RWTH AACHEN UNIVERSITY

Topological Quantum Error Correcting Codes

Kasper Duivenvoorden

JARA IQI, RWTH Aachen

Topological Phases of Quantum Matter – 4 September 2014

Topology

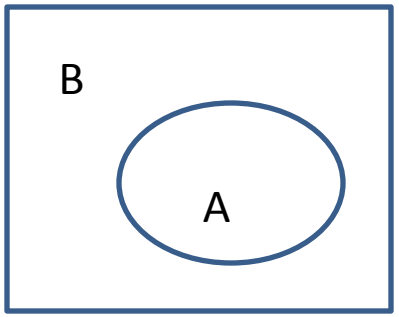
Physical errors are **local** → Store information **globally**

Gapped **topological phase** with ground space degeneracy:

$$\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \ldots |\phi_n\rangle\}$$

Definition: locally ground states are the same

$$\rho_i^A = \mathrm{tr}_B |\phi_i\rangle\langle\phi_i|$$



Use ground space manifold of a topological phase to encode quantum information

Overview

- **Homological codes**

  A formalism using emphasizing topology

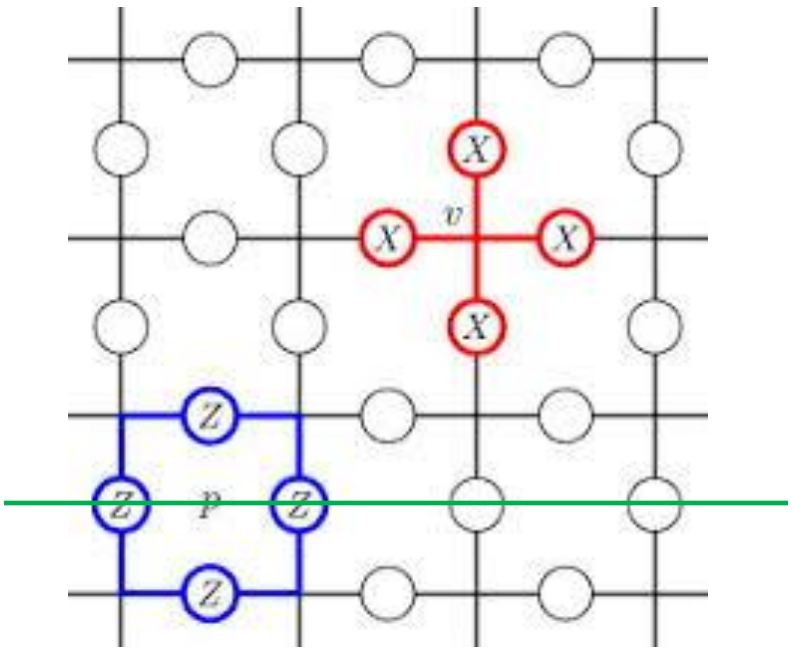
- **Fractal codes**

  Possibly thermally stable codes

- **Chamon code**

  Work in progress

Example: Toric Code



$= X \otimes X \otimes \cdots \otimes X$

$$H = -\sum_v A_v - \sum_p B_p$$

Ground states:

$$A_v|\phi\rangle = |\phi\rangle \quad , \qquad B_p|\phi\rangle = |\phi\rangle$$

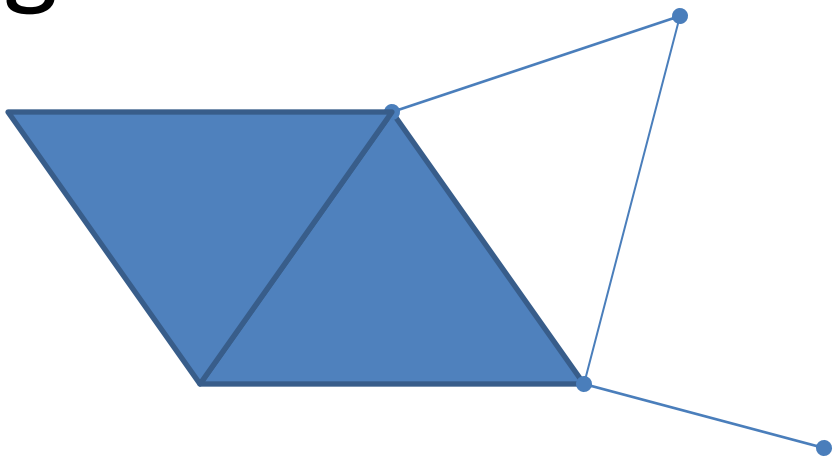
Logicals:

preserves the code space

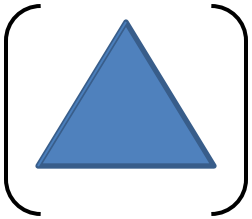
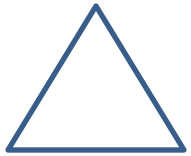
$$[L, A_v] = 0 \ , \qquad [L, B_p] = 0$$

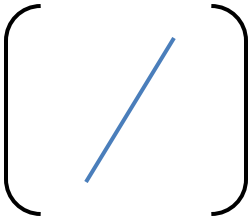
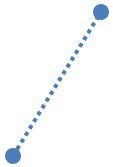
Homological codes

<u>Lattice → CW-complex</u>
Point → 0-cells
Line → 1-cells
Surface → 2-cells
... `→ k-cells

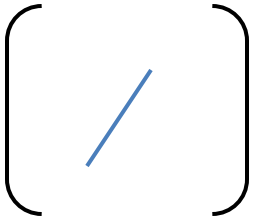
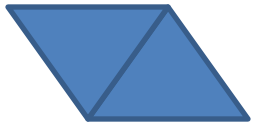


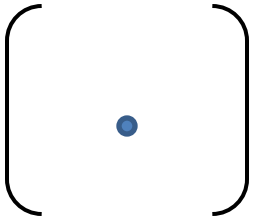
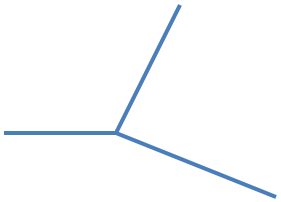
<u>Boundary map</u> ∂

∂  = 

∂  = 

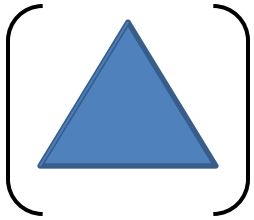
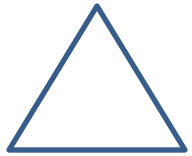
<u>Co-Boundary map</u> δ

δ  = 

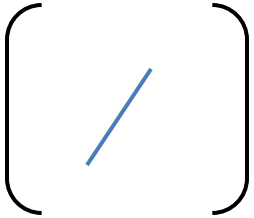
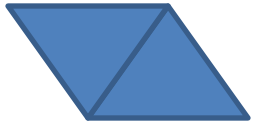
δ  = 

Homological codes

<u>Boundary map</u> ∂

∂  = 

<u>Co-Boundary map</u> δ

δ  = 

<u>Properties</u>

1: $\partial \cdot \partial = 0$

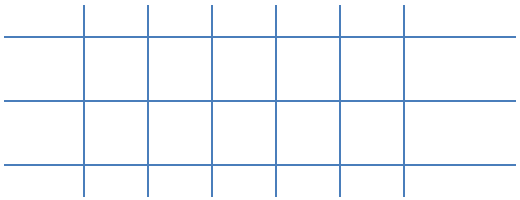
$(\sigma_1, \sigma_2) =$ overlap between cells modulo 2

 $= 1$

2: $(\partial\sigma_1, \sigma_2) = (\sigma_1, \delta\sigma_2)$

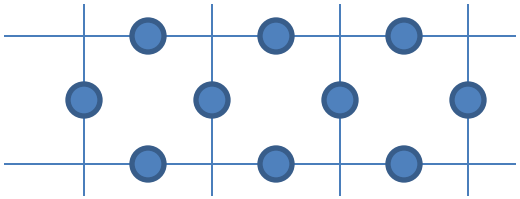
Homological codes

<u>Step 1: CW-complex</u>



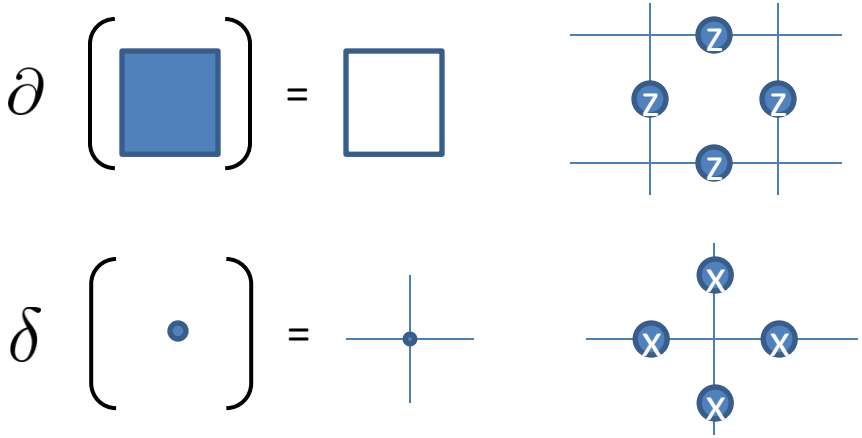
<u>Step 2: Spins on every k-cells</u>

Example: k=1



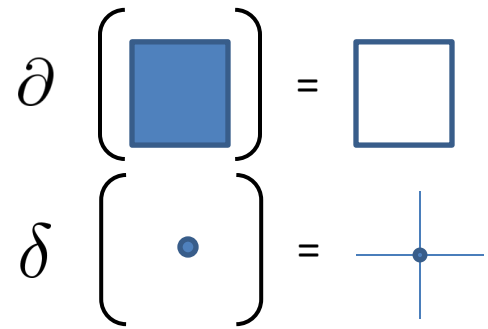
<u>Step 3: Hamiltonian</u>

$$S_z = \partial(\sigma_{k+1})$$
$$S_x = \delta(\sigma_{k-1})$$



Homological codes

$$S_z = \partial(\sigma_{k+1})$$
$$S_x = \delta(\sigma_{k-1})$$



<u>Logicals</u>

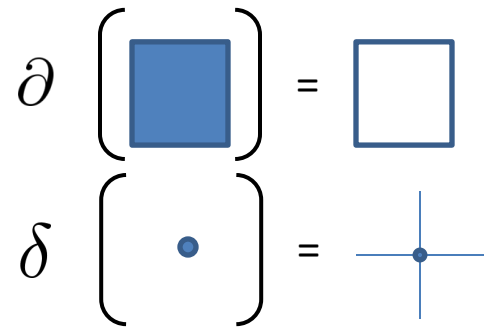
$$0 = (L_Z, \delta(\sigma_{k-1})) = (\partial L_Z, \sigma_{k-1})$$
$$\Rightarrow \partial L_Z = 0$$

$$O_X O_Z = (-1)^{(\sigma_x, \sigma_z)} O_Z O_X$$

$$|\phi\rangle = |\phi\rangle$$

Homological codes

$$S_z = \partial(\sigma_{k+1})$$
$$S_x = \delta(\sigma_{k-1})$$



Logicals

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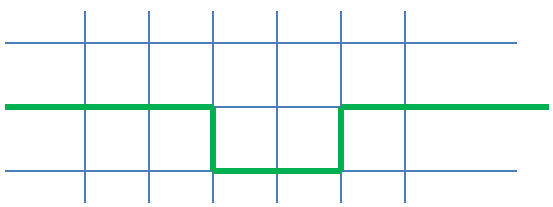
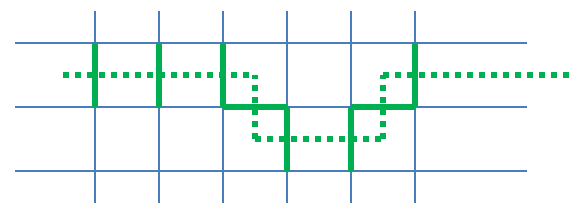
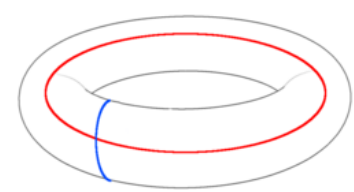


$$\delta L_X = 0$$



Z logicals: $H_k \quad \Rightarrow \quad k$ dimensional
X logicals: $H^k \quad \Rightarrow \quad D-k$ dimensional

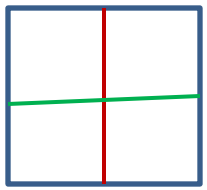
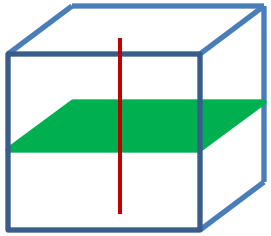
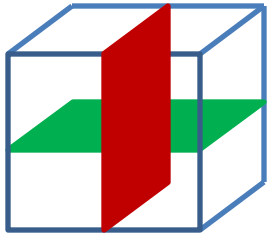


Toric Codes

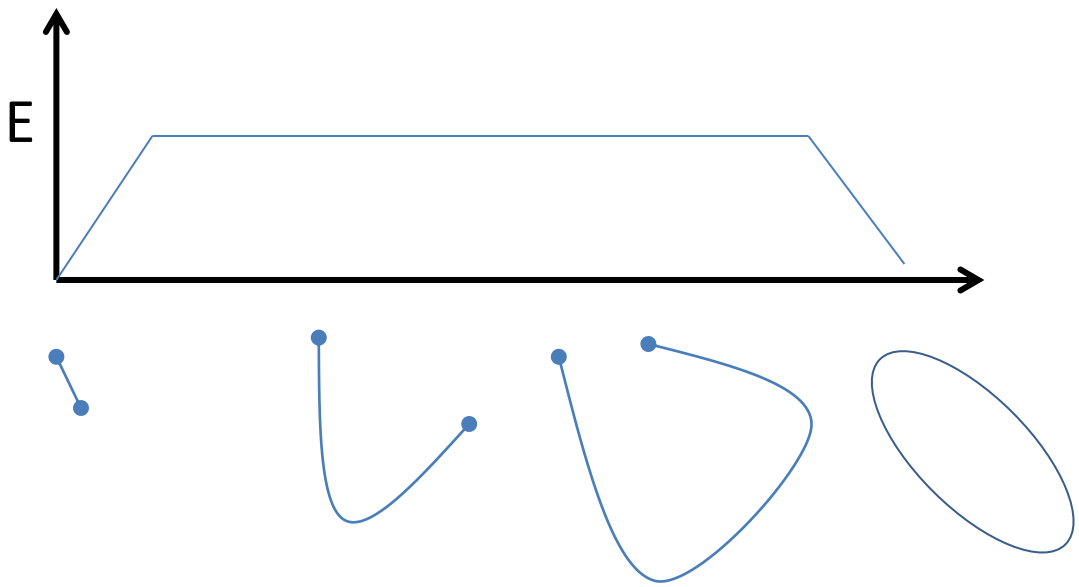
D – dimensional lattice, qubits on k cells

→ **k** and **D-k** dimensional logicals

Dimensionality of logicals

		Z	**X**
2D		1	1
3D		1	2
4D		2	2

Thermal Stability



Movement of anyons

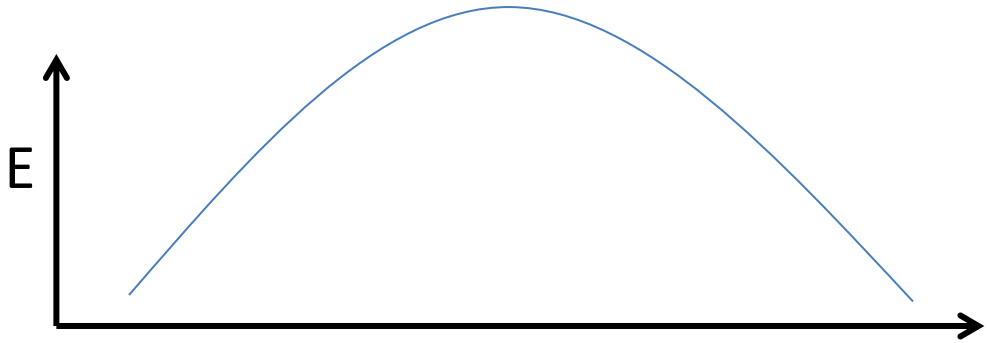
At T=0
Tunneling

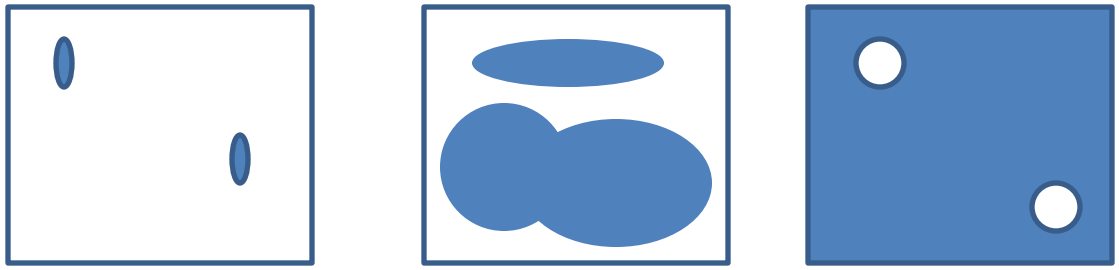
p = exp(-L)

At T>0
Thermal excitation

p = exp(-E/kT)
No stability!

Thermal Stability





At T>0

Thermal excitation

p = exp(-E/kT)

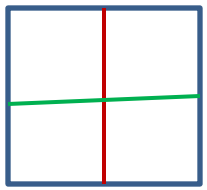
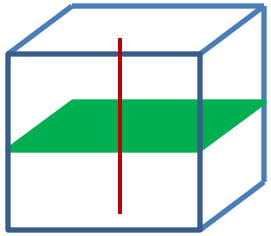
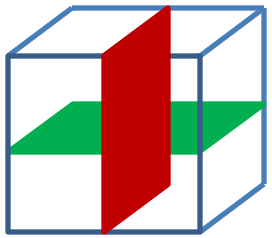
Stability!

Chesi, Loss, Bravyi, Terhal, 2010

Toric Codes

D – dimensional lattice, qubits on k cells

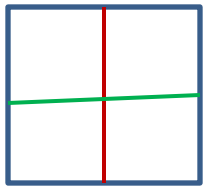
→ **k** and **D-k** dimensional logicals

		Dim. of logicals		Stable Memory
		Z	**X**	
2D		1	1	No
3D		1	2	
4D		2	2	Quantum

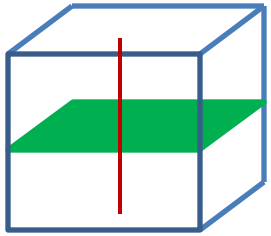
Toric Codes

D – dimensional lattice, qubits on k cells
→ **k** and **D-k** dimensional logicals

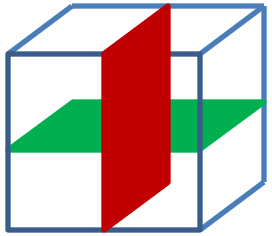
2D



3D

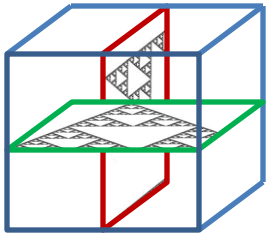


4D



<u>Drawbacks</u>
- Constant information
- We only have 3 dimensions

→ Fractal codes



Overview

- **Homological codes**

  A formalism using emphasizing topology

- **Fractal codes**

  Possibly thermally stable codes

- **Chamon code**

  Work in progress

Yoshida, 2013

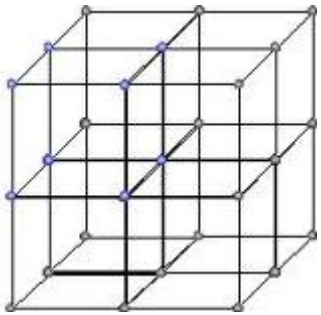
Fractal Codes

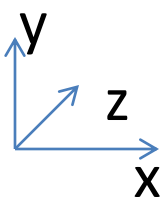
<u>Algebraic representation of Z-type operators</u>

$$Z_1 Z_2 Z_3 \quad \Longrightarrow \quad Z[x + x^2 + x^3]$$

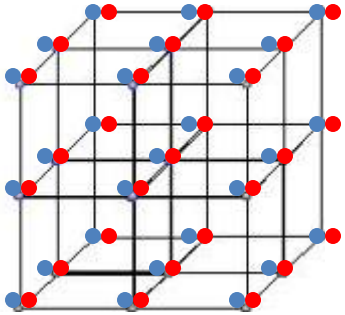
- Generalize to higher dimensions:

$$Z[f(x, y, \dots)]$$





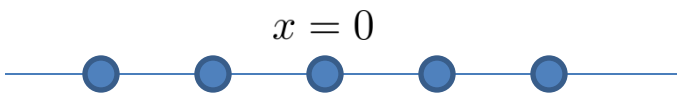
- More spins per site: $Z\left[\begin{pmatrix} f(x, y, \dots) \\ g(x, y, \dots) \end{pmatrix}\right]$



Yoshida, 2013

Fractal Codes

<u>Commutation relations</u>



$$\cdots \mathbb{I} \otimes Z \otimes Z \cdots \qquad Z[f] \quad f = 1 + x$$

$$\cdots X \otimes X \otimes \mathbb{I} \cdots \qquad X[g] \quad g = 1 + x^{-1}$$

$$f\bar{g} = (1+x)(1+x) = 1 + x^2$$

$\bar{g}$: termwise inverse

<u>Interpretation:</u> shift X[g] by 0 , Z[f] and X[g] anticommute

shift X[g] by 2 , Z[f] and X[g] anticommute

Yoshida, 2013

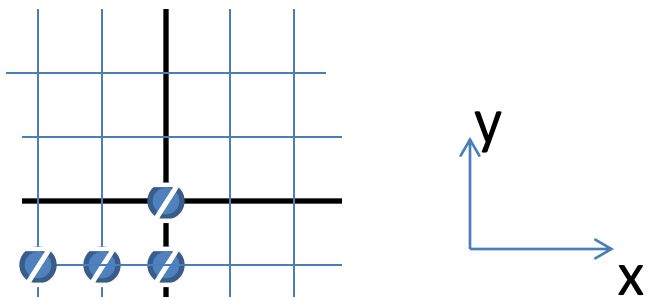
Fractal Codes

<u>Hamiltonian</u>

$$H = \sum_{ij} Z\left[x^i y^j \bar{\alpha}\right] \qquad \alpha = 1 + f(x)y$$

Example:

$$f(x) = 1 + x + x^2$$



Logical X[g] → g(1+fy) = 0 modulo 2

$$g = 1 + fy + (fy)^2 + (fy)^3 + \ldots$$

Assumptions:

$$x^L = 1, \quad y^L = 1$$

$$f^L = 1$$

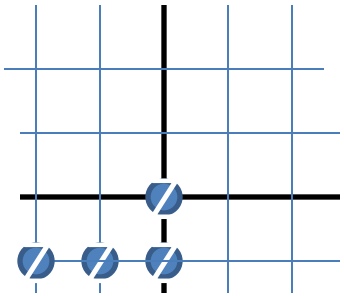
Fractal Codes

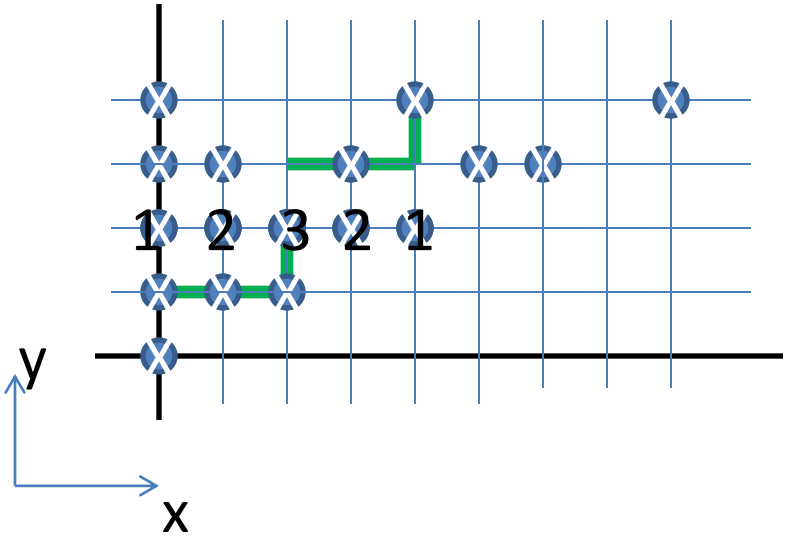
<u>Logical:</u> X[g]

$$g = 1 + fy + (fy)^2 + (fy)^3 + \dots$$

$$f(x) = 1 + x + x^2$$

Hamiltonian terms:





$$f^2 = (1 + x + x^2)f = f + xf + x^2 f$$

$$f = 1 + x + x^2$$

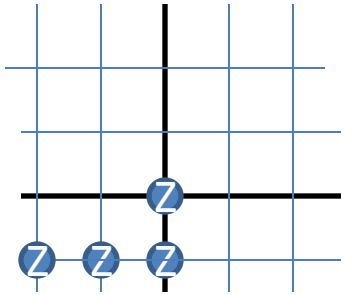
$$1$$

<u>Set of Logicals:</u>

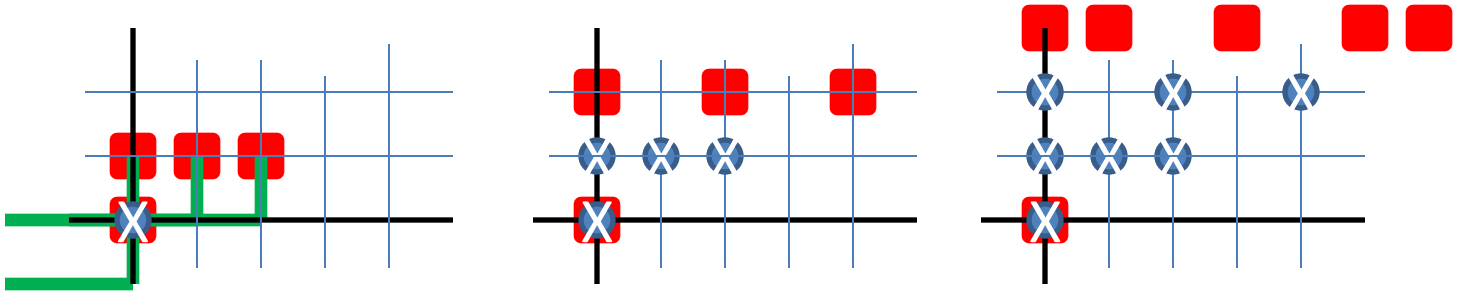
$X[x^i(1 + fy + \cdots)]$ fractal like

$Z[x^i]$ point like

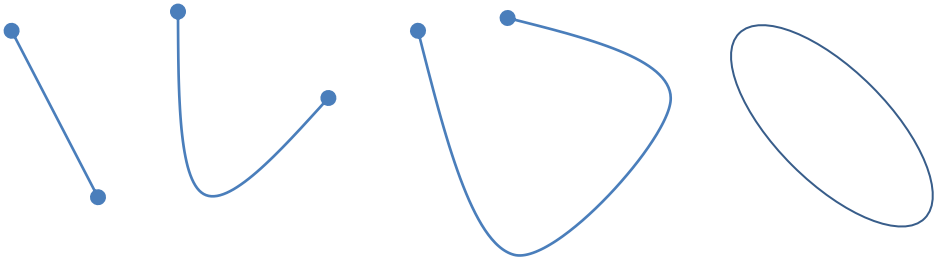
Fractal Codes



Excitations



Compare with Toric code:



→ Possible energy barrier

Yoshida, 2013

Quantum Fractal Codes

<u>Now in 3 dimensions:</u>

$$H = \sum_{ijk} Z\left[x^i y^j z^k \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}\right] + X\left[x^i y^j z^k \begin{pmatrix} \beta \\ \alpha \end{pmatrix}\right]$$

Terms commute since: $\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \cdot \overline{\begin{pmatrix} \beta \\ \alpha \end{pmatrix}} = \bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha} = 0$

$$\alpha = 1 + f(x)y$$
$$\beta = 1 + g(x)z$$

Fractal logicals in x-y plane and in x-z plane

Yoshida, 2013

Quantum Fractal Codes

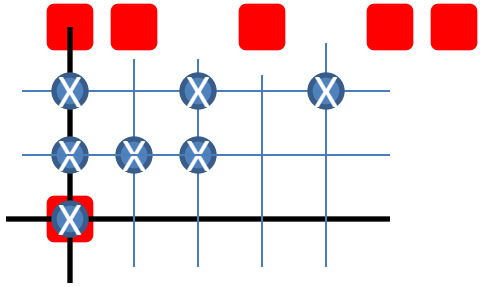
Excitations:

$$H = \sum_{ijk} Z\left[x^i y^j z^k \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}\right] + X\left[x^i y^j z^k \begin{pmatrix} \beta \\ \alpha \end{pmatrix}\right] \quad \begin{array}{l} \alpha = 1 + f(x)y \\ \beta = 1 + g(x)z \end{array}$$

Example:

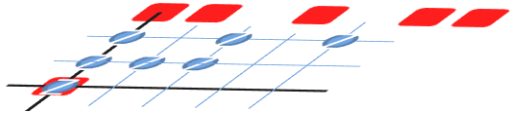
$$f(x) = g(x) = 1 + x + x^2$$

First spin



propagation in y direction

Second spin



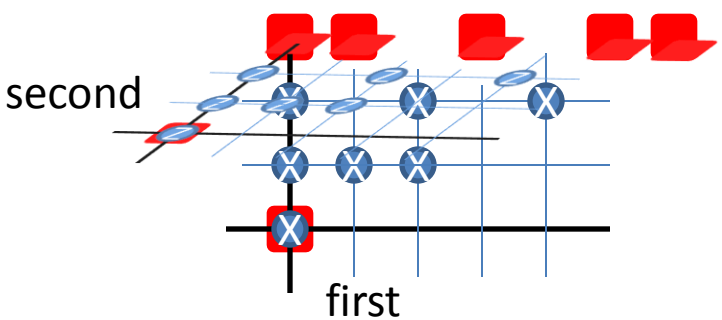
propagation in z direction

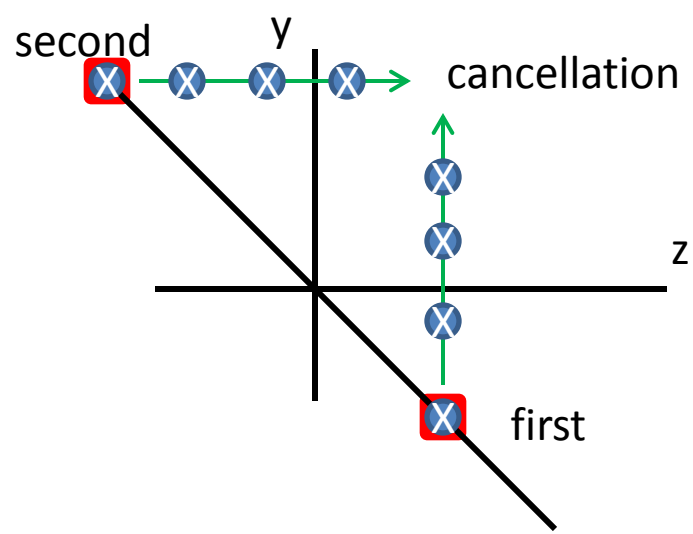
Yoshida, 2013

Quantum Fractal Codes

<u>Excitations:</u>

$$H = \sum_{ijk} Z\left[x^i y^j z^k \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right] + X\left[x^i y^j z^k \begin{pmatrix} \bar{\beta} \\ \bar{\alpha} \end{pmatrix}\right] \qquad \begin{aligned} \alpha &= 1 + f(x)y \\ \beta &= 1 + g(x)z \end{aligned}$$





Excitations can propagate freely in the z-y direction

Due to algebraic dependence of g and f

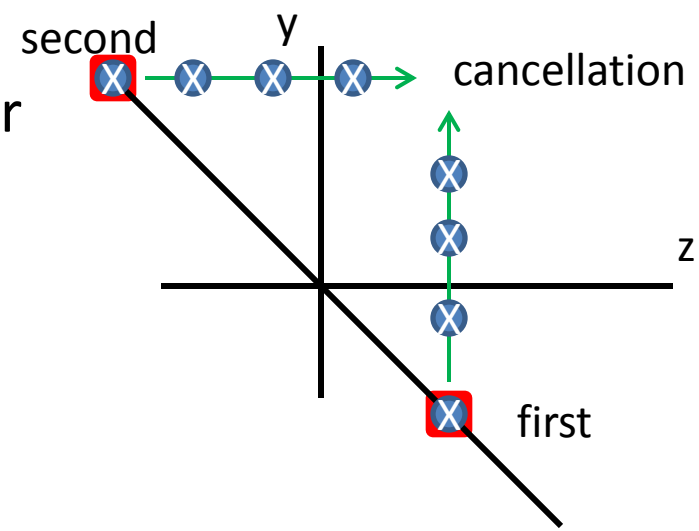
Yoshida, 2013

Quantum Fractal Codes

No algebraic dependence

→ (at least) logarithmic energy barrier

Bravyi, Haah, 2011



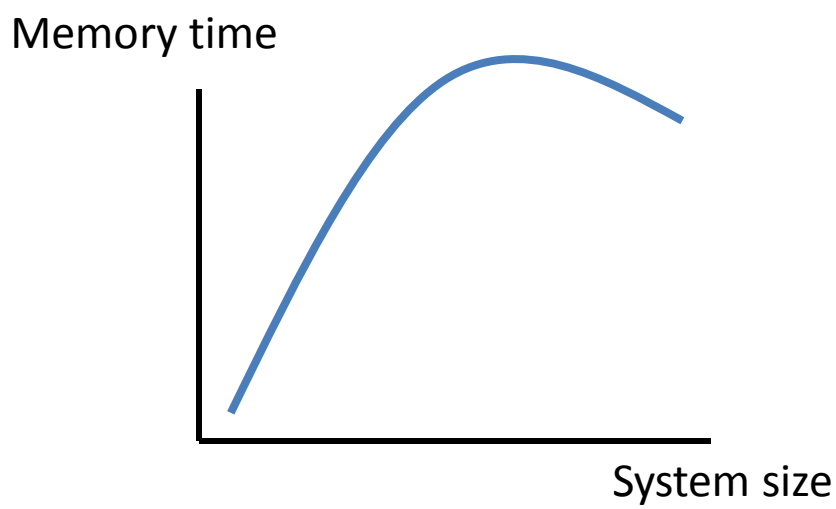
$E \approx \mathrm{Log}(L)$

$p \approx \exp(-E/kT)$

→ Polynomial rate

→ Optimal system size

Bravyi, Haah, 2013



Overview

- **Homological codes**

  A formalism using emphasizing topology

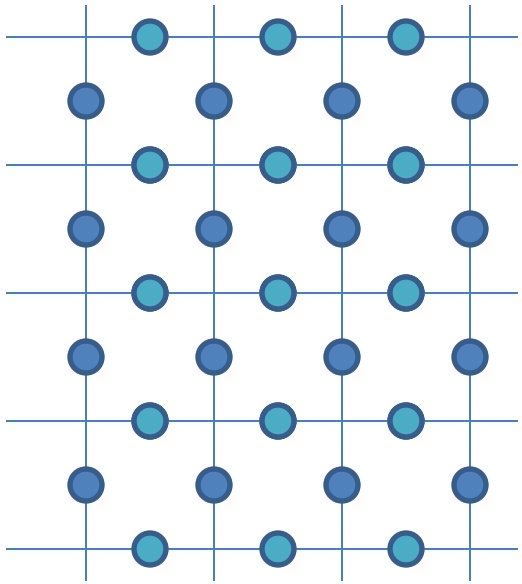
- **Fractal codes**

  Possibly thermally stable codes

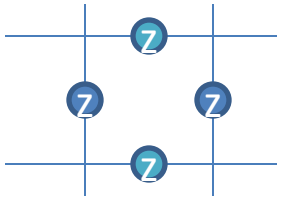
- **Chamon code**

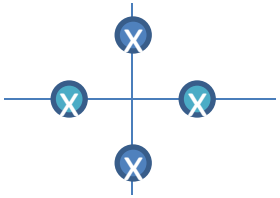
  Work in progress

Chamon Code

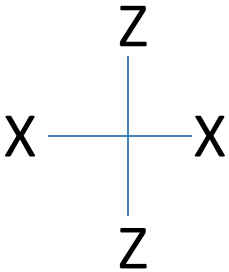


● X ↔ Z

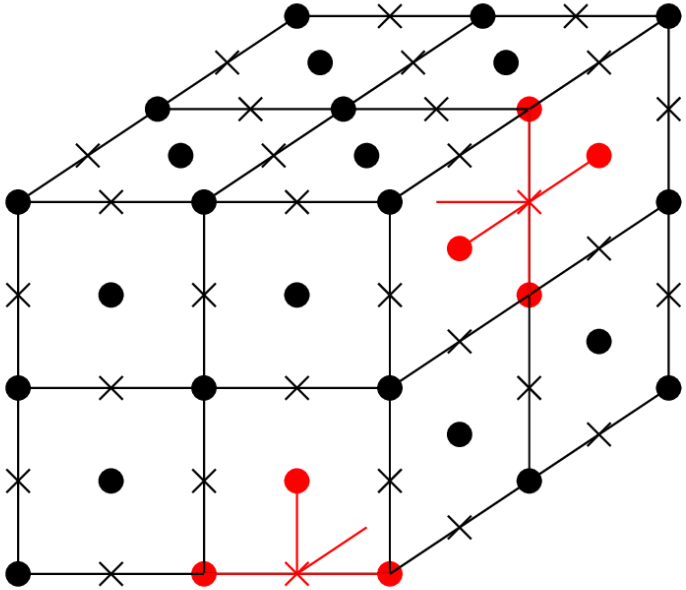




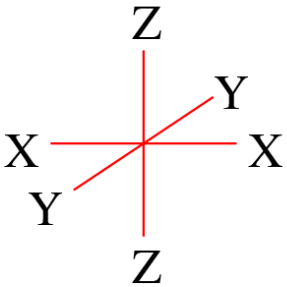
→



... to 3D. (Chamon code, [1])

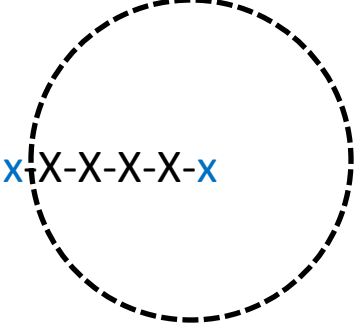


× Stabilizers



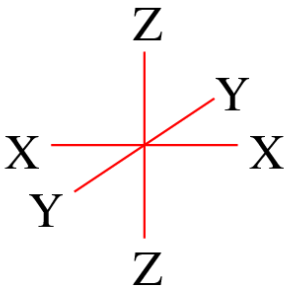
Chamon, 2005

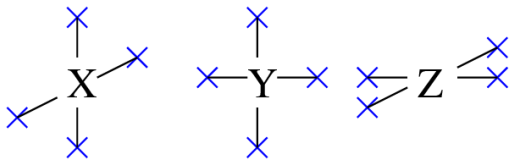
Chamon Code

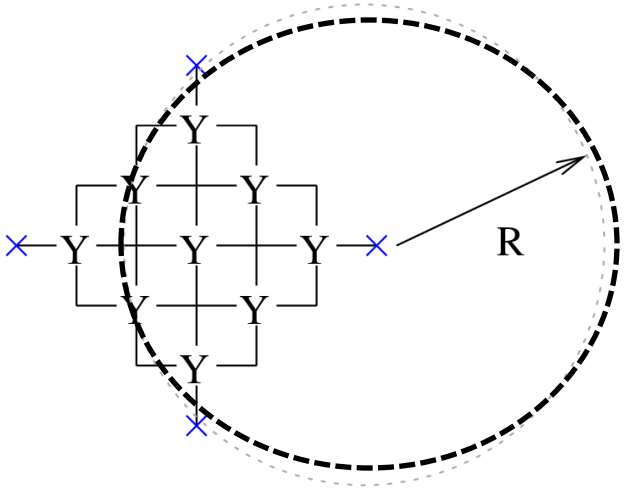


Elementary excitations:

Quadrupoles

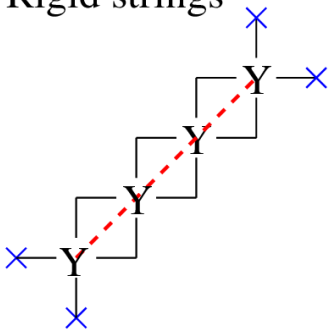






Strings:

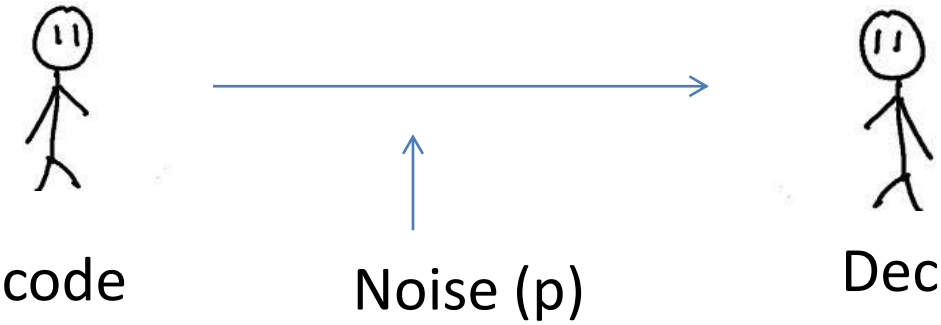
Rigid strings

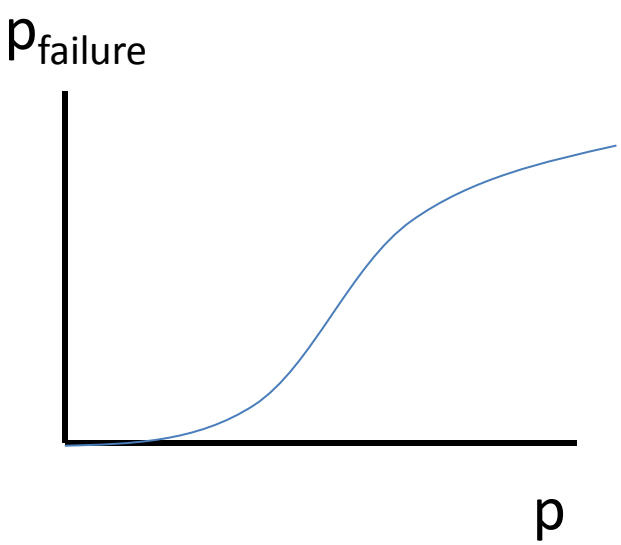


Bravyi, Leemhuis, Terhal, 2011

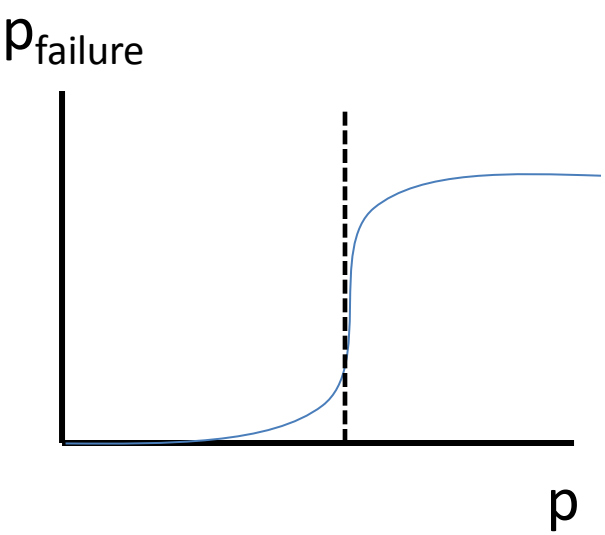
Chamon Code

<u>Error threshold</u>





→ Increased system size



Ben-Or, Aharonov, 1999

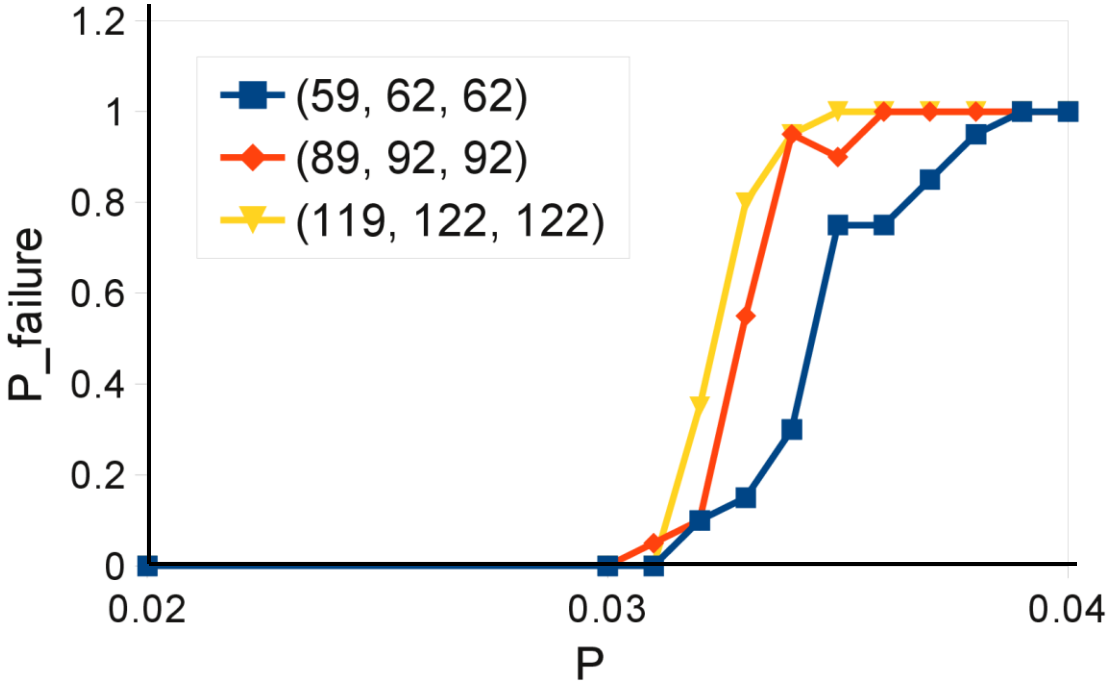
Chamon Code

<u>Error threshold</u>

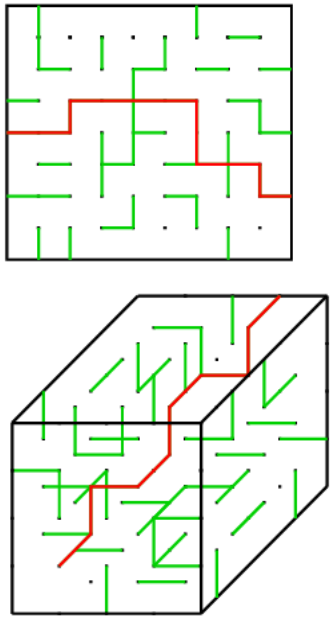
Encode

Noise (p)

Decode → $p_{failure}$



Can be related to percolation



Chamon Code

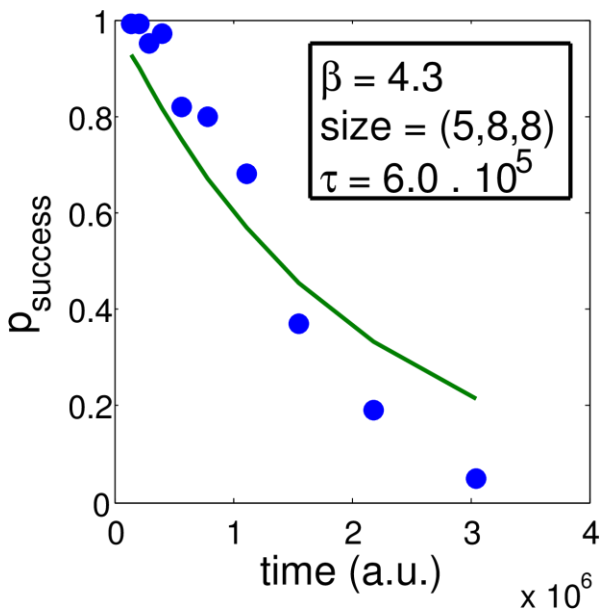
<u>Memory Time</u>

Encode

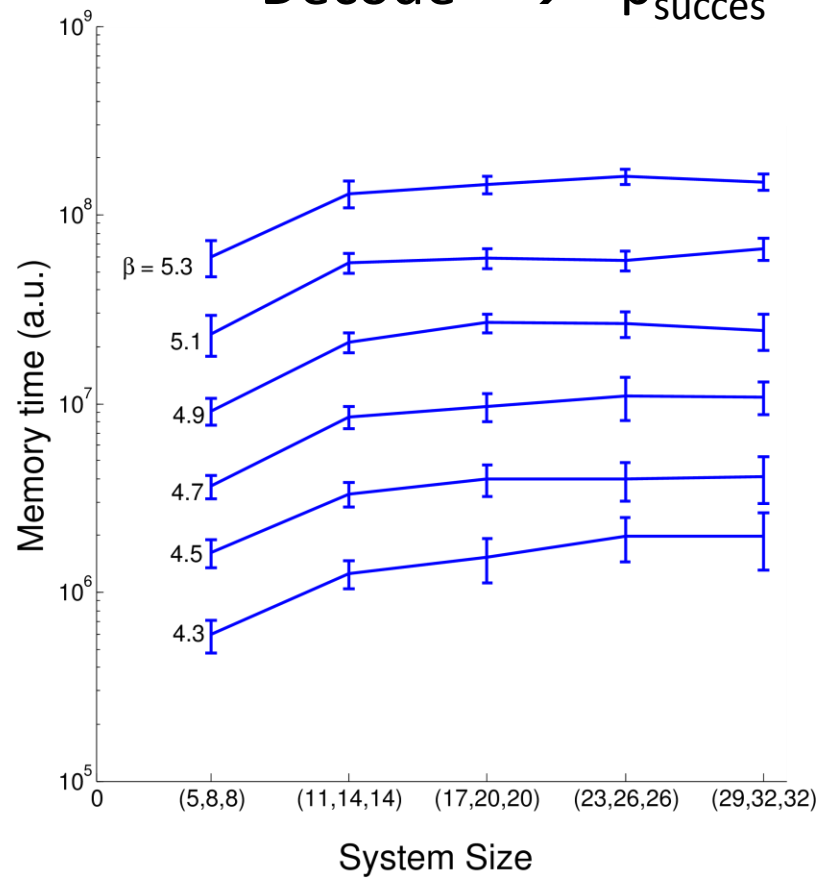
Bath (T)
Time (t)

Decode → p_{succes}

$$p = e^{-t/\tau}$$



→



Future Work

- **Homological codes**

  Determine a more general condition for thermal stability in terms of Hamiltonian properties

- **Fractal codes**

  Understand relation between fractal codes and topological order

- **Chamon code**

  Consider better (but computational more demanding) decoders

Conclustion: Existence of a quantum memory in 3D is still open

Fractal Codes

$$H = \sum_{ijk} Z\left[x^i y^j z^k \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right] + X\left[x^i y^j z^k \begin{pmatrix} \bar{\beta} \\ \bar{\alpha} \end{pmatrix}\right] \qquad \begin{aligned} \alpha &= 1 + f(x)y \\ \beta &= 1 + g(x)z \end{aligned}$$

<u>Logicals:</u> Z

$$Z^{(1)}\left[\begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix}\right] \implies h^{(1)}(1+gz) = 0$$

$$h^{(1)} = 1 + gz + (gz)^2 + (gz)^3 + \dots$$

<u>Logicals:</u> X

$$X^{(1)}\left[\begin{pmatrix} \overline{h^{(2)}} \\ 0 \end{pmatrix}\right] \implies h^{(2)}(1+fy) = 0$$

$$h^{(1)} = 1 + fy + (fy)^2 + (fy)^3 + \dots$$

<u>Similarly:</u>

$$Z^{(2)}\left[\begin{pmatrix} 0 \\ h^{(2)} \end{pmatrix}\right], \qquad X^{(2)}\left[\begin{pmatrix} 0 \\ \overline{h^{(1)}} \end{pmatrix}\right]$$

Yoshida, 2013

Fractal Codes

<u>Logicals:</u> Z

$$Z^{(1)}\left[\begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix}\right] \quad \Longrightarrow \quad h^{(1)} = 1 + gz + (gz)^2 + (gz)^3 + \dots$$

<u>Logicals:</u> X

$$X^{(1)}\left[\begin{pmatrix} \overline{h^{(2)}} \\ 0 \end{pmatrix}\right] \quad \Longrightarrow \quad h^{(1)} = 1 + fy + (fy)^2 + (fy)^3 + \dots$$

<u>Commutation relations</u>

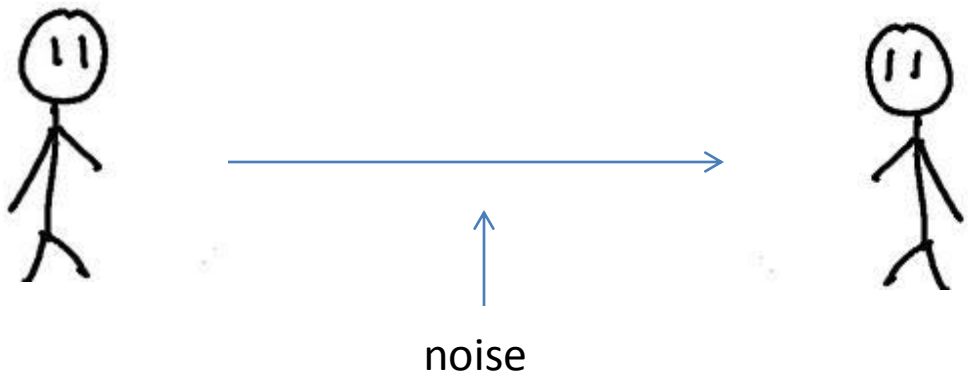
$$\begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix} \cdot \overline{\begin{pmatrix} \overline{h^{(2)}} \\ 0 \end{pmatrix}}\Big|_{x=y=z=0} = 1$$

<u>Both fractal like!</u>

Yoshida, 2013

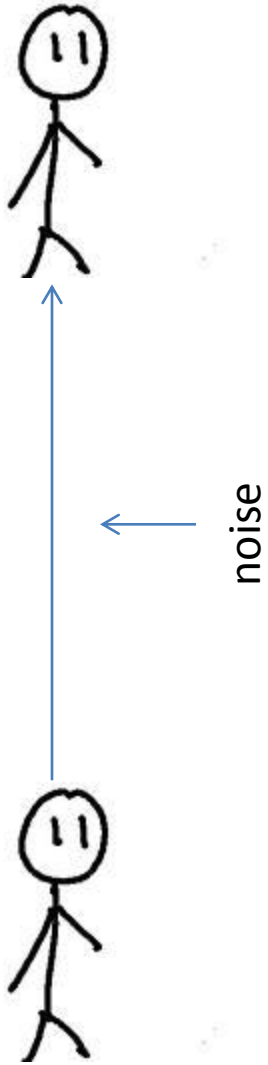
Error Correction

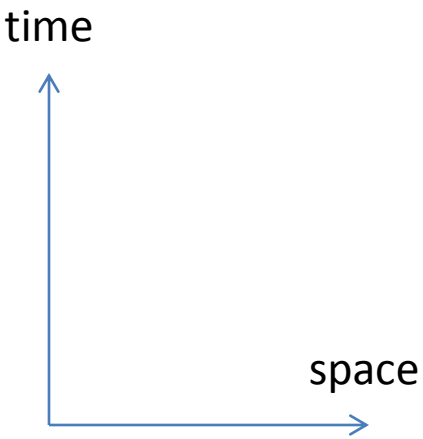
Communication



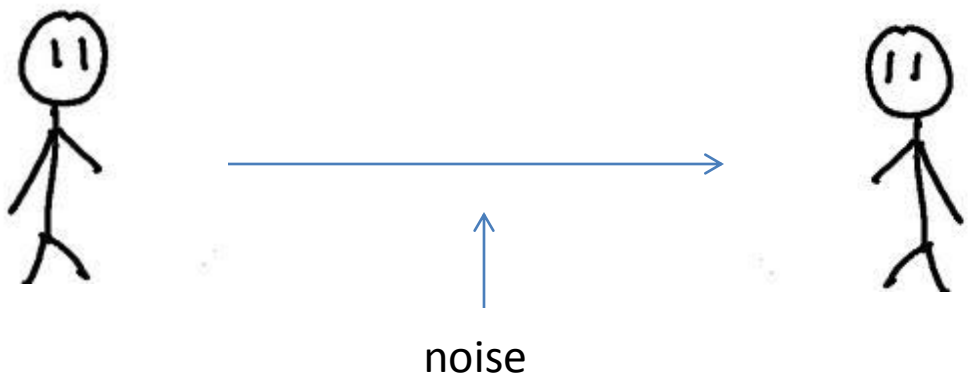
Error Correction

Storage





Error Correction



Solution: Build in Redundancy

Gottesman, PRA 1996

Stabilizer Codes

$H = -\sum_i S_i$

Stabilizers: $S_i \quad [S_i, S_j] = 0$

Ground states: $S_i|\phi_g\rangle = |\phi_g\rangle$

Error: $\rightarrow |\phi_e\rangle = E|\phi_g\rangle \quad (ES_j = -S_jE)$

Exitated states: $S_j|\phi_e\rangle = -|\phi_e\rangle$

- Allow for **fault tollerant** computations: errors do not accumalate when correcting
- **Overhead** independent of computational time

Logicals / Symmetries: $L \quad [L, S_i] = 0$

Gottesman, PRA 1998
Gottesman, 1310.2984

Trade off

Information k (number of qubits) $\Leftrightarrow$ **Stability d** (weight of a logical)

Memory time

$$\|\rho_t - \rho_\beta\| \approx e^{-t/\tau}$$

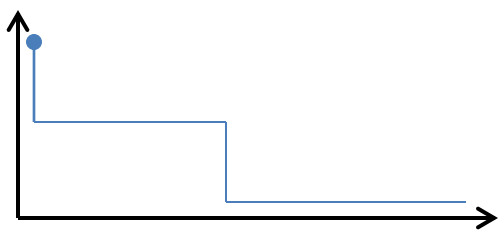
Relation to topological order

Topological order at T > 0 ↔ Stable quantum memory at T > 0

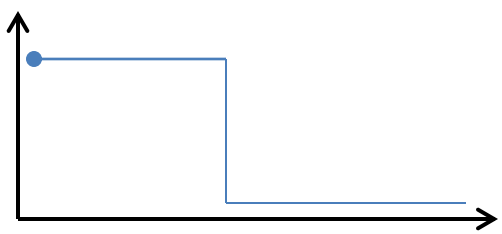
<u>Topological Entropy</u>

2D

3D



4D



Mazac, Hamma, 2012
Caselnovo, Chamon , 2007/2008

<u>Adiabatic Evolution</u>

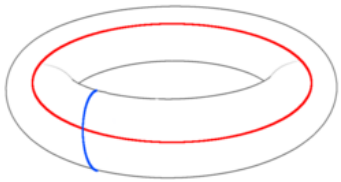
$$U\rho U^\dagger \approx \sum P(a)|\phi_{\text{trivial}}(a)\rangle\langle\phi_{\text{trivial}}(a)|$$

→ No quantum memory in 2D

Hastings, 2011

Homological codes

<u>What can we learn</u>



k: stored information, related to genus
d: distance, related to systole
n: number of qubits, related to volume

Intuitively... $(\text{sys})^2 \leq \text{volume}$

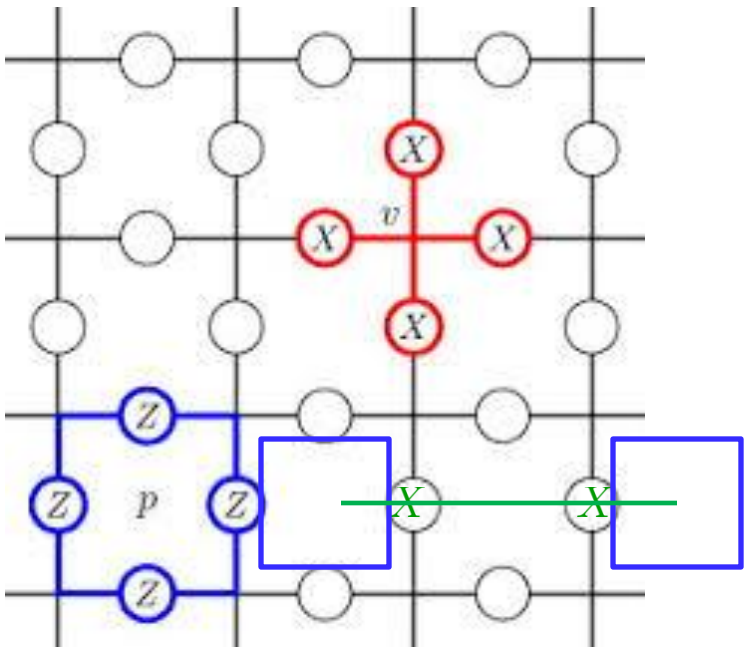
... or better $\text{genus} \times (\text{sys})^2 \leq \text{volume}$

In general $\text{genus} / \log^2(\text{genus}) \times (\text{sys})^2 \leq \text{volume}$

Gromov, 1992
Delfosse, 1301.6588

$$\boxed{k / \log^2(k) \times (d)^2 \leq n}$$

Example: Toric Code



$$H = -\sum_v A_v - \sum_p B_p$$

Ground states:

$$A_v|\phi\rangle = |\phi\rangle \quad , \quad B_p|\phi\rangle = |\phi\rangle$$

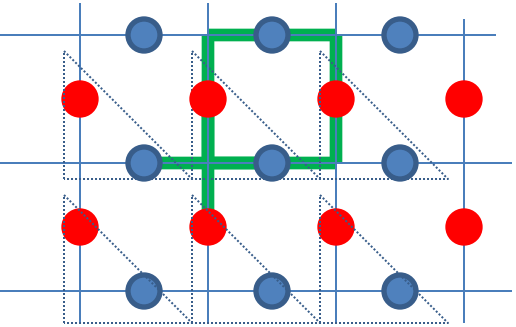
Error:

$$XB_p = -B_pX$$

→ Excitations

$$B_pX|\phi\rangle = -X|\phi\rangle$$

Example: Toric Code



Plaquette: $Z\left[\begin{pmatrix} 1+y \\ 1+x \end{pmatrix}\right]$

Star: $X\left[\begin{pmatrix} 1+x^{-1} \\ 1+y^{-1} \end{pmatrix}\right]$

$$\begin{pmatrix} 1+y \\ 1+x \end{pmatrix} \cdot \overline{\begin{pmatrix} 1+x^{-1} \\ 1+y^{-1} \end{pmatrix}}$$

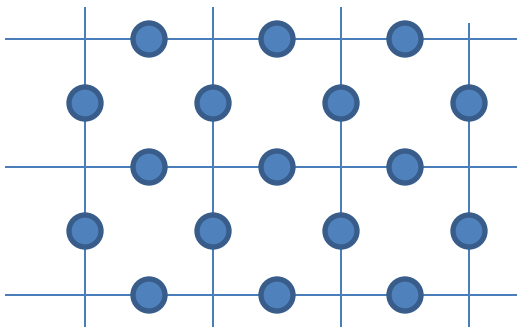
$$= (1+y)(1+x) + (1+x)(1+y)$$

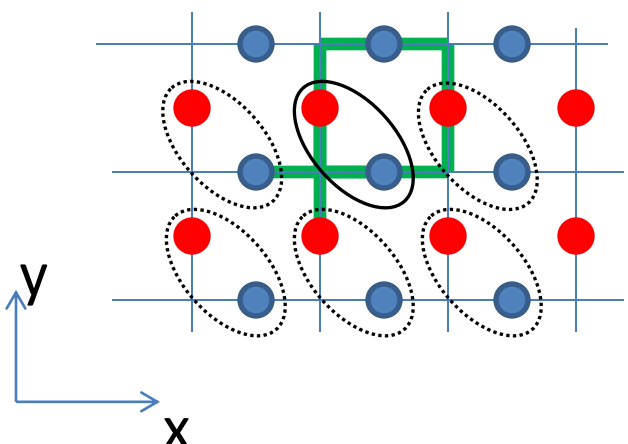
$$= 0$$

Fractal Codes

<u>Algebraic representation of Hamiltonian</u>

Example: toric code





Plaquette: $Z\left[\left(\begin{array}{c} 1+y \\ 1+x \end{array}\right)\right]$

Star: $X\left[\left(\begin{array}{c} 1+x^{-1} \\ 1+y^{-1} \end{array}\right)\right]$

Yoshida, 2013