Topological Quantum Error Correcting Codes

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Topology

Physical errors are **local** \(\rightarrow\) Store information **globally**

Gapped **topological phase** with ground space degeneracy:

\[
\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \ldots |\phi_n\rangle\}
\]

Definition: locally ground states are the same

\[
\rho_i^A = \text{tr}_B |\phi_i\rangle\langle \phi_i|
\]

Use ground space manifold of a topological phase to encode quantum information
Overview

• Homological codes
  A formalism using emphasizing topology

• Fractal codes
  Possibly thermally stable codes

• Chamon code
  Work in progress
Example: Toric Code

\[ H = - \sum_v A_v - \sum_p B_p \]

Ground states:
\[ A_v |\phi\rangle = |\phi\rangle, \quad B_p |\phi\rangle = |\phi\rangle \]

Logicals:
preserves the code space
\[ [L, A_v] = 0, \quad [L, B_p] = 0 \]

\[ = X \otimes X \otimes \cdots \otimes X \]
Homological codes

Lattice $\rightarrow$ CW-complex
Point $\rightarrow$ 0-cells
Line $\rightarrow$ 1-cells
Surface $\rightarrow$ 2-cells
... `$\rightarrow$ k-cells

Boundary map $\partial$

\[
\partial \left( \begin{array}{c} \text{Triangle} \\ \end{array} \right) = \begin{array}{c} \text{Triangle} \\ \end{array}
\]

\[
\partial \left( \begin{array}{c} \text{Line} \\ \end{array} \right) = \begin{array}{c} \text{Line} \\ \end{array}
\]

Co-Boundary map $\delta$

\[
\delta \left( \begin{array}{c} \text{1-cell} \\ \end{array} \right) = \begin{array}{c} \text{1-cell} \\ \end{array}
\]

\[
\delta \left( \begin{array}{c} \text{0-cell} \\ \end{array} \right) = \begin{array}{c} \text{0-cell} \\ \end{array}
\]
Homological codes

Boundary map \( \partial \) \[
\partial \left( \begin{array}{c}
\text{triangle}
\end{array} \right) = \begin{array}{c}
\text{triangle}
\end{array}
\]

Co-Boundary map \( \delta \) \[
\delta \left( \begin{array}{c}
\text{line}
\end{array} \right) = \begin{array}{c}
\text{rectangle}
\end{array}
\]

Properties

1: \( \partial \cdot \partial = 0 \)

(\( \sigma_1, \sigma_2 \)) = overlap between cells modulo 2

\[
\left( \begin{array}{c}
\sigma_1, \sigma_2
\end{array} \right) = 1
\]

2: \( (\partial \sigma_1, \sigma_2) = (\sigma_1, \delta \sigma_2) \)
Homological codes

Step 1: CW-complex

Step 2: Spins on every k-cells
Example: k=1

Step 3: Hamiltonian

\[ S_z = \partial (\sigma_{k+1}) \]
\[ S_x = \delta (\sigma_{k-1}) \]
Homological codes

\[ S_z = \partial (\sigma_{k+1}) \]
\[ S_x = \delta (\sigma_{k-1}) \]

\[ \partial \left( \begin{array}{c} \text{blue square} \end{array} \right) = \begin{array}{c} \text{white square} \end{array} \]
\[ \delta \left( \begin{array}{c} \text{dot} \end{array} \right) = \begin{array}{c} \text{vertical line} \end{array} \]

Logicals

\[ 0 = (L_Z, \delta(\sigma_{k-1})) = (\partial L_Z, \sigma_{k-1}) \]
\[ \Rightarrow \partial L_Z = 0 \]

\[ O_X O_Z = (-1)^{\sigma_x, \sigma_z} O_Z O_X \]
\[ \begin{array}{c} \text{white square} \end{array} |\phi\rangle = |\phi\rangle \]
Homological codes

\[ S_z = \partial(\sigma_{k+1}) \]
\[ S_x = \delta(\sigma_{k-1}) \]

**Logicals**

\[ 0 = (L_Z, \delta(\sigma_{k-1})) = (\partial L_Z, \sigma_{k-1}) \]
\[ \Rightarrow \partial L_Z = 0 \]

\[ \delta L_X = 0 \]

*Z* logicals: \( H^k \) \( \Rightarrow \) \( k \) dimensional

*X* logicals: \( H^k \) \( \Rightarrow \) \( D - k \) dimensional
Toric Codes

D – dimensional lattice, qubits on k cells
→ k and D-k dimensional logicals

Dimensionality of logicals

<table>
<thead>
<tr>
<th>Dimensionality of logicals</th>
<th>Z</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3D</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4D</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Thermal Stability

At $T=0$

Tunneling

$p = \exp(-L)$

At $T>0$

Thermal excitation

$p = \exp(-E/kT)$

No stability!
Thermal Stability

At $T > 0$

Thermal excitation

$p = \exp(-E/kT)$

Stability!

Chesi, Loss, Bravyi, Terhal, 2010
Toric Codes

D – dimensional lattice, qubits on k cells
→ k and D-k dimensional logicals

<table>
<thead>
<tr>
<th>Dim. of logicals</th>
<th>Stable Memory</th>
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<tr>
<td>Z</td>
<td>X</td>
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<td>1</td>
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<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
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</tr>
</tbody>
</table>

2D

3D

4D
Toric Codes

D – dimensional lattice, qubits on k cells
→ k and D-k dimensional logicals

Drawbacks
- Constant information
- We only have 3 dimensions

→ Fractal codes
Overview

• Homological codes
  A formalism using emphasizing topology

• Fractal codes
  Possibly thermally stable codes

• Chamon code
  Work in progress
Fractal Codes

Algebraic representation of Z-type operators

\[ Z_1 Z_2 Z_3 \implies Z[x + x^2 + x^3] \]

- Generalize to higher dimensions:
  \[ Z[f(x, y, \ldots)] \]

- More spins per site:
  \[ Z \left[ \begin{pmatrix} f(x, y, \ldots) \\ g(x, y, \ldots) \end{pmatrix} \right] \]

Yoshida, 2013
Fractal Codes

Commutation relations

\[ x = 0 \]

\[ \ldots \mathbb{I} \otimes Z \otimes Z \ldots \]

\[ \ldots X \otimes X \otimes \mathbb{I} \ldots \]

\[ Z[f] \quad f = 1 + x \]

\[ X[g] \quad g = 1 + x^{-1} \]

\[ f \bar{g} = (1 + x)(1 + x) = 1 + x^2 \]

\[ \bar{g}: \text{termwise inverse} \]

Interpretation:

shift \( X[g] \) by 0,

shift \( X[g] \) by 2,

\( Z[f] \) and \( X[g] \) anticommute

\( Z[f] \) and \( X[g] \) anticommute

Yoshida, 2013
Fractal Codes

Hamiltonian

\[ H = \sum_{i,j} Z \left[ x^i y^j \bar{\alpha} \right] \quad \alpha = 1 + f(x)y \]

Example:

\[ f(x) = 1 + x + x^2 \]

Logical X[g] \[\rightarrow\] \[g(1+fy) = 0 \mod 2\]

\[ g = 1 + fy + (fy)^2 + (fy)^3 + \ldots \]

Assumptions:

\[ x^L = 1, \quad y^L = 1 \]

\[ f^L = 1 \]
Fractal Codes

**Logical:** $X[g]$

$$g = 1 + f y + (fy)^2 + (fy)^3 + \ldots$$

$$f(x) = 1 + x + x^2$$

**Hamiltonian terms:**

$$f^2 = (1 + x + x^2) f = f + xf + x^2 f$$

$$f = 1 + x + x^2$$

*Set of Logicals:*  

$$X[x^i(1 + fy + \cdots)]$$

*fractal like*

$$Z[x^i]$$

*point like*
Fractal Codes

Excitations

Compare with Toric code:

→ Possible energy barrier

Yoshida, 2013
Quantum Fractal Codes

Now in 3 dimensions:

\[ H = \sum_{ijk} Z \left[ x^i y^j z^k \left( \begin{array}{c} \bar{\alpha} \\ \bar{\beta} \end{array} \right) \right] + X \left[ x^i y^j z^k \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) \right] \]

Terms commute since:

\[ \left( \begin{array}{c} \bar{\alpha} \\ \bar{\beta} \end{array} \right) \cdot \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) = \bar{\alpha} \beta + \bar{\beta} \alpha = 0 \]

\[ \alpha = 1 + f(x)y \]
\[ \beta = 1 + g(x)z \]

Fractal logicals in x-y plane and in x-z plane

Yoshida, 2013
Quantum Fractal Codes

Excitations:

\[ H = \sum_{ijk} Z \left[ x^i y^j z^k \left( \frac{\bar{\alpha}}{\bar{\beta}} \right) \right] + X \left[ x^i y^j z^k \left( \frac{\beta}{\alpha} \right) \right] \]

\[ \alpha = 1 + f(x)y \]
\[ \beta = 1 + g(x)z \]

Example:

\[ f(x) = g(x) = 1 + x + x^2 \]

First spin propagation in y direction

Second spin propagation in z direction

Yoshida, 2013
Quantum Fractal Codes

Excitations:

\[ H = \sum_{i,j,k} Z \left[ x^i y^j z^k \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \right] + X \left[ x^i y^j z^k \left( \frac{\beta}{\alpha} \right) \right] \]

\[ \alpha = 1 + f(x)y \]
\[ \beta = 1 + g(x)z \]

Excitations can propagate freely in the \( z-y \) direction

Due to algebraic dependence of \( g \) and \( f \)
Quantum Fractal Codes

No algebraic dependence
→ (at least) logarithmic energy barrier

\[ E \approx \log(L) \]
\[ p \approx \exp(-E/kT) \]

→ Polynomial rate
→ Optimal system size
Overview

• **Homological codes**
  
  *A formalism using emphasizing topology*

• **Fractal codes**
  
  *Possibly thermally stable codes*

• **Chamon code**
  
  *Work in progress*
Chamon Code

... to 3D. (Chamon code, [1])
Chamon Code

Elementary excitations:
Quadrupoles

Strings:
Rigid strings

Bravyi, Leemhuis, Terhal, 2011
Chamonon Code

Error threshold

Encode

Noise (p)

Decode → $p_{\text{failure}}$

$p_{\text{failure}}$ → Increased system size

$p_{\text{failure}}$ →

$p$

Ben-Or, Aharonov, 1999
Chamon Code

Error threshold

Encode → Noise (p) → Decode

Can be related to percolation

\begin{align*}
\text{Can be related to} & \quad \text{percolation} \\
\end{align*}
Chamon Code

Memory Time

Encode

Bath (T) Time (t)

Decode $\rightarrow p_{\text{success}}$

$p = e^{-t/\tau}$

$\beta = 4.3$

size = (5,8,8)

$\tau = 6.0 \times 10^5$

Memory time (a.u.)

System Size

$\beta = 5.3$

5.1

4.9

4.71

4.5

4.3

(5,8,8) (11,14,14) (17,20,20) (23,26,26) (29,32,32)
Future Work

• Homological codes
  Determine a more general condition for thermal stability in terms of Hamiltonian properties

• Fractal codes
  Understand relation between fractal codes and topological order

• Chamon code
  Consider better (but computational more demanding) decoders

Conclusion: Existence of a quantum memory in 3D is still open
Fractal Codes

\[ H = \sum_{i,j,k} Z \left[ x^i y^j z^k \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right] + X \left[ x^i y^j z^k \begin{pmatrix} \bar{\beta} \\ \bar{\alpha} \end{pmatrix} \right] \]

\[ \alpha = 1 + f(x) y \]
\[ \beta = 1 + g(x) z \]

Logicals: \( Z \)

\[ Z^{(1)} \left[ \begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix} \right] \Rightarrow h^{(1)} (1 + g z) = 0 \]

\[ h^{(1)} = 1 + g z + (g z)^2 + (g z)^3 + \ldots \]

Logicals: \( X \)

\[ X^{(1)} \left[ \begin{pmatrix} \bar{h}^{(2)} \\ 0 \end{pmatrix} \right] \Rightarrow h^{(2)} (1 + f y) = 0 \]

\[ h^{(1)} = 1 + f y + (f y)^2 + (f y)^3 + \ldots \]

Similarly:

\[ Z^{(2)} \left[ \begin{pmatrix} 0 \\ h^{(2)} \end{pmatrix} \right], \quad X^{(2)} \left[ \begin{pmatrix} 0 \\ \frac{1}{h^{(1)}} \end{pmatrix} \right] \]

Yoshida, 2013
Fractal Codes

Logicals: $Z$

\[
Z^{(1)} \begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix} \quad \implies \quad h^{(1)} = 1 + gz + (gz)^2 + (gz)^3 + \ldots
\]

Logicals: $X$

\[
X^{(1)} \begin{pmatrix} h^{(2)} \\ 0 \end{pmatrix} \quad \implies \quad h^{(1)} = 1 + fy + (fy)^2 + (fy)^3 + \ldots
\]

Commutation relations

\[
\begin{pmatrix} h^{(1)} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \overline{h^{(2)}} \\ 0 \end{pmatrix} \big|_{x=y=z=0} = 1
\]

Both fractal like!
Error Correction

Communication

noise
Error Correction

- Storage
- Noise
- Time
- Space
Error Correction

Solution: Build in Redundancy

Encoding

Decoding
Stabilizer Codes

\[ H = - \sum_i S_i \]

Stabilizers: \( S_i \) \( [S_i, S_j] = 0 \)

Ground states: \( S_i |\phi_g\rangle = |\phi_g\rangle \)

Error: \( \to |\phi_e\rangle = E|\phi_g\rangle \quad (ES_j = -S_j E) \)

Exited states: \( S_j |\phi_e\rangle = -|\phi_e\rangle \)

- Allow for **fault tolerant** computations: errors do not accumulate when correcting
- **Overhead** independent of computational time

Logicals / Symmetries: \( L \) \( [L, S_i] = 0 \)

<table>
<thead>
<tr>
<th>Information ( k ) (number of qubits)</th>
<th>Trade off</th>
<th>Stability ( d ) (weight of a logical)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Leftrightarrow )</td>
<td></td>
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</tbody>
</table>

Memory time

\[ \|\rho_t - \rho_\beta\| \approx e^{-t/\tau} \]
Relation to topological order

Topological order at $T > 0$ $\iff$ Stable quantum memory at $T > 0$

**Topological Entropy**

- 2D
- 3D
- 4D

**Adiabatic Evolution**

$$U \rho U^\dagger \approx \sum P(a)|\phi_{\text{trivial}}(a)\rangle\langle\phi_{\text{trivial}}(a)|$$

$\rightarrow$ No quantum memory in 2D

Mazac, Hamma, 2012
Caselnovo, Chamon, 2007/2008
Hastings, 2011
Homological codes

What can we learn

\[ k: \text{stored information,}\quad \text{related to genus} \]
\[ d: \text{distance,}\quad \text{related to systole} \]
\[ n: \text{number of qubits,}\quad \text{related to volume} \]

Intuitively...
\[ (\text{sys})^2 \leq \text{volume} \]
... or better
\[ \text{genus} \times (\text{sys})^2 \leq \text{volume} \]

In general
\[ \text{genus} / \log^2 (\text{genus}) \times (\text{sys})^2 \leq \text{volume} \]

\[ k / \log^2 (k) \times (d)^2 \leq n \]

Gromov, 1992
Delfosse, 1301.6588
Example: Toric Code

\[ H = - \sum_v A_v - \sum_p B_p \]

Ground states:
\[ A_v |\phi\rangle = |\phi\rangle \quad , \quad B_p |\phi\rangle = |\phi\rangle \]

Error:
\[ XB_p = -B_p X \]

⇒ Excitations
\[ B_p X |\phi\rangle = -X |\phi\rangle \]
Example: Toric Code

Plaquette: $Z \left[ \begin{pmatrix} 1+y \\ 1+x \end{pmatrix} \right] \cdot \left( \begin{pmatrix} 1+y \\ 1+x \end{pmatrix} \right) = (1+y)(1+x) + (1+x)(1+y)$

Star: $X \left[ \begin{pmatrix} 1+x^{-1} \\ 1+y^{-1} \end{pmatrix} \right] = 0$
Fractal Codes

Algebraic representation of Hamiltonian
Example: toric code

Plaquette: $Z \begin{pmatrix} 1 + y \\ 1 + x \end{pmatrix}$

Star: $X \begin{pmatrix} 1 + x^{-1} \\ 1 + y^{-1} \end{pmatrix}$

Yoshida, 2013