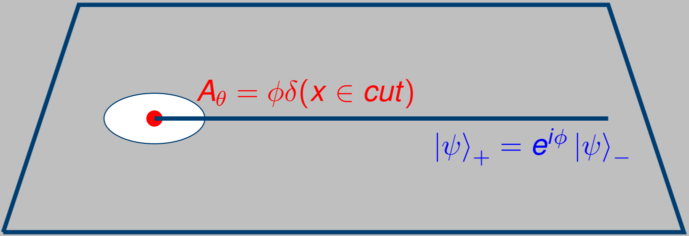
- $$\Phi_0 = 2\pi \underbrace{\frac{\hbar}{e}}_{\textit{fundamental}} = 2\pi$$

- Flux through a micro-organism



AB Periodicity

Flux tube modifies boundary condition:



b.c. 2π periodic in ϕ:

$$|\psi\rangle_+ = e^{i\phi}\,|\psi\rangle_-$$

AB periodicity

$$H(\phi + 2\pi) = UH(\phi)U^*$$

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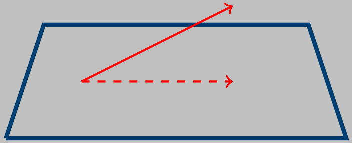
Projections: P

- Orthogonal projections

$$\underbrace{P^2 = P}_{\textit{projection}}, \quad \underbrace{P = P^*}_{\textit{orthogonal}}$$

- $$P = \sum_1^d |\psi_j\rangle \langle\psi_j| , \quad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$



- $$\underbrace{P_\perp = \mathbb{1} - P}_{\textit{complementary}}$$

- $$PP_\perp = P_\perp P = 0$$

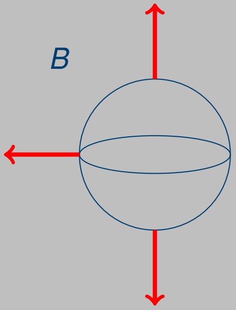
Family of projections

Paradigm

Berry: Spin in magnetic field B

$$P(B) = \frac{\mathbb{1} + H(\hat{B})}{2}, \quad \hat{B} = \frac{B}{|B|}$$

$$H(B) = B \cdot \sigma = \frac{1}{2}\begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}$$



$P(B)$ sick at $B = 0 \Leftrightarrow H(0)$ degenerates

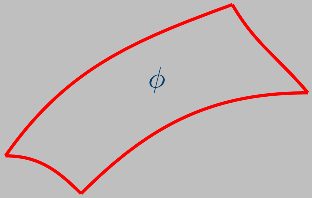
Family of projections

Parameter=control space

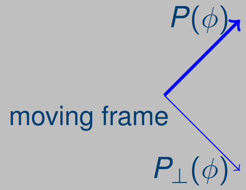
$\phi \in$ parameter space=control space

$$P(\phi) : (\textit{parameter space}) \mapsto \textit{smooth projections}$$

Parameter space



Hilbert space



dP

Motion of projections

$$\underbrace{P^2 = P}_{matrices}$$

$$P\,dP + dP\,P = dP$$

Corrolary

$$dP\,P = (\mathbb{1} - P)dP = P_\perp\,dP$$

$$P\,dP\,P = \underbrace{P\,P_\perp}_{=0}\,dP = 0$$

Kato

$$P\,dP\,P = 0$$

Kato evolution

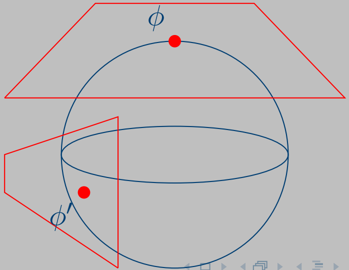
Unitary evolution in evolving subspaces

Kato's unitary evolution

$$\underbrace{P = U\, P_0\, U^*}_{Notion\ of\ parallel\ transport} \quad , \quad U = U(\phi), \quad P = P(\phi)$$

Who generates U?

$$i\, dU = \underbrace{\mathcal{A}}_{generator}\; U$$



Kato's evolution

Commutator equation

Generator satisfies commutator equation

$$dP = i[\mathcal{A}, P]$$

Proof:

$$P_0 = U^* \, P \, U \Longrightarrow 0 = (dU^*) \, P \, U + U^* \, dP \, U + U^* \, P \, dU$$

$$0 = \underbrace{U(dU^*)}_{-(dU)U^*} \, P + dP + P \, \underbrace{(dU)U^*}_{-i\mathcal{A}}$$

$\mathcal{A}$: Not unique!

Ambiguity: commutant (P)

Kato's evolution

Generator

Commutator equation for $\mathcal{A}$:

$$dP = i[\mathcal{A}, P]$$

A Generator

$$\mathcal{A} = \underbrace{i(dU)U^*}_{Definition} = \underbrace{-i[dP, P]}_{Generator}$$

Verify:

$$i[\mathcal{A}, P] = \big[[dP, P], P\big] = (dP)P - 2\underbrace{P(dP)P}_{=0} + P\,dP = dP$$

Parallel transport

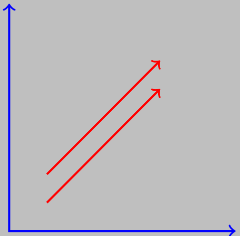
Connection

Parallel transport: No motion in P

$$\underbrace{|\psi\rangle = P\,|\psi\rangle}_{vector \in P}, \quad \underbrace{0 = P\,d\,|\psi\rangle}_{no-motion}$$

$$d\,|\psi\rangle = d\big(P\,|\psi\rangle\big) = (dP)\,|\psi\rangle + \underbrace{Pd\,|\psi\rangle}_{=0}$$

$$= (dP)\,P\,|\psi\rangle = \underbrace{[dP, P]}_{i\mathcal{A}}\,|\psi\rangle$$



Covariant derivative:

$$D = \big(d - i\mathcal{A}\big), \quad D\,|\psi\rangle = 0 \Leftrightarrow Pd\,|\psi\rangle = 0$$

Parallel transport

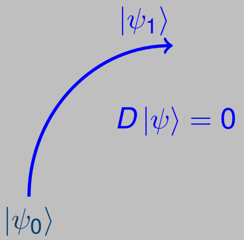
Berry's phase

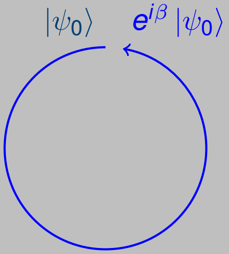
- 1-D projection: $P = |\psi\rangle\langle\psi|$
- Parallel transport:
  $0 = P\,d|\psi\rangle = |\psi\rangle\langle\psi|d\psi\rangle$
- Parallel transport
  $\Longrightarrow$ No local Berry's phase

$$0 = \langle\psi|d\psi\rangle - \frac{1}{2}d\,\underbrace{(\langle\psi|\psi\rangle)}_{=1}$$

$$= \frac{\langle\psi|d\psi\rangle - \langle d\psi|\psi\rangle}{2}$$

$$= i\,\underbrace{Im\,\langle\psi|d\psi\rangle}_{Berry's\ phase}$$

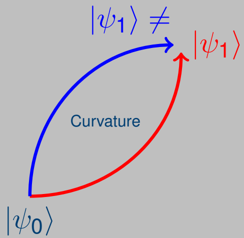




Curvature

Failure of parallel transport

Parallel transport is path dependent:



Curvature=Failure of parallel transport

$$\Omega_{jk} = \underbrace{i\,[D_j, D_k]}_{definition} = \underbrace{\left(\partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j\right) - i[\mathcal{A}_j, \mathcal{A}_k]}_{Non-abelian\ magnetic\ fields}$$

Curvature for projections

$P(dP)(dP)P$

Curvature=$iP(dp)(dP)P$

$$\Omega_{jk} = \underbrace{i\,[D_j, D_k]}_{definition} = i\,[\partial_j P, \partial_k P]$$

Proof:

$$\begin{aligned}\left[D_j, D_k\right] P &= [Pd_j, Pd_k]P \\ &= P(\partial_j P)(\partial_k P) - P(\partial_k P)(\partial_j P) \\ &= P[\partial_j P, \partial_k P]\end{aligned}$$

Curvature

1-D projection

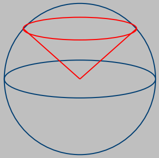
1-D: $P = |\psi\rangle\langle\psi|$

$$\Omega_{jk} P = i\,[\partial_j P, \partial_k P] P = \overbrace{i\big(\langle\partial_j\psi|\partial_k\psi\rangle - \langle\partial_k\psi|\partial_j\psi\rangle\big)}^{Berry's\ curvature} P$$

Example: Spin 1/2

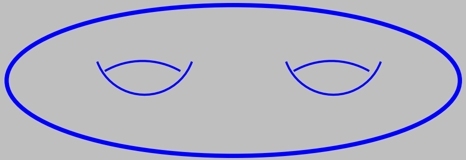
$$H = \Phi\cdot\sigma, \qquad P = \frac{\mathbb{1} + \hat{H}}{2}$$

$$\Omega_{jk} P = \underbrace{\varepsilon_{jk\ell}\frac{\Phi_\ell d\Phi_j d\Phi_k}{4|\Phi|^3}}_{1/2\ spherical\ angle} P$$



Gauss Bonnet

Geometry meets topology



Gauss-Bonnet: Gaussian curvature & genus

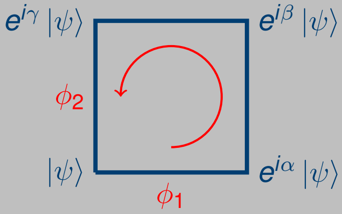
$$\frac{1}{2\pi}\int \underbrace{\Omega}_{Curvature} dS = 2(1 - genus)$$

Chern numbers

Proof for torus (TKNN)

- $P(\phi_1, \phi_2)$ periodic
- $|\psi(\phi_1, \phi_2)\rangle$ periodic up to phase:

$$\begin{aligned} |\psi(0,0)\rangle &= e^{-i\alpha} |\psi(2\pi, 0)\rangle \\ &= e^{-i\gamma} |\psi(0, 2\pi)\rangle \\ &= e^{-i\beta} |\psi(2\pi, 2\pi)\rangle \end{aligned}$$



- Angle counted mod 2π

$$(\alpha - 0)_{mod\ 2\pi} + (\beta - \alpha)_{mod\ 2\pi} + (\gamma - \beta)_{mod\ 2\pi} + (0 - \gamma)_{mod\ 2\pi} = 0$$

Chern numbers

$$i \int_T \langle d\psi | d\psi \rangle = i \oint_{\partial T} \langle \psi | d\psi \rangle \in 2\pi\mathbb{Z}$$

Chern numbers

Projections

Chern numbers

$$Chern(P,\mathcal{M}) = \frac{i}{2\pi}\int_{\mathcal{M}} Tr\, P[\partial_j P, \partial_k P]\, d\Phi_j d\Phi_k \in \mathbb{Z},$$

- $\mathcal{M}$: 2-D compact manifold (no bdry= $\partial\mathcal{M} = 0$)
- $Chern(P,\mathcal{M})$ invariant under smooth deformations of P
- P singular at eigenvalue crossing–dim P jumps

Chern numbers

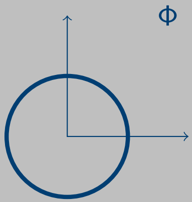
Facts

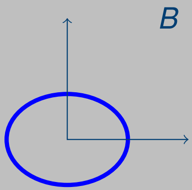
- $Chern(0, \mathcal{M}) = Chern(\mathbb{1}, \mathcal{M}) = 0$
- $Chern(P_1 \oplus P_2, \mathcal{M}) = Chern(P_1, \mathcal{M}) + Chern(P_2, \mathcal{M})$

Chern numbers

From sphere to ball

- $H = B \cdot \sigma, \quad \underbrace{B = (B_x, B_y, B_z)}_{3-D\ space}$
- Linear map of parameter space:
  $B = g\,\Phi, \qquad \Phi = (\Phi_x, \Phi_y, \Phi_z)$
- $\det g \neq 0$





Chern

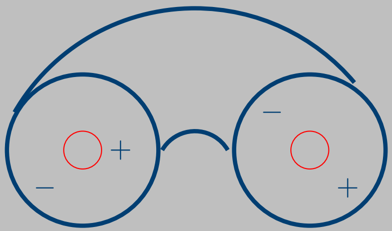
$$H(\Phi) = \sum_{j,k=1}^{3} g_{jk} \Phi^k \sigma^j, \quad P(\Phi) = \frac{\mathbb{1} + \hat{H}}{2}$$

$$\mathit{Chern}(P) = \operatorname{sgn} \det g$$

Chern numbers

What is counted?

Contracting into the solid torus

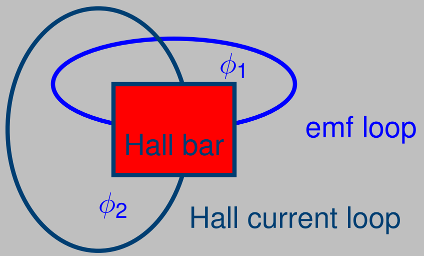


Simon

$$Chern(P,T)=\underbrace{\sum \operatorname{sgn}\det g(\Phi_d)}_{degeneracies}$$

QHE

Driving and response

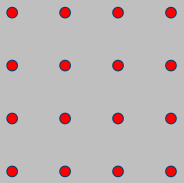


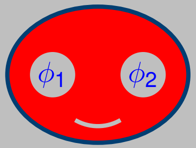
Platonic

- Driving: emf $= \dot{\phi}_1$
- Response: Hall current $= \frac{\partial H}{\partial \phi_2}$
- $H(\phi_1, \phi_2)$: Periodic, nondegenerate, hermitian matrix

Variations on a theme

Bloch momenta & controls





- Periodic
- (k_1, k_2) conserved
- Bloch momenta
- Brillouin Zone
- ∞ noninteracting (gapped) fermions

- Multiply connected
- (ϕ_1, ϕ_2) controls
- Fluxes
- Aharnonov-Bohm period
- Interacting (finite)

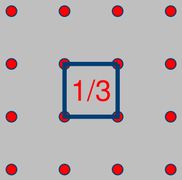
Example: 3×3 matrix function

Hofstadter Butterfly with flux $1/3$

$$H(\phi) = e^{i\phi_1} \underbrace{T}_{\textit{translation}} + e^{i\phi_2} \underbrace{S}_{\textit{shift}} + h.c.$$

$$\underbrace{T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{\textit{lattice translation}}, \quad \underbrace{S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}}_{S = FTF^*}$$

$$\omega = e^{2\pi i/3}$$



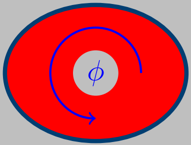
Hofstadter Model $B = 1/3$

Virtual work

Q-observable

- *H : (parameter space ϕ) $\mapsto$ Hamiltonian*
- Virtual work

$$\delta H = \underbrace{\frac{dH(\phi)}{d\phi}}_{\textit{observable}} \delta\phi$$



- Loop current:
  Virtual work of Aharonov-Bohm flux

$$I = \frac{dH}{d\phi}$$

Charge transport

Time dependent Feynman-Hellman

Virtual work=Rate of Berry's phase

$$\underbrace{\langle\psi|\,\partial_\phi H\,|\psi\rangle}_{Virtual\ \ work} = \partial_t \underbrace{\left(i\langle\psi|\partial_\phi\psi\rangle\right)}_{Berry's\ \ phase}$$

- Schrödinger

$$i\partial_t\,|\psi\rangle = H(\phi)\,|\psi\rangle$$

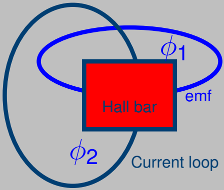
- Pf:

$$\langle\psi|\,\partial_\phi H\,|\psi\rangle = \partial_\phi\,\langle\psi|\,H\,|\psi\rangle \overbrace{-\,\langle\partial_\phi\psi|\underbrace{H\,|\psi\rangle}_{i|\partial_t\psi\rangle} - \underbrace{\langle\psi|\,H}_{-i\langle\partial_t\psi|}\,|\partial_\phi\psi\rangle}^{time-independent=0}$$

$$= \partial_t\left(i\langle\psi|\partial_\phi\psi\rangle\right)$$

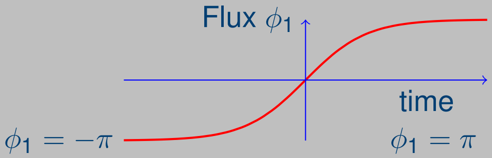
Hall conductance

Definition



Def: Flux averaged Hall conductance:

$$2\pi\sigma = \underbrace{\frac{1}{2\pi}\int_{\pi}^{\pi} d\phi_2 \int_{\infty}^{\infty} dt \, \langle\psi_t|\, \partial_2 H \,|\psi_t\rangle \, dt}_{\textit{flux average charge transport}}$$



Control averaging $\Longleftrightarrow$ Filled band.
Averaging: Gets rid of persistent currents

Chern=Kubo

Geometry of transport

In adiabatic limit: flux average transport=Chern

$$2\pi\sigma \xrightarrow[adiabatic]{} \underbrace{\frac{i}{2\pi}\int_{\pi}^{\pi} d\phi_1 d\phi_2 \, Tr \, P[\partial_1 P, \partial_2 P]}_{Chern}$$

- Loop currents: $\langle\psi|\,\partial_\phi H\,|\psi\rangle = \partial_t\big(i\langle\psi|\partial_\phi\psi\rangle\big)$
- Adiabatic limit: $|\psi\rangle \mapsto |\psi_A\rangle\,, \quad H \mapsto \mathcal{A}$
- Adiabatic loop currents: $\partial_t\big(i\langle\psi_A|\partial_\phi\psi_A\rangle\big) = \langle\psi_A|\,\partial_\phi\mathcal{A}\,|\psi_A\rangle$

$$i\,\langle\psi_A|\,\partial_\phi[\dot{P}, P]\,|\psi_A\rangle = iTr\,P\partial_\phi[\partial_t P, P] = iTr\,P[\partial_\phi P, \partial_t P]$$

Critique

Why Platonic?

- $\phi =$ control :Too general
- $\phi =$ Bloch momenta: Too special
- Gap condition: Too strong–localization
- Where is 2-D?
- Where is thermodynamic limit
- Why average?
- Sample appear to be less important than connecting circuit
- What about fractions?