QUANTUM THEORY OF LARGE SYSTEMS OF NON-RELATIVISTIC MATTER

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Contents

1 Introduction

These are some notes to the course taught by J. Fröhlich at the 1994 summer school at Les Houches. While teaching and being taught at that school – and the hiking, early in the morning and, sometimes, during week ends – were extremely pleasant and rewarding, the preparation of publishable notes superceeding those more than three hundred pages of handwritten lecture notes prepared and distributed during the school turned out to be a very painful process. It constantly collided with more urgent duties (which ETH pays us for).

The reasons why we finally managed to produce some notes are, first of all, the infinite patience of François David and Paul Ginsparg, and, second, the circumstance that, in these times of computers and of TEX, we could adapt and modify TEXfiles of papers written for other purposes.

1.1 Sources, and acknowledgements

Here are the main sources for this first part of our notes:

J. Fröhlich, R. Götschmann, and P.-A. Marchetti, J. Phys. A **28**, 1169 (1995): for Chapter 5.

J. Fröhlich, R. Götschmann, and P.-A. Marchetti, "The Effective Gauge Field Action of a System of Non-Relativistic Electrons", to appear in Commun. Math. Phys. (1995): for Chapter 5.

J. Fröhlich, T. Kerler, U.M. Studer, and E. Thiran, "Structuring the Set of Incompressible Quantum Hall Fluids", submitted to Nucl. Phys. B: for Chapter 8.

J. Fröhlich and U.M. Studer, in: "New Symmetry Principles in Quantum Field Theory", J. Fröhlich, G. t'Hooft, A. Jaffe, G. Mack, P.K. Mitter, and R. Stora (eds), New York: Plenum Press 1992 (page 195): for Chapters 2, 4 and 6.

J. Fröhlich and U.M. Studer, Rev. Mod. Phys. **65**, 733 (1993): for Chapters 2, 3, 4, 6 and 7.

J. Fröhlich, U.M. Studer, and E. Thiran, "A Classification of Quantum Hall Fluids", to appear in J. Stat. Phys. (1995): for Chapter 8.

R. Götschmann, T. Kerler and P.-A. Marchetti deserve our thanks for having contributed their ideas to some of the work underlying these notes. We are also indebted to L. Michel for having guided us through high-dimensional lattices, a mathematical theory that is important for the material in Chapter 8. We thank J. Avron for his encouraging interest in our work. J. Fröhlich is very grateful to U.M. Studer, who spent many hours in front of a computer screen putting these notes in shape. We also thank A. Schultze for her help in adding final touches to these notes. J. Fröhlich sincerely thanks the organizers, François and Paul, for having invited him to present these lectures, for having organized such a marvellous school, and for their patience.

1.2 Topics not treated in these notes

The "*standard model*" of the physics of atoms, molecules and condensed matter, at energies per particle below, say, a tenth of its relativistic rest energy and usually much smaller, describes non-relativistic, quantum-mechanical electrons and nuclei coupled to the quantized radiation field (cutoff in the ultraviolet) and to additional, external electromagnetic fields. It was planned to include a chapter about fundamental properties of the standard model, including stability, non-existence of the ultraviolet limit of quantum electrodynamics (QED) with non-relativistic matter, its "cutoff independence", etc.; (for a recent paper on some aspects of QED with non-relativistic matter see: V. Bach, J. Fröhlich, and I.M. Sigal, "Mathematical Theory of Non-Relativistic Matter and Radiation", to appear in Lett. Math. Phys. (1995), and refs. given there). We planned to sketch the answer to the question why a condensed matter physicist studying transport properties of three-dimensional electron gases can usually forget the radiation field and the fact that electrons are coupled to it. We also planned to explain why if the world were two-dimensional the radiation field would most likely have a drastic effect on the low-energy properties of electron gases; (emergence of a marginal Fermi liquid). A physical system that appears to mimic two-dimensional QED is a quantum Hall fluid at a filling factor of $\frac{1}{2}$ (or $\frac{1}{4}$). It would have been desirable to include an analysis of this system in Chapter 6.

Unfortunately, time did not permit us to include any of this basic material in our notes; (some of it was included in the handwritten notes available at the school). We strongly encourage the reader to study E.H. Lieb's selecta, "The Stability of Matter: From Atoms to Stars" (Berlin, Heidelberg, New York: Springer-Verlag 1991), for fundamental results on non-relativistic quantum theory.

Another topic, originally planned to appear in these notes, concerns the *magnetic properties* of non-relativistic matter. There was a chapter on magnetism in the handwritten notes; but we were unable to include it here. Various forms of magnetism are intimately related to electron spin and the $SU(2)$-gauge invariance of non-relativistic quantum theory. The $U(1) \times SU(2)$-gauge invariance of non-relativistic quantum theory and some of its consequences *are* discussed in our notes at some length, but, for example, the extension of the approach sketched in Chapter 5 for the electric ($U(1)$) current density to the *spin* ($SU(2)$) *current density* had to be omitted. In this connection, we recommend that the reader consult the papers by Cattaneo et al. and by Leutwyler and the book by Fradkin quoted in the references.

Of course, a study of magnetic properties of various tight-binding models and Heisenberg type models had to be omitted, too. Rigorous results in this area are scarce; but there is steady progress, at the heuristic and at the mathematically rigorous level, that we would have liked to review, had time permitted us to do so.

We hope there will be another opportunity to make up for these omissions. They leave the picture of non-relativistic quantum theory drawn in these notes rather incomplete. Nevertheless we hope that, in these notes, we are laying out, with adequate precision, sufficiently many pieces of the puzzle to enable the reader to guess what would fit into some of the holes we leave. We also hope that we are able to convey to the reader some of the excitement we felt when working on the problems reviewed in our notes. We believe that our analysis of the $U(1) \times SU(2)$-gauge invariance of non-relativistic quantum theory, of the quantum-mechanical Larmor theorem and of their general consequences in condensed matter physics, of the generating functions of (connected) current Green functions (effective actions), and our approach towards classifying states of non-relativistic matter ("gauge theory of thermodynamic phases") and, as a special example, of incompressible quantum Hall fluids will turn out to be of some general usefulness. These topics *are* treated fairly carefully in these notes.

We hope that, during less hectic times, we shall be able to iron out various imperfections and complete the puzzle.

2 The Pauli Equation and its Symmetries

In this section we describe non-relativistic electrons and other non-relativistic particles with spin in an external electromagnetic field. In a one-particle language, the wave functions of such particles satisfy the Pauli equation. We show that the Pauli equation exhibits a basic $U(1)_{em} \times SU(2)_{spin}$ gauge symmetry. This symmetry is, for example, a corner stone of our analysis of incompressible quantum fluids presented in Sects. 6–8. In order to provide a first illustration of the *general* usefulness of this symmetry in quantum theory, we include, at the end of this section, discussions of the Aharonov-Bohm effect and its $SU(2)_{spin}$-cousin, the Aharonov-Casher effect. Further applications of gauge invariance to quantum-mechanical effects will be discussed in Sect. 4.

2.1 Gauge-Invariant Form of the Pauli Equation

In order to describe non-relativistic particles of arbitrary spin s, mass m and charge q, we have to find the correct Zeeman and spin-orbit terms in their Hamiltonian. We recall that Bargmann, Michel and Telegdi (1959) have found a relativistic description of the motion of a classical spin $\mathbf{S}$ in a (slowly varying) external electromagnetic field ($\mathbf{E}$, $\mathbf{B}$). Expanding their result in powers of $\frac{v}{c}$, one obtains the equation of motion already found by Thomas (1927; see also Jackson, 1975)

$$\frac{d\mathbf{S}}{dt} \;=\; \frac{q}{mc}\,\mathbf{S} \times \left[\frac{g}{2}\,\mathbf{B} - \left(\frac{g}{2} - \frac{1}{2}\right)\frac{\mathbf{v}}{c} \times \mathbf{E}\right] + \mathcal{O}\left((\tfrac{v}{c})^2\right) \;, \tag{2.1}$$

where $\mathbf{v}$ is the velocity of the spin (with respect to the laboratory frame), and g is its gyromagnetic ratio. The spin-orbit term (second term in (2.1)) consists of two contributions: The terms proportional to $g/2$ describe the precession of the spin (or *magnetic moment*) in the magnetic field in the rest frame of the spin. The remaining term describes the purely kinematical effect of the *Thomas precession* which is a consequence of the acceleration a *charged* particle experiences in an electric field. Recalling the Poisson bracket relations, $\{S_i, S_j\} = \varepsilon_{ijk}\, S_k$, for a classical spin $\mathbf{S}$, we find that Eq. (2.1) is a hamiltonian equation of motion corresponding to the Hamilton function

$$H_{cl} \;=\; -\mathbf{S} \cdot \left[\frac{q}{mc}\frac{g}{2}\,\mathbf{B} - \frac{q}{mc}\left(\frac{g}{2} - \frac{1}{2}\right)\frac{\mathbf{v}}{c} \times \mathbf{E}\right] . \tag{2.2}$$

If we accept (2.1) as the appropriate non-relativistic Heisenberg equation of motion for the (quantum-mechanical) spin operator $\mathbf{S}$ (in the spin-s representation) then the wave function $\psi^{(s)}$ of the particle, a $(2s+1)$-component complex spinor, satisfies the following *Pauli equation*

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\psi^{(s)} \;&=\; q\Phi\,\psi^{(s)} - \boldsymbol{\mu}_{spin}\cdot\mathbf{B}\,\psi^{(s)} + \frac{1}{2m}\,\mathbf{\Pi}^2\psi^{(s)} \\
-\frac{1}{2mc}&\Big[\mathbf{\Pi}\cdot\Big((\boldsymbol{\mu}_{spin} - \frac{q}{2mc}\mathbf{S})\times\mathbf{E}\Big) + \Big((\boldsymbol{\mu}_{spin} - \frac{q}{2mc}\mathbf{S})\times\mathbf{E}\Big)\cdot\mathbf{\Pi}\Big]\psi^{(s)}\,, \qquad (2.3)
\end{aligned}
$$

where the (intrinsic) magnetic moment of the particle is defined by

$$
\boldsymbol{\mu}_{spin} \;:=\; \frac{g\,\mu}{\hbar}\,\mathbf{S}\;, \qquad (2.4)
$$

and, in the spin-s representation, the spin operator $\mathbf{S}$ is given by

$$
\mathbf{S} \;:=\; \frac{\hbar}{2}\,\mathbf{L}^{(s)} \;=\; \frac{\hbar}{2}\,(L_1^{(s)},\, L_2^{(s)},\, L_3^{(s)})\;. \qquad (2.5)
$$

Here $(L_A^{(s)})_{A=1}^3$ are hermitian generators of the Lie algebra $su(2)$ in the spin-s representation normalized such that $L_A^{(1/2)} = \sigma_A$, where σ_1, σ_2 and σ_3 are the usual Pauli matrices. Furthermore, for charged particles, we have that $\mu = \frac{q\hbar}{2mc}$. In particular, for an *electron,* $-\mu = \frac{e\hbar}{2m_oc} =: \mu_B = 5.79\times 10^{-9}\,\mathrm{eV/Gauss}$, the Bohr magneton, and $g = 2$. Other examples are the *proton* and the *neutron,* where $\mu = \frac{e\hbar}{2mc}$, with m the proton mass, and the g-factors are given by $g = 5.59$ and $g = -3.83$, respectively.

Next, we show that, by "completing the square" in the Pauli equation (2.3), we obtain an equation with an astonishingly rich symmetry, namely with a *local* $U(1)_{em}\times SU(2)_{spin}$ *symmetry*. Since the modification needed is a term of order $\mathcal{O}(1/m^3)$, this symmetry should really be viewed as a fundamental property of non-relativistic quantum mechanics.

Let $x^0 := ct$, and $x := (x^\mu) = (x^0, \mathbf{x})$, where $\mathbf{x} := (x^1, x^2, x^3) \in \mathbb{E}^3$ (the three-dimensional euclidean space). We introduce the *covariant derivative* in the μ-direction by setting

$$
D_\mu \;:=\; \frac{\partial}{\partial x^\mu} + i a_\mu(x) + \rho_\mu(x)\;, \qquad \mu = 0,\ldots,3\;, \qquad (2.6)
$$

where the real-valued functions a_μ are given by

$$
a_0(x) \;:=\; \frac{q}{\hbar c}\Phi(x)\;, \quad \text{and} \quad a_k(x) \;:=\; -\frac{q}{\hbar c}A_k(x)\;, \quad k = 1,2,3\;, \qquad (2.7)
$$

and where the $su(2)$-valued functions ρ_μ are defined by

$$
\rho_\mu(x) \;:=\; i\sum_{A=1}^{3}\rho_{\mu A}(x)\,L_A^{(s)}\;, \qquad \mu = 0,\ldots,3\;. \qquad (2.8)
$$

The coefficients $\rho_{\mu A}$ are given by

$$\rho_{0A}(x) \;:=\; -\frac{g\,\mu}{2\hbar c}\, B_A(x)\;, \qquad A = 1,2,3\;, \tag{2.9}$$

and

$$\rho_{kA}(x) \;:=\; \Big(-\frac{g\,\mu}{2\hbar c} + \frac{q}{4mc^2}\Big) \sum_{B=1}^{3} \varepsilon_{kAB}\, E_B(x)\;, \qquad A,\, k = 1,2,3\;, \tag{2.10}$$

where ε_{kAB} is the sign of the permutation $(k\,A\,B)$ of $(1\,2\,3)$.

With the help of the covariant derivatives D_μ we are able to write the Pauli equation (2.3) in the compact form

$$i\hbar c\, D_0\, \psi^{(s)}(x) \;=\; -\frac{\hbar^2}{2m} \sum_{k=1}^{3} D_k D_k\, \psi^{(s)}(x)\;, \tag{2.11}$$

where a term of order $\mathcal{O}(\rho^2)$ has been added. For an electron it can be seen to be equal to half of $\frac{e^2\hbar^2}{8m_o^3c^4}\,\mathbf{E}^2\,\psi$, which can be absorbed into a one-body potential acting on ψ.

The form (2.11) of the Pauli equation shows that non-relativistic quantum mechanics has a $U(1)_{em}\times SU(2)_{spin}$ *gauge symmetry*. The *gauge transformations* are defined as follows:

$$\begin{aligned} U(1)_{em}: \quad & a_\mu(x) \;\mapsto\; {}^\chi a_\mu(x) \;:=\; a_\mu(x) + (\partial_\mu\, \chi)(x) \\ & \psi^{(s)}(x) \;\mapsto\; {}^\chi\psi^{(s)}(x) \;:=\; e^{-i\chi(x)}\, \psi^{(s)}(x)\;, \end{aligned} \tag{2.12}$$

where χ is an arbitrary, real-valued function on space-time $\mathbb{R}\times\mathbb{E}^3$, and

$$\begin{aligned} SU(2)_{spin}: \quad & \rho_\mu(x) \;\mapsto\; {}^g\rho_\mu(x) \;:=\; g(x)\, \rho_\mu(x)\, g^{-1}(x) + g(x)\, (\partial_\mu\, g^{-1})(x) \\ & \psi^{(s)}(x) \;\mapsto\; {}^g\psi^{(s)}(x) \;:=\; g(x)\, \psi^{(s)}(x)\;, \end{aligned} \tag{2.13}$$

where g is (the spin-s representation of) an arbitrary $SU(2)$-valued function on $\mathbb{R}\times\mathbb{E}^3$. Note that, for constant gauge transformations g, ρ_μ transforms according to the adjoint action of $SU(2)$ (on its Lie algebra $su(2)$) which, by (2.9) and (2.10), (in an active interpretation) corresponds to global rotations of the fields $\mathbf{E}$ and $\mathbf{B}$ in physical space. For space-time dependent gauge transformations, there appears an additional inhomogeneous term in (2.13). A full geometrical interpretation of the $SU(2)$ gauge symmetry will be given in the next section. See also the work by Anandan (1989, 1990) for related observations.

We note that Eq. (2.11) can be thought of as the *Euler-Lagrange equation* corresponding to the following $U(1)\times SU(2)$ *gauge-invariant* action functional

$$S(\psi^{(s)*},\, \psi^{(s)};\, a,\, \rho) \;=\; \int dt\, d^3\mathbf{x} \Big[i\hbar c\, \psi^{(s)*}(x)(D_0\, \psi^{(s)})(x) - \frac{\hbar^2}{2m} \sum_{k=1}^{3} (D_k \psi^{(s)})^*(x)\, (D_k \psi^{(s)})(x) \Big] \,. \tag{2.14}$$

This action functional and generalizations thereof provide a convenient starting point for a functional integral formulation of non-relativistic many-body theory.

We illustrate the formalism described so far by reviewing some basic effects in non-relativistic quantum mechanics from the point of view of its $U(1) \times SU(2)$ gauge symmetry.

2.2 Aharonov-Bohm Effect

A key effect reflecting Weyl's $U(1)_{em}$ gauge principle realized in quantum theory is the *Aharonov-Bohm effect* (Aharonov and Bohm, 1959): Consider the scattering of quantum-mechanical particles at a magnetic solenoid. [The wave functions of the particles are required to vanish inside the solenoid.] Then the diffraction pattern seen on a screen depends non-trivially on the magnetic flux, Φ, through the solenoid. The dependence is periodic with period $\frac{hc}{q}$, where q is the charge of the particles. This is a consequence of the fact that the vector potential $\mathbf{A}$ outside the solenoid cannot be gauged away *globally*, in spite of the fact that there is no electromagnetic field, thus leading to *non-integrable* $U(1)$ *phases* of quantum-mechanical wave functions which change interference patterns.

In formulae, we have that $F_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu = 0$ outside the solenoid. Thus, locally, $a_\mu = \partial_\mu \chi$, with $\chi(\mathbf{x}) = \frac{-q}{\hbar c} \int_*^{\mathbf{x}} \mathbf{A} \cdot d\boldsymbol{l}$, where $d\boldsymbol{l}$ denotes the line element along some path of integration from an arbitrary point $*$ to $\mathbf{x}$. The phase factors affecting the interference patterns are then given by $\exp\Big[2\pi i \frac{q}{hc} \oint_\Gamma \mathbf{A} \cdot d\boldsymbol{l}\Big] = \exp\Big[2\pi i \frac{q\Phi}{hc}\Big]$, where Γ is a closed path enclosing the solenoid.

The Aharonov-Bohm effect explains the possibility of *fractional* (or θ- or *abelian braid-*) *statistics* of anyons (Leinaas and Myrheim, 1977; Goldin, Menikoff, and Sharp, 1980, 1981, 1983; Wilczek, 1982a, 1982b; for a review, see Fröhlich, 1990) in two-dimensional systems: Anyons are particles carrying both electric charge q and magnetic flux Φ $(= \sigma_H^{-1} q$ where σ_H is a "Hall conductivity") and hence give rise to Aharonov-Bohm phases which one can interpret as statistical phases; see Subsect. 6.3.

2.3 Aharonov-Casher Effect

One might wonder whether there is a similar interference effect due to the $SU(2)_{spin}$ gauge symmetry of non-relativistic quantum mechanics. The answer is yes: It is the *Aharonov-Casher effect* (Aharonov and Casher, 1984). Consider a system of quantum-mechanical particles with spin s, electric charge 0, but with a magnetic moment $\mu_{spin} \neq 0$, moving in a plane or in three-dimensional space. [The particles could be neutrons, or neutral atoms, ...] Following Aharonov and Casher, we study the influence of a (static) external electric field on the dynamics of such particles. As a consequence of relativistic effects, the moving particles will, in their rest frame, feel a magnetic field that interacts with their magnetic moment. Up to order $\mathcal{O}(v/c)$, this is taken into account by the *spin-orbit term* in the Pauli equation (2.3); see also (2.1).

In the formalism developed above, this effect should be described as follows: The $SU(2)$ gauge potential ρ_μ is defined in Eqs. (2.8)–(2.10), and we find that

$$\rho_{0A}(\mathbf{x}) \;=\; 0\,, \quad \text{and} \quad \rho_{kA}(\mathbf{x}) \;=\; -\frac{g\,\mu}{2\hbar c}\sum_{B=1}^{3}\varepsilon_{kAB}\,E_B(\mathbf{x})\;, \qquad A,\,k=1,2,3\;. \tag{2.15}$$

For general electric fields, the $SU(2)$ curvature, defined by

$$G^A_{\mu\nu} \;:=\; \partial_\mu\rho_{\nu A} - \partial_\nu\rho_{\mu A} - 2\sum_{B,C=1}^{3}\varepsilon_{ABC}\,\rho_{\mu B}\,\rho_{\nu C}\;, \qquad A=1,2,3,\ \text{and}\ \mu,\nu=0,\ldots,3\,,$$

does not vanish on full-measure sets of space, and so we are not surprised to find that the electric field $\mathbf{E}$ causes non-trivial spin-orbit interactions. However, if we consider a system of particles confined to the (x,y)-plane in $\mathbb{E}^3$ which move in the electric field of a charged wire placed along the z-axis, with constant charge Q per unit of length, we encounter an *$SU(2)$ version of the Aharonov-Bohm effect*: Here, the electric field $\mathbf{E}$ is given by $\mathbf{E}(\mathbf{x}) = \frac{Q}{2\pi r^2}(x,\,y,\,0)$, where $r=\sqrt{x^2+y^2}$, and, with (2.15), we find

$$\boldsymbol{\rho}(\mathbf{x}) \;:=\; (\rho_{13}(\mathbf{x}),\,\rho_{23}(\mathbf{x})) \;=\; \frac{g\,\mu\,Q}{4\pi\hbar c\,r^2}\,(y,\,-x)\;, \tag{2.16}$$

and $\rho_{i1}=\rho_{i2}=0$, for $i=1,2$. Note that ρ_{3A} – which does *not* vanish for $A=1,2$ – does not enter the dynamics of a system confined to the (x,y)-plane. One then easily checks that, for a two-dimensional system confined to the (x,y)-plane, the only component of the $SU(2)$ curvature which does not vanish identically is given by

$$G^3_{12}(\mathbf{x}) \;=\; -\frac{g\,\mu}{2\hbar c}\,Q\,\delta(\mathbf{x})\;.$$

The distribution G^3_{12} is supported at the origin, i.e., the $SU(2)$ connection ρ is "flat" outside the wire. Thus, locally, it is possible to write ρ as a pure gauge, i.e., $\rho_k = g\,\partial_k g^{-1}$, with $g=\exp\!\left[-i\int_*^{\mathbf{x}}\boldsymbol{\rho}\cdot d\boldsymbol{l}\;L_3^{(s)}\right]$, where $d\boldsymbol{l}$ is as above. However, the scattering of the particles at the charged wire depends on its charge per unit length, Q, because, although ρ is flat except at the origin, it cannot be gauged away *globally*. Therefore, ρ gives rise to "*non-integrable*

$SU(2)$ *phase factors*" in the wave functions of the particles which affect their interference patterns. These phase factors are given by $\exp\left[i \oint_\Gamma \boldsymbol{\rho} \cdot d\boldsymbol{l}\right] = \exp\left[2\pi i \frac{g\mu}{2hc} Q\right]$, where Γ is a path enclosing the wire, and the patterns are periodic in Q with a period given by $\frac{2hc}{g\mu}$.

This effect was first described by Aharonov and Casher (1984) in a somewhat more classical language. A general discussion of this effect, much along the lines of thought sketched above, can be found in Anandan (1989, 1990).

We recall that the Aharonov-Bohm effect explains why two-dimensional quantum theory can describe anyons with fractional statistics, namely particles carrying charge and flux. It is natural to ask whether the Aharonov-Casher effect also has something to do with exotic statistics in two-dimensional quantum theory. The answer is yes! The Aharonov-Casher effect is closely related to the existence of particles in two-dimensional quantum theory with *non-abelian braid statistics* (Fröhlich, 1988; Fredenhagen, Rehren, and Schroer, 1989; Fröhlich and Gabbiani, 1990; Fröhlich, Gabbiani, and Marchetti, 1990; Fröhlich and Marchetti, 1991). Such particles have topological interactions that can be described by an $SU(2)$ Knizhnik-Zamolodchikov connection (Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987). Consider, for example, a two-dimensional chiral spin liquid made of particles with spin $s_o \geq 1$ (if such systems exist). An incompressible chiral spin liquid of this type will most likely exhibit excitations of arbitrary spin $s = \frac{1}{2}, \ldots, s_o$. The claim is that an excitation of non-zero spin $s < s_o$ exhibits non-abelian braid statistics, (as pointed out in Fröhlich, Kerler, and Marchetti (1992)). This will be discussed further in Subsect. 6.3.

3 Gauge Invariance in Non-Relativistic Quantum Many-Particle Systems

In this section, we build upon and generalize the formalism outlined in the previous section. It is our aim to describe non-relativistic quantum-mechanical systems in (one), two, and three space dimensions, composed of particles with arbitrary spin coupled to external electromagnetic fields, variable background metrics, and affine spin connections on spaces of non-vanishing curvature and torsion.

The energy-mass distribution of the background gives rise to curvature which describes gravitational fields; (torsion is assumed to vanish in gravity). Torsion and curvature can also provide an effective description of crystalline backgrounds with dislocations and disclinations. Such a geometric description of the background is reasonable, provided the energy of a quantum-mechanical particle is so small that the lattice structure of the background cannot be resolved, and the background may be treated as a “smooth” manifold, i.e., provided the typical wave length of particles is much larger than the crystal lattice spacing. Furthermore, non-trivial background metrics can account for off-diagonal disorder in the systems and for a variable effective mass.

We begin this section by reviewing a geometrical framework which is well suited to describe all these phenomena; (for more mathematical background, see, e.g., Eguchi, Gilkey, and Hanson, 1980, Bleecker, 1981, and de Rham, 1984; for a brief summary of basic notions in differential geometry, see also Sect. 2 in Alvarez-Gaumé and Ginsparg, 1985). Apart from describing possible physical effects related to curvature and torsion, the purpose of the general formalism developed here is to elucidate the geometrical meaning and origin of the $U(1)_{em} \times SU(2)_{spin}$ gauge invariance of non-relativistic quantum mechanics.

Since we are interested in time-dependent many-body systems, it will be convenient to work in a Feynman-Berezin path integral formalism (see, e.g., Negele and Orland, 1987, Fradkin, 1991, and Feldman, Knörrer, and Trubowitz, 1992). We show that the action functionals governing such systems are $U(1) \times SU(2)$ gauge-invariant. This gives rise to powerful Ward identities which are central in a general treatment of incompressible quantum fluids in two dimensions and their generalized Hall effects; see Sect. 6 below, and Fröhlich and Studer (1993b). At the end of this section, we present a quantum-mechanical version of Larmor's theorem, including spin degrees of freedom. Further applications of the general formalism developed in this section to basic effects in quantum many-body theory will be given in Sect. 4.

3.1 Differential Geometry of the Background

As stated above, the main reasons for introducing the following geometrical framework are an elucidation of the geometrical meaning of the $U(1)_{em} \times SU(2)_{spin}$ gauge invariance of non-relativistic quantum mechanics and a preparation for treating such systems in “moving coordinates”; see Subsect. 3.3.

Under the condition of low energy described above, physical space is a ($d = 2$ or 3)-dimensional manifold, M, with a possibly time-dependent riemannian metric, and space-time is given by $N := \mathbb{R} \times M$. The system is confined to the interior of a space-time cylinder $\Lambda \subset N$. The intersection of Λ with a fixed-time slice is denoted by Ω_t, where t is time. In local coordinates, points in M are denoted by $\mathbf{x}, \mathbf{y}, \ldots$, points in N by $x := (t, \mathbf{x})$, $y := (t, \mathbf{y}), \ldots$. The riemannian metric on M is denoted by $g_{ij}(t, \mathbf{x})$, and space-time N carries the "lorentzian" metric $\eta_{\mu\nu}(x)$, where $\eta_{00}(x) = 1$, $\eta_{0i}(x) = \eta_{i0}(x) = 0$, $\eta_{ij}(x) = -g_{ij}(t, \mathbf{x})$, where the indices range over $i, j = 1, \ldots, d$, and $\mu, \nu = 0, \ldots, d$. In the tangent space at a point $\mathbf{x} \in M$ we also have the flat cartesian metric, δ_{AB}, with $A, B = 1, \ldots, d$. [Similarly, in the tangent space at a space-time point $x \in N$ we have the usual "minkowskian" metric $\eta^0_{\alpha\beta}$, with $\alpha, \beta = 0, \ldots, d$.]

If the dimension of M is two, we imagine that M is a surface embedded in a three-dimensional riemannian manifold L with metric also denoted by $g_{ij}(t, \mathbf{x})$, and the metric on M is the induced one.

Since, in *non-relativistic* quantum mechanics, time is merely a *parameter* we temporarily omit it from our notations and focus our attention on the description of the "spatial" geometry of M or L, respectively. In order to be able to describe particles of *arbitrary spin* $s = 0, \frac{1}{2}, 1, \ldots$ moving in M, it is necessary to make use of the *(co)tangent frame* - or *dreibein formalism*. This formalism, involving *local* bases in the (co)tangent spaces (orthonormal frames), naturally incorporates two *local* symmetry groups: the group of coordinate reparametrizations of the manifold (diffeomorphisms) and the group of local frame rotations ($SO(3)$ gauge transformations). Wave functions of particles of half-integral spin will transform under spinor representations of the frame rotation group.

In the cotangent bundle to L, $T^*(L)$, we choose (smooth) sections of 1-forms, $(e^A)_{A=1}^3$, with the property that they form an orthonormal basis (or *orthonormal frame*) in the cotangent spaces $T^*_{\mathbf{x}}(L)$, $\mathbf{x} \in L$. The components of the orthonormal frame $(e^A(\mathbf{x}))_{A=1}^3$ in the coordinate basis $(dx^j)_{j=1}^3$ of $T^*_{\mathbf{x}}(L)$ are denoted by $e^A{}_i(\mathbf{x})$ and are called *dreibein (fields)*. If $\dim M = 2$ we choose $(e^A(\mathbf{x}))_{A=1}^3$ such that, for $\mathbf{x} \in M \subset L$, $e^3(\mathbf{x})$ is orthogonal to $T^*_{\mathbf{x}}(M)$ in the metric of $T^*_{\mathbf{x}}(L)$. The metric on L can be expressed in terms of the dreibein as follows:†

$$g_{ij}(\mathbf{x}) \;=\; \delta_{AB}\, e^A{}_i(\mathbf{x})\, e^B{}_j(\mathbf{x}) \;. \tag{3.1}$$

If $\dim M = 2$, we choose local coordinates on L in a neighbourhood of M such that the metric on M at a point $\mathbf{x}$ is given by

$$g_{ij}(\mathbf{x}) \;=\; \sum_{A,B=1}^{2} \delta_{AB}\, e^A{}_i(\mathbf{x})\, e^B{}_j(\mathbf{x}) \;, \qquad i, j = 1, 2\,, \tag{3.2}$$

†Throughout this work, if not stated explicitly, the Einstein summation convention over repeated indices is understood.

i.e., the coordinate x^3 is transversal to M. In the following we focus on the geometry of L, thinking of the "background manifold" M as being identified with L (for $d=3$), or as being a proper submanifold of L (for $d=2$) embedded in L in the way just explained.

The inverse of the dreibein $e^A_{\ i}(\mathbf{x})$ is given by

$$\mathcal{E}_A^{\ i}(\mathbf{x}) \ := \ \delta_{AB}\, g^{ij}(\mathbf{x})\, e^B_{\ j}(\mathbf{x}) \ , \tag{3.3}$$

where $(g^{ij}(\mathbf{x}))$ is the inverse matrix of $(g_{ij}(\mathbf{x}))$. Clearly,

$$\mathcal{E}_A^{\ i}(\mathbf{x})\, e^B_{\ i}(\mathbf{x}) \ = \ \delta^B_A \ , \quad \text{and} \quad g^{ij}(\mathbf{x}) \ = \ \delta^{AB}\, \mathcal{E}_A^{\ i}(\mathbf{x})\, \mathcal{E}_B^{\ j}(\mathbf{x}) \ . \tag{3.4}$$

To summarize, the dreibein $e^A_{\ i}(\mathbf{x})$ is the matrix which transforms the coordinate basis (dx^i) of 1-forms in $T^*_{\mathbf{x}}(L)$ to an orthonormal basis of 1-forms (orthonormal frame), $(e^A(\mathbf{x}))$, in $T^*_{\mathbf{x}}(L)$, i.e.,

$$e^A(\mathbf{x}) \ = \ e^A_{\ i}(\mathbf{x})\, dx^i \ , \quad \text{and} \quad g^{ij}(\mathbf{x})\, e^A_{\ i}(\mathbf{x})\, e^B_{\ j}(\mathbf{x}) \ = \ \delta^{AB} \ . \tag{3.5}$$

Similarly, $\mathcal{E}_A^{\ i}(\mathbf{x})$ transforms the basis $(\frac{\partial}{\partial x^i})$ of vector fields in $T_{\mathbf{x}}(L)$ to an orthonormal basis of vector fields, $(\mathcal{E}_A(\mathbf{x}))$, in $T_{\mathbf{x}}(L)$, i.e.,

$$\mathcal{E}_A(\mathbf{x}) \ = \ \mathcal{E}_A^{\ i}(\mathbf{x})\, \frac{\partial}{\partial x^i} \ = \ \mathcal{E}_A^{\ i}(\mathbf{x})\, \partial_i \ , \quad \text{and} \quad g_{ij}(\mathbf{x})\, \mathcal{E}_A^{\ i}(\mathbf{x})\, \mathcal{E}_B^{\ j}(\mathbf{x}) \ = \ \delta_{AB} \ . \tag{3.6}$$

From Eq. (3.1) it follows that the dreibien $e^A_{\ i}$ is a "square root" of the metric (g_{ij}). This "square root", however, is not unique. It is only defined up to *local frame rotations*. [Note that the dreibein $e^A_{\ i}$ has 9 independent components while the metric (g_{ij}) has only 6. It is the group of local frame rotations which accounts for the mismatch: $\dim SO(3) = 3$.] Thus every cotangent space $T^*_{\mathbf{x}}(L)$, $\mathbf{x} \in L$, carries a three-dimensional (spin-1) representation, $R(\mathbf{x}) \in SO(3)$, of the rotation group. The rotations $R(\mathbf{x})$ act on the dreibein $e^A_{\ i}(\mathbf{x})$ as "*gauge transformations*" in the following way:

$$\begin{aligned} e^A_{\ i}(\mathbf{x}) \ \mapsto \ {}^R e^A_{\ i}(\mathbf{x}) \ &:= \ R(\mathbf{x})^A_{\ B}\, e^B_{\ i}(\mathbf{x}) \ , \quad \text{or} \\ e(\mathbf{x}) \ \mapsto \ {}^R e(\mathbf{x}) \ &:= \ R(\mathbf{x})\, e(\mathbf{x}) \ . \end{aligned} \tag{3.7}$$

In order to define *parallel transport* on L in the (co)tangent frame formalism, one introduces the notion of an *affine spin connection*, $\omega^A_{\ B}$. This connection is an $so(3)$-valued

1-form on L. It can be expanded in the coordinate basis (dx^i), or in the orthonormal frame $(e^A(\mathbf{x}))$ of $T^*_{\mathbf{x}}(L)$:

$$\omega^A{}_B(\mathbf{x}) = \omega^A{}_{Bi}(\mathbf{x})\,dx^i = \omega^A{}_{BC}(\mathbf{x})\,e^C{}_i(\mathbf{x})\,dx^i = \omega^A{}_{BC}(\mathbf{x})e^C(\mathbf{x}) \ . \tag{3.8}$$

Notice that, with the help of the dreibein and its inverse (see (3.5) and (3.6)), the indices of any tensor can be changed at will from coordinate indices, $i, j, \ldots$, to frame indices, $A, B, \ldots$. Geometrically, the connection $\omega^A{}_B\,(\xi; x)$ determines an isomorphism from $T^*_x(L)$ to $T^*_{x+\xi}(L)$, where $\boldsymbol{\xi} = (\xi^i)$ is an infinitesimal vector, and

$$\omega^A{}_B(\boldsymbol{\xi}\,;\, x) := \omega^A{}_{Bi}(x)\xi^i(x) \ . \tag{3.9}$$

A tensor important to characterize the affine spin connection $\omega^A{}_B$ is the *torsion 2-form*, $\mathcal{T}^A$, associated to e^A and $\omega^A{}_B$. It is defined through *Cartan's first structure equation*

$$\begin{aligned} \mathcal{T}^A(\mathbf{x}) &= \mathcal{T}^A{}_{ij}(\mathbf{x})\,dx^i \wedge dx^j \\ &= \mathrm{d}e^A(\mathbf{x}) + \omega^A{}_B(\mathbf{x}) \wedge e^B(\mathbf{x}) \ , \end{aligned} \tag{3.10}$$

where d denotes exterior differentiation (given in local coordinates by $\mathrm{d} = dx^i\,\frac{\partial}{\partial x^i}$) and $\wedge$ stands for the (totally antisymmetric) exterior product; see, e.g., Eguchi, Gilkey, and Hanson (1980), Bleecker (1981), and de Rham (1984).

It is customary to decompose the affine spin connection into two parts:

$$\omega^A{}_B(\mathbf{x}) = \lambda^A{}_B(\mathbf{x}) + \kappa^A{}_B(\mathbf{x}) \ , \tag{3.11}$$

where $\lambda^A{}_B$ is the *Levi-Civita connection* and $\kappa^A{}_B$ is the so-called *contorsion field.* The Levi-Civita connection plays a prominent role in general relativity, and hence it is important if we wish to study quantum mechanics in a curved space-time. On any riemannian manifold $(L,\, g_{ij})$, it is uniquely determined by requiring that its torsion vanishes, i.e., if one replaces $\omega^A{}_B$ by $\lambda^A{}_B$ in Eq. (3.10) the resulting expression has to vanish. The components $\lambda^A{}_{Bi}$ can be expressed purely in terms of the dreibein $e^A{}_i$, its derivatives, and its inverse $\mathcal{E}_A{}^i$ (Eguchi, Gilkey, and Hanson, 1980; Alvarez-Gaumé and Ginsparg, 1985).

$$\begin{aligned} \lambda^A{}_{Bi}(\mathbf{x}) = \ & \frac{1}{2}\Big[\,\mathcal{E}_B{}^k(\mathbf{x})\left(\partial_k e^A{}_i - \partial_i e^A{}_k\right)(\mathbf{x}) + \delta^{AC}\delta_{BD}\,\mathcal{E}_C{}^k(\mathbf{x})\left(\partial_i e^D{}_k - \partial_k e^D{}_i\right)(\mathbf{x}) \\ & + \delta^{AC}\delta_{DE}\,e^D{}_i(\mathbf{x})\mathcal{E}_B{}^k(\mathbf{x})\mathcal{E}_C{}^l(\mathbf{x})\left(\partial_k e^E{}_l - \partial_l e^E{}_k\right)(\mathbf{x})\,\Big] \ . \end{aligned} \tag{3.12}$$

The contorsion field $\kappa^A{}_B$ contains additional information about the affine geometry of L. This information is relevant if one considers the motion of *spinning* particles in L; see Subsect. 3.2.

As a last geometrical notion we introduce the *curvature 2-form* $\mathcal{R}^A{}_B(\mathbf{x})$ of the connection $\omega^A{}_B$ on L. It is defined through *Cartan's second structure equation*

$$\begin{aligned} \mathcal{R}^A{}_B(\mathbf{x}) &= \mathcal{R}^A{}_{Bij}(\mathbf{x})\, dx^i \wedge dx^j \\ &= \mathrm{d}\omega^A{}_B(\mathbf{x}) + \omega^A{}_C(\mathbf{x}) \wedge \omega^C{}_B(\mathbf{x}) \ . \end{aligned} \tag{3.13}$$

It is easy to deduce from Eqs. (3.10) and (3.13) how ω and $\mathcal{R}$ transform under the gauge transformations (3.7) of the dreibein:

$$\begin{aligned} \omega(\mathbf{x}) \ \mapsto \ {}^R\omega(\mathbf{x}) &:= R(\mathbf{x})\, \omega(\mathbf{x})\, R^T(\mathbf{x}) + R(\mathbf{x})\, \mathrm{d}R^T(\mathbf{x}) \ , \\ \mathcal{R}(\mathbf{x}) \ \mapsto \ {}^R\mathcal{R}(\mathbf{x}) &:= R(\mathbf{x})\, \mathcal{R}(\mathbf{x})\, R^T(\mathbf{x}) \ , \end{aligned} \tag{3.14}$$

where the superscript ^{T} denotes transposition of a matrix. Furthermore, the following transformation properties follow from Eq. (3.14) and from the decomposition of ω into the two parts λ and κ, as given in (3.11):

$$\begin{aligned} \lambda(\mathbf{x}) \ \mapsto \ {}^R\lambda(\mathbf{x}) &:= R(\mathbf{x})\, \lambda(\mathbf{x})\, R^T(\mathbf{x}) + R(\mathbf{x})\, \mathrm{d}R^T(\mathbf{x}) \ , \\ \kappa(\mathbf{x}) \ \mapsto \ {}^R\kappa(\mathbf{x}) &:= R(\mathbf{x})\, \kappa(\mathbf{x})\, R^T(\mathbf{x}) \ . \end{aligned} \tag{3.15}$$

The contorsion field κ transforms *homogeneously* under gauge transformations, i.e., according to the adjoint action of the gauge group.

We end this subsection with some remarks about the physical relevance of the geometrical notions introduced above in connection with crystalline backgrounds exhibiting dislocations and disclinations. We summarize results contained, e.g., in Kleinert (1989; see also Katanaev and Volovich, 1992) where more details can be found.

Let $\mathbf{y}_n \in \mathbb{E}^3$ denote the lattice sites of a perfect crystalline background. If the crystal suffers some distortion, the original lattice sites get shifted to $\mathbf{x}_n$, where $\mathbf{x}_n = \mathbf{y}_n + \mathbf{u}(\mathbf{x}_n)$, and *defects* may form. In order to study these defects in the framework of differential geometry, one assumes that the crystalline background can be treated as a continuous (isotropic) medium. Then $\mathbf{u}(\mathbf{x})$ is called the *total distortion field*. It describes a *singular* infinitesimal transformation of euclidean space $\mathbb{E}^3$ (with metric δ_{ij}) into a space L containing *defects* or, from a geometrical point of view, into a manifold with non-vanishing *curvature* and *torsion*.

Densities of different types of defects in L can be expressed in terms of (derivatives of) the distortion field $u_i(\mathbf{x})$ as follows:
The *density of dislocations* (or translational defects) is given by

$$\alpha_{ij}(\mathbf{x}) \;:=\; \varepsilon_i^{\;hk}\,\partial_h\partial_k u_j(\mathbf{x}) \;, \tag{3.16}$$

the *density of disclinations* (or rotational defects) by

$$\Theta_{ij}(\mathbf{x}) \;:=\; \frac{1}{2}\,\varepsilon_i^{\;hk}\varepsilon_j^{\;mn}\,\partial_h\partial_k\partial_m u_n(\mathbf{x}) \;, \tag{3.17}$$

and the general *defect density* by

$$\eta_{ij}(\mathbf{x}) \;:=\; \frac{1}{2}\,\varepsilon_i^{\;hk}\varepsilon_j^{\;mn}\,\partial_h\partial_m\Big(\partial_k u_n + \partial_n u_k\Big)(\mathbf{x}) \;. \tag{3.18}$$

These expressions are non-vanishing, in general, because the distortion field $u_i(\mathbf{x})$ is *singular*! In the presence of a single defect line Γ, the dislocation and disclination densities are both proportional to a δ-function along the line Γ; see Kleinert (1989).

The geometric properties of the manifold L are coded into its metric $g_{ij}(\mathbf{x})$ and contorsion field $\kappa_{jhk}(\mathbf{x})$. In terms of the distortion field $u_i(\mathbf{x})$ they are given, in linear approximation, by

$$g_{ij}(\mathbf{x}) \;=\; \delta_{ij} - \Big(\partial_i u_j + \partial_j u_i\Big)(\mathbf{x}) \;, \tag{3.19}$$

and

$$\kappa_{jhk}(\mathbf{x}) \;=\; \frac{1}{2}\Big[\,\partial_k\Big(\partial_j u_h - \partial_h u_j\Big)(\mathbf{x}) - \partial_j\Big(\partial_h u_k + \partial_k u_h\Big)(\mathbf{x}) + \partial_h\Big(\partial_j u_k + \partial_k u_j\Big)(\mathbf{x})\,\Big] \;, \tag{3.20}$$

where $\kappa_{jhk} = \delta_{AC}\, e^C_{\;j}\, e^B_{\;h}\, \kappa^A_{\;Bk}$; see (3.1) and (3.11).

This allows for a comparison of the defect densities of L, given in (3.16) to (3.18), with the expressions for torsion and curvature of the manifold L; see (3.10)–(3.13). The following relations hold:

$$\alpha_{ij}(\mathbf{x}) \;=\; \varepsilon_i^{\;hk}\kappa_{jhk}(\mathbf{x}) \tag{3.21}$$

$$\Theta_{ij}(\mathbf{x}) \;=\; \mathcal{R}_{ij}(\mathbf{x}) - \frac{1}{2}\,g_{ij}(\mathbf{x})\,\mathcal{R}(\mathbf{x}) \;, \quad \text{and} \tag{3.22}$$

$$\eta_{ij}(\mathbf{x}) \;=\; \overset{\mathrm{LC}}{\mathcal{R}}_{ij}(\mathbf{x}) - \frac{1}{2}\,g_{ij}(\mathbf{x})\,\overset{\mathrm{LC}}{\mathcal{R}}(\mathbf{x}) \;, \tag{3.23}$$

where $\mathcal{R}_{ij} := \mathcal{R}^{A}{}_{ijA} = \mathcal{E}_A{}^k\, e^B{}_i\, \mathcal{R}^A{}_{Bjk}$ is the *Ricci tensor*, and $\mathcal{R} := g^{ij}\, \mathcal{R}_{ij}$ is the *scalar curvature* of the affine spin connection $\omega^A{}_{Bj}$. Similarly, $\overset{\mathrm{LC}}{\mathcal{R}}_{ij}$ and $\overset{\mathrm{LC}}{\mathcal{R}}$ denote the Ricci tensor and scalar curvature, respectively, of the Levi-Civita connection $\lambda^A{}_{Bj}$, the torsion-free part of the affine spin connection; see Eq. (3.11).

For quantum-mechanical particles moving in a crystalline background with defects, the following assumption appears to be reasonable: If the energy of the particles is so small that the lattice structure of the background cannot be resolved, then a (metrically non-trivial) riemannian manifold provides an effective description of the background when studying the *orbital* motion of the particles. [Technically, the Laplacian in the Schrödinger-Pauli equation should be replaced by the Laplace-Beltrami operator associated with the riemannian metric on the manifold.] Moreover, the formalism presented above is well adapted to describing the orbital motion of particles confined (e.g., by some potential) to a curved surface in $\mathbb{E}^3$.

The question of the motion of the spin degrees of freedom, however, is more subtle and will be addressed in the next subsection.

3.2 Systems of Spinning Particles Coupled to External Electromagnetic and Geometric Fields

We start this section by showing how to describe systems of *spinning* particles moving in a geometrically non-trivial background and coupled to an external electromagnetic field.

We assume that the manifold L admits a *spin structure*. Then we may introduce *spinor bundles* over L (associated to the cotangent bundle $T^*(L)$ over L). Let $s = 0, \frac{1}{2}, 1, \ldots$ denote the spin of the particles, i.e., $2s+1$ is the dimension of an irreducible representation of $SU(2) = \widetilde{SO}(3)$ with spin s. The fiber of the spin-s spinor bundle, $E^{(s)}(L)$, over L is isomorphic to the $(2s+1)$-dimensional Hilbert space, $\mathcal{D}^{(s)}$, carrying the spin-s representation of $SU(2)$. *Sections* of the spin-s spinor bundle are denoted by $\psi^{(s)}(\mathbf{x})$. From now on, we choose the gauge transformations $R(\mathbf{x})$ to be $SU(2)$-valued. The action of these gauge transformations on the cotangent bundle $T^*(L)$ is given by their adjoint (spin-1) representation, also denoted by $R(\mathbf{x})$; see (3.7).‡ Under a gauge transformation $R(\mathbf{x})$, a section $\psi^{(s)}(\mathbf{x})$ of $E^{(s)}(L)$ transforms as follows:

$$\psi^{(s)}(\mathbf{x}) \mapsto {}^{R}\psi^{(s)}(\mathbf{x}) := U^{(s)}(R(\mathbf{x}))\, \psi^{(s)}(\mathbf{x}) \;, \tag{3.24}$$

where $U^{(s)}$ is the spin-s representation of $SU(2)$. The transition functions of the spin-s spinor bundle $E^{(s)}(L)$ – which must be specified if the topology of the base manifold L is non-trivial – are inherited from the transition functions of the cotangent bundle $T^*(L)$ by lifting them to the spin-s representation of $SU(2)$. [Since we have assumed that L admits a spin structure, this is possible even if s is half-integer!]

‡There is little danger of confusion.

Physically, what is meant by "spin up" or "spin down" is now a *local notion*, depending on the point $\mathbf{x} \in L$ at which the spin is located and determined by the local frames $(e^A(\mathbf{x}))_{A=1}^3$; see Subsect. 3.1.

The spaces of wave functions in non-relativistic, one-particle quantum mechanics are Hilbert spaces of sections of these spinor bundles. In non-relativistic quantum mechanics, wave functions are complex-valued. We therefore tensor the fiber space $\mathcal{D}^{(s)}$ – real when s is an integer – by $\mathbb{C}$. The structure group of the resulting complexified bundle, denoted by $E_{\mathbb{C}}^{(s)}(L)$, is then $U(1) \times SU(2)$. The factor $U(1)$ (phase transformations of $\psi^{(s)}$) is connected to electromagnetism, as recognized by Weyl more than sixty years ago (Weyl, 1928; see also Weyl, 1918).

In order to keep our notation simple, it is advantageous to formulate quantum mechanics of many-particle systems in the language of second quantization. The sections $\psi^{(s)}(\mathbf{x})$ of $E_{\mathbb{C}}^{(s)}(L)$ over L are then interpreted as *operator-valued distributions* acting on Fock space and subject to *canonical equal-time commutation or anti-commutation relations*

$$
\begin{aligned}
\left[\psi_\alpha^{(s)}(\mathbf{x}),\, \psi_\beta^{(s)*}(\mathbf{y})\right]_\pm &= \frac{1}{\sqrt{g(\mathbf{x})}}\,\delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y})\ , \qquad \text{and} \\
\left[\psi_\alpha^{(s)\sharp}(\mathbf{x}),\, \psi_\beta^{(s)\sharp}(\mathbf{y})\right]_\pm &= 0\ , \qquad \alpha,\beta = 1,\ldots,2s+1\ ,
\end{aligned} \tag{3.25}
$$

where $[\ ,\ ]_+$ denotes the anticommutator and $[\ ,\ ]_-$ the usual commutator, $\psi^{(s)\sharp} = \psi^{(s)}$ or $\psi^{(s)*}$; $\psi^{(s)*}$, the *creation operator*, is the adjoint (on Fock space) of $\psi^{(s)}$, the *annihilation operator*; $g(\mathbf{x})$ denotes the determinant of the metric $(g_{ij}(\mathbf{x}))$ on L. The usual connection between spin and statistics is to choose anticommutators in (3.25), corresponding to Fermi statistics, when s is half-integer, and commutators, corresponding to Bose statistics, when s is integer.

Our purpose is to specify some non-relativistic dynamical laws governing the time-evolution of the operators $\psi^{(s)\sharp}$ in the *Heisenberg picture*. Let $\psi^{(s)\sharp}(x) = \psi^{(s)\sharp}(t,\mathbf{x})$ denote the Heisenberg picture creation - and annihilation operators with initial conditions $\psi^{(s)\sharp}(0,\mathbf{x}) = \psi^{(s)\sharp}(\mathbf{x})$. Geometrically, these operators are sections of a trivially extended spin-s spinor bundle, $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$, over the space-time manifold, $\mathbb{R}\times L$. In order to formulate local dynamical laws for $\psi^{(s)\sharp}(x)$, we need to be able to differentiate these fields in t and $\mathbf{x}$. This necessitates introducing a notion of *parallel transport* in $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$. Parallel transport in the spinor bundle $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$ is defined with the help of a $U(1)\times SU(2)$ *connection*, i.e., by a *vector potential* with values in $\mathbb{R}\oplus su(2)$. Once such a connection is fixed, derivatives of sections $\psi^{(s)\sharp}(x)$ are defined as *covariant derivatives*. Setting $x^0 := ct$, and $x := (x^\mu) = (x^0, \mathbf{x})$, where $\mathbf{x}\in L$, the covariant derivative in the μ-direction is defined by

$$
\mathcal{D}_\mu := \frac{\partial}{\partial x^\mu} + i a_\mu(x) + w_\mu^{(s)}(x)\ , \qquad \mu = 0,\ldots,3\ , \tag{3.26}
$$

where the real-valued 1-form $a = a_\mu(x)\,dx^\mu$ is the $U(1)$ connection, and the $su(2)$-valued 1-form $w^{(s)} = w^{(s)}_\mu(x)\,dx^\mu$ is the $SU(2)$ connection in the spin-s representation of $su(2)$, i.e.,

$$w^{(s)}_\mu(x) \;=\; i \sum_{A=1}^{3} w_{\mu A}(x)\, L^{(s)}_A \;, \tag{3.27}$$

where we have adopted the same notation as in Eq. (2.8): $(L^{(s)}_A)^3_{A=1}$ are hermitian generators of $su(2)$ in the spin-s representation, normalized such that $L^{(1/2)}_A = \sigma_A$, where σ_1, σ_2 and σ_3 are the standard Pauli matrices.

We shall argue shortly that we can identify a with the electromagnetic vector potential (up to multiplication by physical constants). This is no surprise, given the observations in Subsect. 2.1 (see (2.6) and (2.7)). What about $w^{(s)}$? First, from a geometrical point of view, it is clear that the affine spin connection $\omega^A_{\ \,B}$, introduced in (3.8), enters the definition of $w^{(s)}$ since the spinor bundle $E^{(s)}_{\mathbb{C}}(\mathbb{R}\times L)$ is associated to the cotangent bundle $T^*(\mathbb{R}\times L)$, i.e., it inherits the geometrical structure of $T^*(\mathbb{R}\times L)$. Second, based on the observations in Subsect. 2.1 (see (2.6) and (2.8) to (2.10)), we expect the interaction of the external electromagnetic field with the magnetic moment carried by the particles (Zeeman and spin-orbit couplings) to be described by an additional term, $\rho^{(s)} = \rho^{(s)}_\mu(x)\,dx^\mu$, in the $SU(2)$ connection $w^{(s)}$. Since the sum of ω and $\rho^{(s)}$ must be an $SU(2)$ connection, $\rho^{(s)}$ has to transform under $SU(2)$ gauge transformations according to the adjoint action of the gauge group. We use the following notations:

$$w^{(s)}_\mu(x) \;=\; \omega^{(s)}_\mu(x) + \rho^{(s)}_\mu(x) \;, \tag{3.28}$$

where

$$\omega^{(s)}_\mu(x) \;=\; \frac{i}{2} \sum_{A,B,C=1}^{3} \varepsilon_A^{\ \,BC}\, \omega^A_{\ \,B\mu}(x)\, L^{(s)}_C \;, \tag{3.29}$$

$\varepsilon_A^{\ \,BC} = \varepsilon_{ABC}$ is the sign of the permutation $(A\,B\,C)$ of $(1\,2\,3)$, and

$$\rho^{(s)}_\mu(x) \;=\; i \sum_{A=1}^{3} \rho_{\mu A}(x)\, L^{(s)}_A \;. \tag{3.30}$$

All notions introduced in Subsect. 3.1 – defined over space L and its cotangent bundle $T^*(L)$ – can easily be extended to space-time, $\mathbb{R}\times L$, and its cotangent bundle, $T^*(\mathbb{R}\times L) \simeq \mathbb{R}\times T^*(L)$. In a non-relativistic setting, the space-time metric $\eta_{\mu\nu}(x)$ has the property that $\eta_{0i}(x) = 0 = \eta_{i0}(x)$, (see the beginning of Subsect. 3.1), and, as a consequence, most "temporal" components of the different geometrical fields introduced in Subsect. 3.1 vanish. In (3.29), $\omega^A_{\ \,Bi}(x)$, $i = 1, 2, 3$, is given by (3.8), (3.11) and (3.12), and

$$\omega^A{}_{B0}(x) \;=\; \frac{1}{2}\Big[\, \delta^{AC}\delta_{BD}\, \mathcal{E}_C{}^k(x)\, \partial_0 e^D{}_k(x) - \mathcal{E}_B{}^k(x)\, \partial_0 e^A{}_k(x) \,\Big]\;. \tag{3.31}$$

Under an $SU(2)$ *gauge transformation* $R(x)$, i.e., under *local frame rotations* in the cotangent bundle $T^*(\mathbb{R} \times L)$ (see (3.7)), the different terms of the $SU(2)$ connection $w^{(s)}$ transform as follows:

$$\omega_\mu^{(s)}(x) \;\mapsto\; {}^R\omega_\mu^{(s)}(x) \;:=\; U^{(s)}(R(x))\, \omega_\mu^{(s)}(x)\, U^{(s)}(R(x))^* + U^{(s)}(R(x))\, \partial_\mu U^{(s)}(R(x))^*\;, \tag{3.32}$$

which can be inferred from (3.14), and

$$\rho_\mu^{(s)}(x) \;\mapsto\; {}^R\rho_\mu^{(s)}(x) \;:=\; U^{(s)}(R(x))\, \rho_\mu^{(s)}(x)\, U^{(s)}(R(x))^*\;, \tag{3.33}$$

where * denotes the adjoint of a matrix.

If the metric on L is time-independent $\omega_0^{(s)}(x)$ vanishes; see (3.29) and (3.31). After a time-dependent $SU(2)$ gauge transformation, however, it may be different from zero. Furthermore, the $\rho^{(s)}$-part of the $SU(2)$ connection $w^{(s)}$ will, in general, be different from zero. Geometrically, it corresponds to an additional contorsion field yielding non-vanishing torsion; see (3.10). The physical interpretation of the $\omega^{(s)}$-part of the $SU(2)$ connection $w^{(s)}$, as well as the precise identifications with physical quantities of the $\rho^{(s)}$-part of $w^{(s)}$ and of the $U(1)$ connection a will be given below. (The material in Subsect. 2.1 will serve us as a guide.)

Having introduced a $U(1) \times SU(2)$ connection and defined covariant differentiation of the sections $\psi^{(s)\sharp}$, we are now in a position to formulate *local dynamical laws*. It is convenient to use the *lagrangian formalism*, but we could also work in the hamiltonian formalism; see Fröhlich and Kerler (1991). Let us consider a system of non-relativistic particles of fixed spin s, and, to simplify our notations, we drop the superscript $^{(s)}$ from the field operators $\psi^{(s)\sharp}$ and the $SU(2)$ connection $w^{(s)} = \omega^{(s)} + \rho^{(s)}$. Our ansatz for the action of the system is an obvious generalization of the action (2.14) found in Subsect. 2.1. It reads (with $x := (ct, \mathbf{x})$)

$$\begin{aligned} S_\Lambda(\psi^*, \psi; g, a, w) \;\;:=\;\; & \int_\Lambda \sqrt{g(t,\mathbf{x})}\, dt\, d^3\mathbf{x} \Big[i\hbar c\, \psi^*(x)(\mathcal{D}_0\psi)(x) \\ & - \frac{\hbar^2}{2m}\, g^{kl}(t,\mathbf{x})\, (\mathcal{D}_k\psi)^*(x)\, (\mathcal{D}_l\psi)(x) - U(\psi^*,\, \psi)(x) \Big]\;, \end{aligned} \tag{3.34}$$

where the covariant derivatives are given in (3.26), m is the *effective mass* of the particles (sometimes also denoted by m^*; in common situations of solid state physics, it can be considerably smaller than m_o, the mass of the particles in the vacuum), and $U(\psi^*, \psi)(x)$ is a $U(1) \times SU(2)$-*invariant* functional of ψ and ψ^*, e.g., $(y := (ct, \mathbf{y}))$

$$\begin{aligned} U(\psi^*, \psi)(x) \;&:=\; v(x)\, \psi^*(x)\psi(x) \\ +\, \frac{1}{2} \int_\Lambda \sqrt{g(t, \mathbf{y})}\, d^3\mathbf{y}\; &:\Big(\psi^*(x)\psi(x) - n\Big)\, V(t, \mathbf{x}, \mathbf{y})\, \Big(\psi^*(y)\psi(y) - n\Big): \;. \end{aligned} \tag{3.35}$$

The double colons indicate Wick ordering, v is a possibly time-dependent one-body background potential (depending on the background and on the scalar curvature $\mathcal{R}$ of M), V is some two-body potential (e.g., for charged particles a (possibly screened) Coulomb potential, or for neutral atoms or molecules a van der Waals potential), and n is approximately equal to the background density of the system (related to its chemical potential). We recall that $\Lambda \subset \mathbb{R} \times M$ is a cylindrical region in space-time. At fixed time t, we impose Dirichlet boundary conditions at the boundary, $\partial\Omega_t$, of the region Ω_t to which the system is confined.

The *field equation* (or *Euler-Lagrange equation*) for ψ or ψ^* follows by setting the variation of S_Λ with respect to ψ^* or ψ, equal to zero. The resulting equation is a *generalization* of the Pauli equation (2.11) to a system of spinning particles in a geometrically non-trivial background.

In order to illustrate these matters, let us consider a simple situation: We choose space M to be given by $\mathbb{E}^2$ (the (x, y)-plane in $L = \mathbb{E}^3$) or by $\mathbb{E}^3$; $g_{ij}(t, \mathbf{x}) = \delta_{ij}$, for all times t and all $\mathbf{x} \in M$, $\Lambda = \mathbb{R} \times \Omega$ where Ω is some time-independent open set in M. The field equation for ψ, obtained by varying the action S_Λ defined in (3.34) with respect to ψ^*, then reduces to the Pauli equation given in (2.11). This equation coincides with the usual form given in (2.3) (up to a modification of order $\mathcal{O}(1/mm_o^2)$, see Subsect. 2.1) provided we make the same identifications as in Eqs. (2.7)–(2.10): The components of the $U(1)$ connection a (with respect to the coordinate basis (dx^μ) of the cotangent space $T_x^*(\mathbb{R} \times L)$) are given by the electromagnetic potentials Φ and $\mathbf{A}$,

$$a_0(x) \;=\; \frac{q}{\hbar c}\Phi(x) \;, \qquad \text{and} \qquad a_k(x) \;=\; -\frac{q}{\hbar c}A_k(x) \;, \tag{3.36}$$

where q is the charge of the particles. Furthermore, the components of the ρ-part of the $SU(2)$ connection w are expressed in terms of the electromagnetic field $(\mathbf{E}, \mathbf{B})$ as follows:

$$\rho_{0A}(x) \;=\; -\frac{g\mu}{2\hbar c}\, B_A(x) \;, \tag{3.37}$$

where $B_A(x)$ is the A-component of the magnetic field $\mathbf{B}(x)$ in the basis $(e^A(x))$ of $T_x^*(\mathbb{R} \times L)$, and

$$\rho_{kA}(x) \;=\; \Big(-\frac{g\mu}{2\hbar c} + \frac{q}{4mc^2}\Big) \sum_{B=1}^{3} \varepsilon_{kAB}(x)\, E_B(x) \;, \tag{3.38}$$

where $E_B(x)$ is the B-component of the electric field $\mathbf{E}(x)$, and the symbol $\varepsilon_{kAB}(x)$ is defined by

$$\varepsilon_{kAB}(x) \;:=\; e^{C}_{\ k}(x)\, \varepsilon_{CAB} \;, \tag{3.39}$$

where ε_{CAB} is the sign of the permutation $(C\,A\,B)$ of $(1\,2\,3)$. In Eqs. (3.37) and (3.38), the magnetic moment of the particles enters via $g\mu$; see (2.4). Although the orthonormal frames $e^A(x)$ could be chosen to vary, it is simplest, in the present situation, to choose them as $e^{A}_{\ \mu}(x) = \delta^A_\mu$. Then the "geometrical" part ω of the $SU(2)$ connection w clearly vanishes.

In a general situation, when the background of the system has the structure of an *arbitrary riemannian spin manifold* M, the physical interpretation of the connections a and w is straightforward: The $U(1)_{em}$ *connection* a is still expressed in terms of the *electromagnetic potentials*, as in Eq. (3.36). The $SU(2)_{spin}$ *connection* w has been given in Eq. (3.28) with the "geometric" part ω being specified in terms of the *affine spin connection* $\omega^{A}_{\ B}$ on $\mathbb{R}\times L$; see (3.29). Its ρ-part (describing *Zeeman* and *spin-orbit couplings* of the magnetic moment of particles to the external electromagnetic field) always contains the terms in (3.37)–(3.39). The only difference is that, on a general riemannian manifold, it is not possible to choose the dreibein $e^{A}_{\ \mu}(x)$ to be *constant* on all of $\mathbb{R}\times L$.

Remark. We should comment on the physical status of the ω-part in the $SU(2)$ connection w. In the study of *gravitational fields*, the affine spin connection ω is torsion-free. It is given by the Levi-Civita connection λ which is canonically associated to the gravitational metric field g; see (3.12). Hence, if we consider a quantum-mechanical system in a (strong) external gravitational field then ω enters into the description of the motion of the spin of the particles as a *fundamental* physical field.

At the end of Subsect. 3.1, we have also argued that the geometrical framework of riemannian manifolds provides an effective description for the *orbital* motion of low-energy particles in a *crystalline background with defects* (or of particles confined to a curved surface in $\mathbb{E}^3$). Given the Levi-Civita connection, λ, we must ask whether the corresponding affine spin connection, $\omega = \lambda + \kappa$ (see (3.11)), might provide an effective description of the interaction of the *spin* of the particles with the crystalline background through which they are moving. In general, this is not likely to be so! For example, let us consider a spinning particle with vanishing magnetic moment which moves in a crystalline background. Then, from the point of view of basic one-body quantum mechanics (see Subsect. 2.1), we do not expect that the dynamics of the spin of the particle is coupled to the effective metric associated with the background. (In this situation, the spin can be viewed as an internal degree of freedom.) More generally, in one-body quantum mechanics (in the absence of gravitational fields), the dynamics of the spin of a particle (moving in some background or constrained to a surface in $\mathbb{E}^3$) is completely determined by the Zeeman effect and by spin-orbit coupling, including the kinematical effect of the Thomas precession. The ρ-part of the $SU(2)$ connection w fully accounts for these effects, and ω can be transformed to 0 in a suitable $SU(2)$ gauge.

We emphasize that the main reasons for introducing the geometrical framework have been to elucidate, from a geometrical point of view, the meaning and origin of the $SU(2)$ gauge invariance (i.e., the introduction of an $SU(2)$ connection and of *local* frame rotations), and to prepare for the description of quantum-mechanical systems in "moving coordinates" which will be the subject matter of the following subsection.

Finally, we recall that with the help of the dreibein $e^A_{\ i}$ and its inverse $\mathcal{E}_A^{\ i}$ the components of the electromagnetic field and its vector potential can easily be changed from the form they take in orthonormal frames to the form they take in local coordinates (see (3.5) and (3.6)), e.g.,

$$B_A(x) \;=\; \mathcal{E}_A^{\ k}(x)\, B_k(x)\ , \qquad \text{and} \qquad A_k(x) \;=\; e^C_{\ k}(x)\, A_C(x)\ . \tag{3.40}$$

Note that the expressions (3.37) and (3.38) are consistent with the transformation law (3.33) of $\rho^{(s)}_\mu$ under $SU(2)_{spin}$ gauge transformations. Moreover, defining $U(1)_{em}$ gauge transformations as in (2.12) (with χ an arbitrary, real-valued function on $\mathbb{R} \times L$), the discussion above (see, in particular, Eq. (3.34) for the action S_Λ) proves that:

Non-relativistic quantum mechanics of charged, spinning particles moving in an external electromagnetic field and in a geometrically non-trivial background is $U(1)_{em} \times SU(2)_{spin}$ *gauge-invariant.*

3.3 Moving Coordinates and Quantum-Mechanical Larmor Theorem

We now imagine that the background of the system is *moving* on the manifold M according to some *classical flow* $\phi(t, \cdot)$. Here $\phi(t, \mathbf{y})$ is the position in M of a point particle at time t starting at position $\mathbf{y}$ at time 0. Then, in the x-coordinates ("laboratory coordinates") fixed to $\mathbb{R} \times M$, the one-body potential $v(x)$ and the electric and magnetic fields $\mathbf{E}(x)$ and $\mathbf{B}(x)$ created by the background are *time-dependent*. This implies that, in the (time-independent) x-coordinates on $\mathbb{R} \times M$, the Hamiltonian of the system is *time-dependent* which complicates the mathematical analysis of the system. In particular, it complicates the analysis of its *thermal equilibrium properties*. It is quite clear, physically, that approximate thermal equilibrium in such a system will be reached *locally* in regions moving *with* the background according to the flow $\phi(t, \cdot)$. Thus, we ought to formulate quantum mechanics in "*moving coordinates*", (y^1, y^2, y^3), where

$$\mathbf{x} = \phi(t, \mathbf{y}) , \quad \text{i.e.,} \quad \mathbf{y} = \phi^{-1}(t, \mathbf{x}) . \tag{3.41}$$

In accordance with our non-relativistic treatment of quantum theory, time will not be transformed, and in our calculations only terms to order $\mathcal{O}(f/c)$ are taken into account, where f is the modulus of the velocity field of the moving background, (see below). We shall see that the geometrical formalism introduced in the first part of this section allows for a natural description of the transformation to "moving coordinates", since, from the outset, it incorporates the local symmetry group of coordinate reparametrizations of the manifold, i.e., diffeomorphisms. For a different account of quantum mechanics in moving coordinates (or in non-inertial reference frames), see Schmutzer and Plebański (1977).

In the new coordinates (y^1, y^2, y^3), the one-body potential $v(t, \mathbf{y})$ and the background fields $\mathbf{E}(t, \mathbf{y})$ and $\mathbf{B}(t, \mathbf{y})$ may be expected to be (approximately) *time-independent*. In this situation, the Hamiltonian for spinless particles $(s = 0)$ will be (approximately) time-independent, and we can apply the rules of gibbsian statistical mechanics to study thermal equilibrium.

Unfortunately, for spinning particles $(s = \frac{1}{2}, 1, \ldots)$, the situation is not quite as neat, because, in the y-coordinates, the dreibein $\tilde{e}^A{}_i(y)$ is *time-dependent*,

$$\tilde{e}^A{}_k(y) := e^A{}_j(t, \phi(t, \mathbf{y})) \frac{\partial}{\partial y^k} \phi^j(t, \mathbf{y}) = e^A{}_j(t, \phi(t, \mathbf{y})) \frac{\partial x^j}{\partial y^k} . \tag{3.42}$$

In order to eliminate as much of this undesirable time-dependence as possible, we attempt to find a suitable $SU(2)$ *gauge transformation* of the new dreibein $\tilde{e}^A{}_k(y)$; see (3.7). What is the optimal choice? The answer is, perhaps, somewhat ambiguous, in general. But the following choice tends to be quite optimal. Let $(f^j(t, \mathbf{x}))$ be the *velocity field* generating the flow $\phi(t, \cdot)$, i.e.,

$$c \frac{\partial x^j}{\partial y^0} = \frac{\partial}{\partial t} \phi^j(t, \mathbf{y}) = f^j(t, \phi(t, \mathbf{y})) , \qquad j = 1, 2, 3 . \tag{3.43}$$

Then the infinitesimal rotation of an orthonormal frame carried along by the flow $\phi(t, \cdot)$, at the point $\mathbf{x} \in M$ and at time t, is given by

$$\Omega^A{}_B(t, \mathbf{x}) := \frac{1}{2} \Big[(\partial_B f^A)(t, \mathbf{x}) - \delta^{AC} \delta_{BD} (\partial_C f^D)(t, \mathbf{x}) \Big] , \tag{3.44}$$

where $\partial_A := \mathcal{E}_A{}^i(x) \frac{\partial}{\partial x^i}$, and $f^A(t, \mathbf{x}) := e^A{}_j(x) f^j(t, \mathbf{x})$; see (3.6) and the remark after (3.8). The vector $\mathbf{\Omega}(t, \mathbf{x})$ dual to the antisymmetric matrix $(\Omega^A{}_B(t, \mathbf{x}))$ is called the *vorticity* or *circulation* of the vector field $\mathbf{f}(t, \mathbf{x})$ and is the local angular velocity of the rotation induced by $\phi(t, \cdot)$ of a frame at the point $\mathbf{x}$, at time t.

We define a rotation matrix $(R^A{}_B(t,\mathbf{x}))$ by setting

$$R^A{}_B(t,\mathbf{x}) \;:=\; T\exp\Big[\gamma\int_0^t dt'\,\Omega(t',\mathbf{x})\Big]^A{}_B\;, \tag{3.45}$$

where "$T\exp$" denotes a time-ordered exponential, and γ is a real constant to be chosen later. (Its physical meaning will become clear at the end of this subsection; see also Subsect. 4.3). The r.h.s. of (3.45) can be defined, for example, by a convergent Dyson series if $\boldsymbol{\Omega}(t,\mathbf{x})$ is uniformly bounded in t. We now define (see (3.7))

$$\hat{e}^A{}_k(y) \;:=\; {}^R\tilde{e}^A{}_k(y) \;=\; R^A{}_B(t,\phi(t,\mathbf{y}))\,\tilde{e}^B{}_k(y)\;, \tag{3.46}$$

where $\tilde{e}^B{}_i(y)$ is given by (3.42). We also define the following transformed quantities:

$$\hat{\psi}(t,\mathbf{y}) \;\;:=\;\; U^{(s)}(t,\mathbf{y})\,\psi(t,\phi(t,\mathbf{y}))\;, \tag{3.47}$$

$$\hat{g}^{kl}(t,\mathbf{y}) \;\;:=\;\; \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\,g^{ij}(t,\phi(t,\mathbf{y}))\;, \tag{3.48}$$

and

$$\begin{aligned}
\hat{a}_0(t,\mathbf{y}) \;&:=\; a_0(t,\phi(t,\mathbf{y})) + \frac{\partial x^j}{\partial y^0}\,a_j(t,\phi(t,\mathbf{y}))\;,\\
\hat{a}_k(t,\mathbf{y}) \;&:=\; \frac{\partial x^j}{\partial y^k}\,a_j(t,\phi(t,\mathbf{y}))\;.
\end{aligned} \tag{3.49}$$

Furthermore,

$$\begin{aligned}
\hat{w}_0^{(s)}(t,\mathbf{y}) \;&:=\; U^{(s)}(t,\mathbf{y})\Big[\,w_0^{(s)}(t,\phi(t,\mathbf{y})) + \frac{\partial x^j}{\partial y^0}\,w_j^{(s)}(t,\phi(t,\mathbf{y}))\,\Big]\,U^{(s)}(t,\mathbf{y})^*\\
&\quad + U^{(s)}(t,\mathbf{y})\,\frac{\partial}{\partial y^0}U^{(s)}(t,\mathbf{y})^*\;, \qquad\qquad \text{and}\\
\hat{w}_k^{(s)}(t,\mathbf{y}) \;&:=\; U^{(s)}(t,\mathbf{y})\Big[\,\frac{\partial x^j}{\partial y^k}\,w_j^{(s)}(t,\phi(t,\mathbf{y}))\,\Big]\,U^{(s)}(t,\mathbf{y})^*\\
&\quad + U^{(s)}(t,\mathbf{y})\,\frac{\partial}{\partial y^k}U^{(s)}(t,\mathbf{y})^*\;,
\end{aligned} \tag{3.50}$$

where

$$U^{(s)}(t,\mathbf{y}) \;:=\; U^{(s)}\Big(R(t,\phi(t,\mathbf{y}))\Big)\;. \tag{3.51}$$

Our aim is to rewrite the action S_Λ introduced in (3.34) and (3.35) in moving coordinates, y, using the transformations (3.46)–(3.51). By (3.47) and (3.51),

$$\begin{aligned}\psi(t,\mathbf{x}) &= U^{(s)}(t,\phi^{-1}(t,\mathbf{x}))^*\,\hat{\psi}(t,\phi^{-1}(t,\mathbf{x})) \\ &= U^{(s)}\Big(R(t,\mathbf{x})\Big)^*\,\hat{\psi}(t,\phi^{-1}(t,\mathbf{x}))\ .\end{aligned} \tag{3.52}$$

Hence, (with $\frac{\partial}{\partial x^0}=\frac{1}{c}\frac{\partial}{\partial t}=\frac{\partial}{\partial y^0}$)

$$\begin{aligned}&U^{(s)}\Big(R(t,\mathbf{x})\Big)\,\frac{\partial}{\partial x^0}\psi(t,\mathbf{x})\Big|_{\mathbf{x}=\phi(t,\mathbf{y})} = \\ &= \frac{\partial}{\partial y^0}\hat{\psi}(t,\mathbf{y}) + U^{(s)}(t,\mathbf{y})\,\frac{\partial}{\partial y^0}U^{(s)}(t,\mathbf{y})^*\,\hat{\psi}(t,\mathbf{y}) \\ &\quad -\frac{1}{c}\hat{f}^k(t,\mathbf{y})\Big[\frac{\partial}{\partial y^k}\hat{\psi}(t,\mathbf{y}) + U^{(s)}(t,\mathbf{y})\,\frac{\partial}{\partial y^k}U^{(s)}(t,\mathbf{y})^*\,\hat{\psi}(t,\mathbf{y})\Big] \\ &= \frac{\partial}{\partial y^0}\hat{\psi}(t,\mathbf{y}) - \frac{1}{c}\hat{f}^k(t,\mathbf{y})\,\frac{\partial}{\partial y^k}\hat{\psi}(t,\mathbf{y}) \\ &\quad -\frac{i\gamma}{4c}\sum_{A,B,C=1}^{3}\varepsilon_A{}^{BC}\,\hat{\Omega}^A{}_B(t,\mathbf{y})\,L^{(s)}_C\,\hat{\psi}(t,\mathbf{y})\ ,\end{aligned} \tag{3.53}$$

where $-\hat{f}^k(t,\mathbf{y}) := -\frac{\partial y^k}{\partial x^j}\,f^j(t,\phi(t,\mathbf{y}))$ is the k^{th} component of the vector field generating $\phi^{-1}(t,\cdot)$ in the y-coordinates, and $(\hat{\Omega}^A{}_B(t,\mathbf{y}))$ is the vorticity of the generating vector field in the y-coordinates with respect to the orthonormal frame $(\hat{e}^A(t,\mathbf{y}))$, given in (3.46). By comparing (3.53) with (3.49)–(3.51) we find that

$$\begin{aligned}&U^{(s)}\Big(R(x)\Big)\,\left(\frac{\partial}{\partial x^0}+ia_0(x)+w_0^{(s)}(x)\right)\,\psi(x)\Big|_{x=(ct,\,\phi(t,\mathbf{y}))} = \\ &= \left(\frac{\partial}{\partial y^0}+i\hat{a}_0(y)+\hat{w}_0^{(s)}(y)\right)\,\hat{\psi}(y) - \frac{1}{c}\hat{f}^k(y)\,\left(\frac{\partial}{\partial y^k}+i\hat{a}_k(y)+\hat{w}_k^{(s)}(y)\right)\,\hat{\psi}(y)\ .\end{aligned} \tag{3.54}$$

We define the transformed covariant derivatives

$$\begin{aligned}\hat{\mathcal{D}}_0 &:= \frac{\partial}{\partial y^0}+i\hat{a}_0(y)+\hat{w}_0^{(s)}(y)\ , \\ \hat{\mathcal{D}}_k &:= \frac{\partial}{\partial y^k}+i\hat{a}_k(y)-i\frac{m}{\hbar}\hat{f}_k(y)+\hat{w}_k^{(s)}(y)\ ,\end{aligned} \tag{3.55}$$

where $\hat{f}_k(y)=\hat{g}_{kl}(y)\hat{f}^l(y)$, and the transformed one-body potential

$$\hat{v}(t, \mathbf{y}) \;\; := \;\; v(t, \phi(t, \mathbf{y})) - \frac{m}{2} \hat{f}_k(y) \hat{f}^k(y) - i \frac{\hbar}{2} \frac{1}{\sqrt{\hat{g}(y)}} \frac{\partial}{\partial y^k} \left(\sqrt{\hat{g}(y)}\, \hat{f}^k(y) \right) \,, \tag{3.56}$$

as well as the transformed two-body potential

$$\hat{V}(t, \mathbf{y}, \mathbf{y}') \;\; := \;\; V(t, \phi(t, \mathbf{y}), \phi(t, \mathbf{y}')) \,. \tag{3.57}$$

After these preparations, one easily verifies the following

Theorem 3.3.1. *In moving coordinates, the action of a non-relativistic system of charged, spinning particles moving in an external electromagnetic field and on a geometrically non-trivial manifold is given by*

$$\begin{aligned} S_\Lambda(\psi^*, \psi; g, a, w) \;&=\; \hat{S}_{\hat{\Lambda}}(\hat{\psi}^*, \hat{\psi}; \hat{g}, \hat{a}, \hat{f}, \hat{w}) \\ := \;& \int_{\hat{\Lambda}} \sqrt{\hat{g}(y)}\, dt\, d^3\mathbf{y} \left[i\hbar c\, \hat{\psi}^*(y) (\hat{\mathcal{D}}_0 \hat{\psi})(y) \right. \\ & \left. - \frac{\hbar^2}{2m}\, \hat{g}^{kl}(y)\, (\hat{\mathcal{D}}_k \hat{\psi})^*(y)\, (\hat{\mathcal{D}}_l \hat{\psi})(y) - \hat{U}(\hat{\psi}^*, \hat{\psi})(y) \right] , \end{aligned} \tag{3.58}$$

where in the definition of $\hat{U}(\hat{\psi}^*, \hat{\psi})$ *(see (3.35)) the potentials* $\hat{v}$ *and* $\hat{V}$ *of (3.56) and (3.57) are used, and* $\hat{\Lambda} := \{(t, \mathbf{y}) \mid (t, \mathbf{x} = \phi(t, \mathbf{y})) \in \Lambda\}$.

To prove (3.58), one expands the r.h.s. of (3.58) in powers of $\hat{f}^k$, integrates by part, and compares the resulting expression to (3.54), (3.34) and (3.35), using (3.55) through (3.57) and the fact that $(U^{(s)}\psi)^*(U^{(s)}\psi) = \psi^*\psi$.

We pause to interpret the result (3.58). By (3.55), $-\frac{m}{\hbar}\hat{f}_k(y)$ enters the action $\hat{S}_{\hat{\Lambda}}$ as a contribution to the $U(1)$ connection. By (3.36), $m\hat{f}_k(y)$ and $\frac{q}{c}\hat{A}_k(y)$ play analogous roles, i.e., (in x-coordinates)

$$m\, \mathbf{f}(x) \;\leftrightarrow\; \frac{q}{c}\, \mathbf{A}(x) \,. \tag{3.59}$$

The vector potential $\mathbf{A}(x)$ gives rise to the *Lorentz force* in the classical limit. The Lorentz force has the same form as the *Coriolis force* if one replaces $\frac{q}{c}\, \mathbf{B}(x)$ by $2m\, \mathbf{\Omega}(x)$, where $\mathbf{\Omega}(x)$ is the local angular velocity which is precisely half the curl of the vector field $\mathbf{f}(x)$; see Eq. (3.44). Thus $\mathbf{f}(x)$ is the vector potential that gives rise to the Coriolis force in the classical limit.

By (3.53) and (3.54), the action $\hat{S}_{\hat{\Lambda}}$ contains a term

$$\gamma\,\hat{\psi}^*(y)\left(\sum_{A=1}^{3}\hat{\Omega}_A(y)\,\frac{\hbar}{2}L_A^{(s)}\right)\hat{\psi}(y)\ , \tag{3.60}$$

where $\hat{\mathbf{\Omega}}(y) = \frac{1}{2}\,curl\,\hat{\mathbf{f}}(y)$, in y-coordinates. It has the same form as the *Zeeman term*

$$\frac{g\mu}{\hbar}\,\hat{\psi}^*(y)\left(\sum_{A=1}^{3}\hat{B}_A(y)\,\frac{\hbar}{2}L_A^{(s)}\right)\hat{\psi}(y)\ , \tag{3.61}$$

which, by (3.50), (3.37), (3.30), and (3.28), also appears in $\hat{S}_{\hat{\Lambda}}$. Recall that the magnetic moment of a particle with spin s has been defined by $\boldsymbol{\mu}_{spin} := \frac{g\mu}{2}\,\mathbf{L}^{(s)}$; see (2.4). Thus $\frac{g\mu}{\hbar}\,\hat{\mathbf{B}}$ is the angular velocity of *spin precession* in the magnetic field $\hat{\mathbf{B}}$.

Next, we analyze the one-body potential $\hat{v}$ in moving coordinates. By (3.56), $\hat{v}$ is *complex-valued, unless*

$$div_{\hat{g}}\,\hat{\mathbf{f}}(y)\ :=\ \frac{1}{\sqrt{\hat{g}(y)}}\,\frac{\partial}{\partial y^k}\Big(\sqrt{\hat{g}(y)}\,\hat{f}^k(y)\Big)\ =\ 0\ , \tag{3.62}$$

i.e., *unless the vector field* $\hat{\mathbf{f}}$ *is divergence-free*. A divergence-free vector field generates a *volume-preserving* flow $\phi(t,\cdot)$, hence

$$\hat{g}(y)\ :=\ \det\left(\hat{g}_{kl}(t,\mathbf{y})\right)\ =\ g(t,\phi(t,\mathbf{y}))\ . \tag{3.63}$$

Thus, for volume-preserving (i.e., *incompressible*) flows, and *only* for such flows, $\hat{v}$ is again *real-valued*. [This is, because if volume is preserved by $\phi(t,\cdot)$ then, by (3.63), the quantum-mechanical time-evolution in the moving coordinate system preserves probabilities with respect to the volume element $\sqrt{g(t,\phi(t,\mathbf{y}))}\ d^3\mathbf{y}$ and hence is generated ba a *hermitian (selfadjoint)* Hamiltonian!] But $\hat{v}$ contains an additional term, $-\frac{m}{2}\hat{f}_k\hat{f}^k$, that was not present in the original one-body potential, see (3.56). What does it correspond to physically? It is the potential of the *centrifugal force*, because $\frac{m}{2}\frac{\partial}{\partial y^l}(\hat{f}_k\hat{f}^k)$ is precisely the l^{th} component of the centrifugal force at the point $\mathbf{y}$, at time t. [Note, incidentally, that $\frac{m}{2}\,\hat{f}_k\hat{f}^k$ is the classical kinetic energy of the particle in the time-independent frame which must be subtracted in the y-coordinates.]

In conclusion, we find that quantum mechanics in moving coordinates is hamiltonian, with a *hermitian* (but possibly still time-dependent) Hamilton operator, if, and only if, the flow $\phi(t,\cdot)$ defining the moving coordinate system is *volume-preserving*, or *incompressible*. Henceforth this property is usually assumed. It is worthwhile recalling that in *two* space dimensions, incompressible flows are automatically symplectic (hamiltonian) flows, because the vector fields generating them are divergence-free and hence are dual to the gradient of some (scalar) Hamilton function. (This is the basis of a mathematical analysis of the two-dimensional Euler equations.)

In order to provide a concrete application illustrating the general formulae given in this subsection, we rewrite the *one-particle Pauli equation in moving coordinates.* The flow $\phi(t,\cdot)$ defining the moving coordinate system (the moving background) is assumed to be volume-preserving; see (3.62). Varying the action (3.58) (without two-body potential term) with respect to $\hat{\psi}^*$, the one-particle Pauli equation reads

$$i\hbar c\,(\hat{\mathcal{D}}_0\hat{\psi})(y) \;=\; -\frac{\hbar^2}{2m}\,\frac{1}{\sqrt{\hat{g}(y)}}\,\hat{g}^{kl}(y)\,\hat{\mathcal{D}}_k(\sqrt{\hat{g}}\,\hat{\mathcal{D}}_l\hat{\psi})(y)+\hat{v}(y)\hat{\psi}(y)\;, \tag{3.64}$$

where

$$\hat{\mathcal{D}}_\mu \;:=\; \frac{\partial}{\partial y^\mu}+i\tilde{a}_\mu(y)+i\sum_{A=1}^{3}\hat{w}_{\mu A}(y)\,L_A^{(s)}\;,\qquad \mu=0,\ldots,3\;, \tag{3.65}$$

with

$$\begin{aligned}
\tilde{a}_0(y) \;&:=\; \hat{a}_0(y) \;=\; \frac{q}{\hbar c}\,\Phi(t,\phi(t,\mathbf{y}))-\frac{q}{\hbar c}\,(\mathbf{f}\cdot\mathbf{A})(t,\phi(t,\mathbf{y}))\;,\\
\tilde{a}_k(y) \;&:=\; \hat{a}_k(y)-\frac{m}{\hbar}\hat{f}_k(y) \;=\; -\frac{q}{\hbar c}\hat{A}_k(y)-\frac{m}{\hbar}\hat{f}_k(y)\;,\\
\hat{w}_{0A}(y) \;&:=\; -\frac{g\mu}{2\hbar c}\,\hat{B}_A(y)-\frac{1}{c}\left(-\frac{g\mu}{2\hbar c}+\frac{q}{4mc^2}\right)\sum_{B,C=1}^{3}\varepsilon_{ABC}\,\hat{f}_B(y)\hat{E}_C(y)\\
&\quad\; -\frac{\gamma}{2c}\,\hat{\Omega}_A(y)+\text{ terms }\propto\,\omega\;,
\end{aligned} \tag{3.66}$$

and $\hat{w}_{kA}(y)$ is given by (3.50) which contains terms proportional to first, $\frac{\partial}{\partial y^k}\phi^j$, and second derivatives, $\frac{\partial}{\partial y^k}\frac{\partial}{\partial y^l}\phi^j$, of the flow $\phi(t,\cdot)$. Furthermore, in (3.64), we have the terms

$$\hat{v}(y) \;:=\; v(t,\phi(t,\mathbf{y}))-\frac{m}{2}\hat{f}_k(y)\hat{f}^k(y)\;, \tag{3.67}$$

and

$$\hat{g}^{kl}(y) \;:=\; \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\,g^{ij}(t,\phi(t,\mathbf{y}))\;, \tag{3.68}$$

where $g_{ij}(t,\mathbf{x})$ is the metric on the manifold M (in the time-independent x-coordinates) on which the background of the one-particle system is moving according to the flow $\phi(t,\cdot)$; $\hat{g}(y)$ is the determinant of the metric $\hat{g}^{kl}$, as defined in (3.63).

An interesting consequence of formulating quantum mechanics in moving coordinates is a quantum-mechanical version of *Larmor's theorem* in which also the spin degrees of freedom of particles moving in a (variable) external magnetic field are taken into account.

For definiteness, let us consider a system of particles with effective mass m, charge q, spin s, and magnetic moment μ_{spin}, (where $\boldsymbol{\mu}_{spin} := \frac{g\mu}{\hbar}\mathbf{S}$, with $\mu = \frac{q\hbar}{2m_o c}$, and m_o is the mass of the particles in the vacuum; see (2.4)). Furthermore, for simplicity, we choose the background manifold M to be euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$, i.e., the metric takes the form $g_{ij}(x) = \delta_{ij}$, and all geometrical contributions to the $SU(2)$ connection $w(x)$ are absent; see (3.28)–(3.30). We now suppose that the system is under the influence of a (variable) external magnetic field, $\mathbf{B}(x)$, and assume that there is no external electric field; $\mathbf{E}(x) = 0$. As we shall see shortly, it is convenient to work in a $U(1)$ gauge where $div\, \mathbf{A}(x) = 0$ (Coulomb gauge). We then have the following result.

Theorem 3.3.2 (Quantum-Mechanical Larmor Theorem). *For the system just described, the effect of an external magnetic field $\mathbf{B}(x)$ can be eliminated to first order by choosing to work in moving coordinates generated by the velocity field $\mathbf{f}(x) = -\frac{q}{mc}\mathbf{A}(x)$, and by performing an $SU(2)$ gauge transformation $U^{(s)}(R(x))$ where the local frame rotation $R(x)$ is given by (3.45) with $\gamma = g\frac{m}{m_o}$.*

Note that the vorticity field $\mathbf{\Omega}(x) := \frac{1}{2}\, curl\, \mathbf{f}(x) = -\frac{q}{2mc}\mathbf{B}(x)$ of the velocity field $\mathbf{f}(x)$ is precisely the so-called *Larmor angular velocity*. Choosing the vector potential $\mathbf{A}(x)$ in the Coulomb gauge renders the flow generated by $\mathbf{f}(x)$ divergence free (i.e., volume preserving).

The proof of this theorem is straightforward. We use that $w_j^{(s)}(x) = 0 = a_0(x)$ by (3.28)–(3.30), and (3.36)–(3.38). Then, adopting the identifications given in the theorem, it follows from (3.55) and (3.56), using (3.49) and (3.50), that in moving coordinates:

$$\begin{aligned} \hat{\mathcal{D}}_0 &= \frac{\partial}{\partial y^0} + \mathcal{O}\Big(\max\big(\,\big|\hat{\mathbf{f}}\big|^2(y),\ \big|\tfrac{\partial}{\partial y^k}\hat{\mathbf{B}}\big|(y)\,\big)\Big)\ , \\ \hat{\mathcal{D}}_k &= \frac{\partial}{\partial y^k} + \mathcal{O}\Big(\big|\tfrac{\partial}{\partial y^k}\hat{\mathbf{B}}\big|(y))\Big)\ , \qquad \text{and} \\ \hat{v}(y) &= v(t, \phi(t, \mathbf{y})) + \mathcal{O}\Big(\big|\hat{\mathbf{f}}\big|^2(y)\Big)\ , \end{aligned} \tag{3.69}$$

see also (3.60) and (3.61). Note that if the external magnetic field is constant (in the moving coordinates), $\hat{\mathbf{B}}(y) = \hat{\mathbf{B}}_o$, then the "tidal vector potential" $\hat{\mathbf{f}}(y)$ may be written as

$$\hat{\mathbf{f}}(y) \;=\; \hat{\mathbf{\Omega}} \times \mathbf{y} \;=\; \frac{q}{2mc}\,\mathbf{y} \times \hat{\mathbf{B}}_o\ . \tag{3.70}$$

Hence, in this situation, the terms proportional to $\frac{\partial}{\partial y^k}\ \hat{\mathbf{B}}(y)$ are absent in (3.69), and $\mathcal{O}\Big(\big|\hat{\mathbf{f}}\big|^2(y)\Big) = \mathcal{O}\Big(\big|\hat{\mathbf{B}}_o\big|^2\Big)$.

The formalism developed in this section applies equally well to (one-), two- and three-dimensional systems.

It often happens in solid state physics, e.g. in two-dimensional heterostructures used in measurements of the quantum Hall effect, that the system exhibits an approximate internal symmetry described by some compact group G. The spinors $\psi^{(s)\sharp}$ then transform according to some non-trivial representation, π, of G. A breaking of this symmetry by an external background might be described as the effect of coupling $\psi^{(s)\sharp}$ to an external gauge field in the representation π_* of the Lie algebra of G. Let us denote this gauge field by z. By modifying the covariant derivatives given in (3.26),

$$\mathcal{D}_\mu \mapsto \mathcal{D}'_\mu := \mathcal{D}_\mu + z_\mu(x) , \tag{3.71}$$

we may easily extend the entire formalism developed in this section to systems with *gauged internal symmetries*. This can be important in applications.

Note that, in this situation, the action S_Λ introduced in Eq. (3.34) is $U(1)\times SU(2)\times G$ *gauge invariant*, i.e., it does not change if, for an arbitrary real-valued fucntion χ, an $SU(2)$-valued function R, and a G-valued function g, all defined over *space-time* $\mathbb{R}\times M$, the following substitutions are made:

$$\begin{aligned}
\psi^{(s)}(x) &\mapsto e^{-i\chi(x)}\, U^{(s)}(R(x)) \otimes \pi(g(x))\, \psi^{(s)}(x) , \\
a_\mu(x) + \frac{m}{\hbar} f_\mu(x) &\mapsto a_\mu(x) + \frac{m}{\hbar} f_\mu(x) + \partial_\mu \chi(x) , \\
w_\mu^{(s)}(x) &\mapsto U^{(s)}(R(x))\, w_\mu^{(s)}(x)\, U^{(s)}(R(x))^* + U^{(s)}(R(x))\, \partial_\mu U^{(s)}(R(x))^* ,
\end{aligned}$$

and

$$z_\mu(x) \mapsto \pi(g(x))\, z_\mu(x)\, \pi(g(x))^* + \pi(g(x))\, \partial_\mu \pi(g(x))^* . \tag{3.72}$$

Thus, barring gauge anomalies, which actually *cannot* appear in systems of *finitely* many non-relativistic particles, the non-relativistic quantum mechanics of such systems is $U(1)\times SU(2)\times G$ gauge invariant. Ward identities expressing this gauge invariance turn out to play an important role in establishing certain universal properties of such systems; see Fröhlich and Studer (1992a–1992d) and Sect. 6 below.

4 Some Key Effects Related to the $U(1) \times SU(2)$ Gauge Invariance of Non-Relativistic Quantum Mechanics

In this section, we describe some effects in quantum mechanics from the point of view of its $U(1)_{em} \times SU(2)_{spin}$ gauge-invariance. We continue and expand the discussion started at the end of Sect. 2. Most of the material reviewed here is well-known, but our perspective, emphasizing gauge-invariance, may be somewhat novel.

4.1 "Tidal" Aharonov-Bohm and "Geometric" Aharonov-Casher Effects

After what we have learned in Subsect. 3.3 on the $U(1)$ vector potential of the Coriolis force, present in moving coordinates (see (3.59)), it is clear that there must exist a "*tidal*" *Aharonov-Bohm effect*: Consider a mass-current conducting superfluid in a large container penetrated by some straight cylindrical tube that excludes the quantum fluid. Now, set the fluid into circular motion around the axis of the tube with velocity field $\mathbf{f}$, where $|\mathbf{f}(r)| = \frac{\kappa}{2\pi r}$ at a distance r from the axis of the tube, and κ is a quantity of dimension cm^2/sec, the total *vorticity* or *circulation*, $\kappa := \oint \mathbf{f} \cdot d\mathbf{l}$. [Note that $\kappa = \frac{2\pi \overline{L_z}}{NM}$, where M is the mass of the particles constituting the quantum fluid, $\overline{L_z}$ is the expectation value of the component of the total angular momentum operator parallel to the tube in the given state of the system, and N is the number of paricles in the system.] Small mass currents excited in this system, scattered at the tube, will exhibit an Aharonov-Bohm effect depending periodically on κ, with period $\frac{h}{m}$, where m is the mass of the particles in the scattering currents; compare to Subsect. 2.2.

While this effect may be somewhat difficult to test experimentally, it is important theoretically: Consider a superfluid film with manifestly (e.g., through rotation) or spontaneously broken time-reversal and reflections-in-lines (i.e., two-dimensional parity) invariance. The formation of such films at a particular superfluid ^3He-A/B interface has been discussed by Salomaa and Volovik (1989). Such a two-dimensional superfluid will, in general, exhibit *vortex excitations* of vorticity $\kappa = n\frac{h}{M}$, $n \in \mathbb{Z}$, where M is the mass of the constituent particles in the superfluid, and *fractional mass* (rather than fractional charge) $\sigma_H \kappa$, where $\sigma_H = \sigma \frac{M^2}{h}$ is a "tidal Hall conductivity". Such excitations give rise to Aharonov-Bohm phases and hence are anyons if σ is not an integer, i.e., if the superfluid shows a *fractional* "tidal" quantum Hall effect; a detailed discussion of this effect can be founed in Subsect. VII.B in Fröhlich and Studer (1993b). Such excitations may be observed experimentally by measuring fluctuations in the longitudinal resistance of superfluid current conduction. See Simmons *et al.* (1989) and Hwang *et al.* (1992) for an analogous experiment in two-dimensional electronic systems exhibiting the fractional quantum Hall effect.

A remarkable experimental observation of a "tidal" Aharonov-Bohm effect has been provided by Werner, Staudenmann, and Colella (1979). Using a neutron interferometer they have detected a quantum-mechanical interference effect due to the rotation of the Earth. For a brief theoretical comment on this interference experiment that is close to our discussion in Subsect. 3.3, see Sakurai (1980).

Other interesting systems where mixed "tidal" and electromagnetic effects play a role are, for example, rotating superconductors. Following an analysis by Semon (1982; see also Schmutzer and Plebański, 1977), an experiment performed by Zimmerman and Mercereau (1965) can be interpreted as the realization of a thought experiment proposed by Aharonov and Carmi (1973): Given a (uniformly) rotating sample that is not simply connected, the "tidal" forces (Coriolis - and centrifugal force) felt by the particles in the moving system (all of the same charge to mass ratio) can be cancelled by an electromagnetic field whose vector potential does not cancel the "tidal" vector potential completely everywhere; see (3.59). This uncancelled "tidal" vector potential then leads to a quantum-mechanical interference effect.

In Subsect. 2.3, we have discussed the Aharonov-Casher effect as an $SU(2)$ version of the Aharonov-Bohm effect. Furthermore, in Subsect. 3.2, systems in geometrically non-trivial backgrounds have been described by including a "geometrical" term in the $SU(2)$ connection w; see (3.28)–(3.31) and (3.11). Here we combine these findings and discuss a "*geometrical version*" *of the Aharonov-Casher effect*: We consider a *two-dimensional* system of particles with non-zero spin on a *cone* with tip at $\mathbf{x} = 0$. Then, although $\rho = 0$ if there are no electromagnetic fields, the $SU(2)$ connection w determined by $\omega^A{}_B$, the affine spin connection on the cone, cannot be gauged away globally, although $\omega^A{}_B$ is *flat* for $\mathbf{x} \neq 0$. The $SU(2)$ connection w has the same form as the electromagnetic part ρ given in Eq. (2.16), but Q now denotes the defect angle of the cone. Scattering of particles at the tip of the cone will yield interference patterns depending on the defect angle Q. This effect is perhaps better known than its electromagnetic cousin. It has attracted attention, for example, in connection with quantum mechanics and quantum field theory in the presence of cosmic strings (Deser and Jackiw, 1988; t'Hooft, 1988; Kay and Studer, 1991).

Do spinless particles "see" the tip of the cone, or is spin important? The answer depends on our choice of a quantum-mechanical state space: We must impose some "boundary conditions" on the wave functions, i.e., $\psi(r, \varphi + 2\pi - Q) = e^{i\theta}\psi(r, \varphi)$, where φ is the polar angle, and θ is some phase to be specified; plus some boundary condition at $r = 0$. But no matter how we choose θ, we can make the tip of the cone "invisible" to spinless particles by threading a magnetic flux through $\mathbf{x} = 0$. If the particles have spin *and* a non-zero magnetic moment then, in addition, we would have to put a charged wire through $\mathbf{x} = 0$, in order to make the tip "invisible".

Finally, we remark that there is an analogue of the Aharonov-Casher effect where $SU(2)$ is replaced by a gauged internal symmetry group G. This effect can, perhaps, be tested in inhomogeneous heterostructures. It is related, both physically and mathematically, to the existence of particles in two-dimensional quantum theory with topological pair interactions described by a G-Knizhnik-Zamolodchikov connection that, just as in the case of $SU(2)$, may give rise to non-abelian braid statistics; see Subsect. 6.3.

4.2 Flux Quantization

A superconductor exhibits the *Meissner-Ochsenfeld effect*: A magnetic field cannot penetrate into the bulk of a superconducting material. However, in a type II superconductor, thin magnetic field tubes can thread through the bulk. They have the property that they carry a magnetic flux Φ which is an integer multiple of $\frac{hc}{q}$, where q is the charge of the particles in the condensate, (e.g., $q = -2e$, for BCS pairs of electrons). These tubes are called *Abrikosov vortices*. The quantization of Φ is explained by requiring that, outside an Abrikosov vortex, the superconducting state of the system remain undisturbed. From what we have said about the Aharonov-Bohm effect in Subsect. 2.2, it follows that this requirement is fulfilled precisely if Φ is an integer multiple of $\frac{hc}{q}$. Experimentally, this effect has been established first in Deaver and Fairbank (1961) and Doll and Näbauer (1961; for a review, see, e.g., Chap. 6 in Tilley and Tilley, 1986).

The discussion of the "tidal" Aharonov-Bohm effect above makes it clear that the Meissner-Ochsenfeld effect and flux quantization in Abrikosov vortices have their counterparts in the theory of superfluidity: Consider a superfluid in some container. Now set the container into uniform rotation. The superfluid inside the container abhors angular velocity which could destroy its superfluidity and does, therefore, *not* follow the rotation of the container's walls. However, just like there can be Abrikosov vortices in a type II superconductor, the superfluid can eventually be set into motion, and the motion is generated by a velocity field $\mathbf{f}$, whose circulation (or vorticity), $\mathbf{\Omega} = \frac{1}{2}\, curl\, \mathbf{f}$ (see (3.44)), is localized along thin tubes. The "tidal" Aharonov-Bohm effect then predicts that the total circulation in such a tube is quantized in integer multiples of $\frac{h}{M}$, where M is the mass of the particles (e.g., ^3He-pairs) constituting the superfluid. [This can also be understood by appealing to the quantization of orbital angular momentum.] If, in such a fluid, one can excite mass currents of quantum-mechanical particles, "dopants", of mass $m < M$ one may be able to test the "tidal" Aharonov-Bohm effect.

Our conclusions agree with another theoretical analysis given by Pines and Nozières (1989). Experimentally, the first verification of the quantization of circulation (in superfluid $He\,II$) has been given by Vinen (1961; for a review, see, e.g., Chap. 6 in Tilley and Tilley, 1986). The phenomena described here may be relevant in the astrophysics of rotating neutron stars (pulsars) which appear to be superfluid (see, e.g., Tsakadze and Tsakadze, 1980).

4.3 Barnett and Einstein - de Haas Effects

The Barnett and Einstein-de Haas effects (see, e.g., Landau and Lifshitz, 1960) find a very natural explanation in the light of the quantum-mechanical Larmor theorem discussed at the end of Subsect. 3.3. Consider a cylinder of iron or some other ferromagnetic material suspended at a wire in such a way that it can freely rotate around its axis. Let us suppose that, initially, the cylinder is at rest and demagnetized. Now, imagine that the cylinder is set into rapid rotation around its axis. As explained in Subsect. 3.3, the quantum mechanics of the electrons in this material should now be described in a *uniformly rotating* coordinate system fixed to the cylindrical background. In this coordinate system the electronic Hamiltonian will be *time-independent*, but it now contains a Zeeman term

$$\hat{\psi}^* \, \hat{\mathbf{\Omega}} \cdot \frac{\hbar}{2} \boldsymbol{\sigma} \, \hat{\psi} \ , \tag{4.1}$$

where $\hat{\mathbf{\Omega}} = \text{const.}$ is the angular velocity of the rotation; (see (3.60) where we have set $\gamma = 1$, i.e., by (3.44)–(3.46), the orthonormal frame in the (co)tangent space ("spin space") rotates with the *same* angular velocity as the rotating coordinate system). Note that, in (4.1), $\hat{\mathbf{\Omega}}$ plays the role of the magnetic field $\mathbf{B}$ in an inertial frame. Furthermore, the Hamiltonian contains a tidal vector potential $\hat{\mathbf{f}} = \hat{\mathbf{\Omega}} \times \mathbf{y}$ in the covariant derivatives $\hat{\mathcal{D}}_k$, and a potential $-\frac{m}{2} \left| \hat{\mathbf{\Omega}} \times \mathbf{y} \right|^2$ of the centrifugal forces; see (3.55) and (3.70), and (3.56), respectively. All the additional terms can be combined into the term

$$\hat{\psi}^* \, \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{J}} \, \hat{\psi} \ , \tag{4.2}$$

where $\hat{\mathbf{J}} = \hat{\mathbf{S}} + \hat{\mathbf{L}}$ is the total angular momentum operator (see also Kerman and Onishi, 1981). The effect of centrifugal forces will be balanced by the chemical potential of the background. Thus the electronic Hamiltonian is essentially equivalent to the one for a cylinder at rest but in a magnetic field $\mathbf{B} = \left(\frac{g\mu_B}{\hbar} \right)^{-1} \mathbf{\Omega}$. The result is, in both situations, that the cylinder is *magnetized*, because the spins of the electrons will align with $\mathbf{\Omega}$ or $\mathbf{B}$, respectively. This is the *Barnett effect*.

Conversely, in the *Einstein-de Haas effect*, one turns on a magnetic field, $\mathbf{B}$, antiparallel to the spontaneous magnetization of a *magnetized* piece of iron at rest, thereby *increasing* the *free energy* of the system. The system reacts to this perturbation by starting to *rotate* around the axis of the external magnetic field so as to offset the effect of $\mathbf{B}$ on the electrons by rotation. It thereby returns to a state corresponding to a local minimum of the free energy. By (3.60) and (3.61), the angular velocity of this rotation, $\mathbf{\Omega}$, is given by $\mathbf{\Omega} = \frac{g\mu_B}{\hbar} \mathbf{B}$ which is precisely the angular velocity of spin precession in the magnetic field $\mathbf{B}$. A similar effect is observed when one tries to magnetize a paramagnet. It would appear interesting to test a *local version* of this effect in a "ferro-fluid". If the magnetic field acting on a highly mobile ferro-fluid, locally in thermal equilibrium, is modified locally the fluid reacts by starting to flow with a velocity field that optimally offsets the change in the magnetic field so as to restore local equilibrium. The particle - and magnetic current densities induced are given by $n \, \mathbf{f}$ and $\mathbf{M} \otimes \mathbf{f}$, respectively, where $\mathbf{f}$ is the velocity field, n the particle density, and $\mathbf{M}$ the magnetization density. A somewhat analogous effect for quantum Hall fluids will be discussed in Subsect. 6.2.

There is another variant (Bell and Leinaas, 1983, 1987) of the Barnett effect: Consider a beam of non-relativistic particles, e.g. heavy ions, with spin, moving in a storage ring with some mean angular velocity $\boldsymbol{\Omega}$. Then they experience a "tidal" Zeeman effect, given by (3.60), in addition to the usual magnetic Zeeman effect, given by (3.61). After relaxation to a steady state, the "tidal" Zeeman energy obviously affects the ratio of "spin-up" to "spin-down" ions in the beam!

Similar considerations are important e.g. in the study of electronic spectra of rotating molecules in the Born-Oppenheimer approximation (Kerman and Onishi, 1981).

4.4 Meissner - Ochsenfeld Effect and London Theory of Superconductivity

Consider a superconducting condensate of charged bosons (e.g., electron pairs) of charge q and mass M, in equilibrium. Imagine that a (weak) magnetic field, $\mathbf{B}_c$, is turned on inside the bulk of this system, resulting in an increase of the free energy of the system (weak form of the Meissner-Ochsenfeld effect). Then the condensate *reacts* to the field $\mathbf{B}_c$ by starting to *flow* according to a velocity field $\mathbf{f}$ in such a way as to cancel $\mathbf{B}_c$ in the moving coordinate system. For, in this way, the free energy in regions of the system moving with the flow is minimal. Neglecting the centrifugal potential, $-\frac{m}{2}\mathbf{f}\cdot\mathbf{f}$, and (in a first step) the magnetic field created by the resulting current, it follows from Eqs. (3.58), (3.59) and (3.62) that the optimal velocity field $\mathbf{f}$ is given by

$$\mathbf{f} \;=\; -\frac{q}{Mc}\,\mathbf{A}_c^T \;,$$

where $\mathbf{A}_c^T$ is the vector potential of $\mathbf{B}_c$ in the Coulomb gauge (i.e., $\mathit{div}\,\mathbf{A}_c^T = 0$). Thus, the system exhibits a supercurrent density, $\mathbf{J}_s$, given in our approximation by

$$\mathbf{J}_s \;\approx\; qn_s\,\mathbf{f} \;=\; -\frac{q^2 n_s}{Mc}\,\mathbf{A}_c^T \;, \tag{4.3}$$

where n_s is the density of the condensate. Of course, this supercurrent $\mathbf{J}_s$ will give rise to an additional vector potential, $\mathbf{A}_s^T$, determined by Maxwell's equation: $\Delta\,\mathbf{A}_s^T = -\frac{1}{c}\,\mathbf{J}_s$. Adding this potential to the r.h.s. of (4.3) we obtain the equation

$$\mathbf{J}_s \;=\; -\frac{q^2 n_s}{Mc}\,\mathbf{A}^T \;, \tag{4.4}$$

where $\mathbf{A}^T := \mathbf{A}_c^T + \mathbf{A}_s^T$ is the total vector potential of the external magnetic field, $\mathbf{B}_c$, and the magnetic field created by the supercurrent. This is the *London equation* characterizing the superconducting state! Relating the external magnetic field $\mathbf{B}_c$ to an external current density, $\mathbf{J}_c$, via Maxwell's equation $\Delta\,\mathbf{A}_c^T = -\frac{1}{c}\,\mathbf{J}_c$, and assuming that the external current

$\mathbf{J}_c$ does not enter into the bulk of the superconductor, i.e., $\mathbf{J}_c = 0$ inside the superconducting region, the London equation (4.4) immediately implies the equation

$$\Delta\,\mathbf{A}^T \;=\; \frac{q^2 n_s}{Mc^2}\,\mathbf{A}^T \;, \tag{4.5}$$

which shows that, in a stationary state, currents and magnetic fields in superconductors can exist only within a surface layer of thickness $\Lambda := \left(\frac{Mc^2}{q^2 n_s}\right)^{1/2}$, the so-called London penetration depth (see, e.g., de Gennes, 1966). This is the *Meissner-Ochsenfeld effect.* Note that by Eq. (4.4) a *supercurrent* $\mathbf{J}_s$ is really a sign for the presence of a *vector potential*, $\mathbf{A}^T$, and thus can be used for experimental tests of the Aharonov-Bohm effect. However, if $\oint_\Gamma \mathbf{A}^T \cdot d\boldsymbol{l} = n\frac{hc}{q}$, $n \in \mathbb{Z}$, for any closed curve Γ contained in the superconducting phase, then the Aharonov-Bohm phase factors of the charged bosons are trivial (see Subsect. 2.2), and no supercurrent results. This is the phenomenon of flux quantization.

The expulsion of a magnetic field from the interior of a superconductor is related to the fact that, inside a superconductor, "photons are massive", i.e., one observes the phenomenon of the *Anderson-Higgs mechanism.* We now show that, in a superconductor, the presence of a "*mass term for the photons*" can also be inferred directly from the London equation (4.4): Since the electric current density, $\mathbf{J}$, and the *free energy*, $F(\mathbf{A})$, of a system in the presence of a (static) electromagnetic field with vector potential $\mathbf{A} := (A_1, A_2, A_3)$, are related by

$$\mathbf{J}(\mathbf{x}) \;=\; -c\,\frac{\delta F(\mathbf{A})}{\delta \mathbf{A}(\mathbf{x})} \;, \tag{4.6}$$

it follows from Eq. (4.4) that, in the bulk of a superconductor,

$$F(\mathbf{A}) \;=\; \frac{1}{2\Lambda^2}\int d^3\mathbf{x}\;\mathbf{A}^T(\mathbf{x})\cdot\mathbf{A}^T(\mathbf{x}) \;+\; \cdots \;, \tag{4.7}$$

where $A_i^T(\mathbf{x}) := \left[\delta_i^j - \partial_i\,\Delta^{-1}\partial^j\right] A_j(\mathbf{x})$, with Δ the three-dimensional Laplacian, and the dots stand for higher derivative terms. Notice that (4.7) is a *non-local* functional of $\mathbf{A}$ which is manifestly *invariant* under $U(1)$ gauge transformations, $\mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla}\chi$. It provides a "mass term for the photons" in the bulk of a superconductor.

4.5 Quantum Hall Effect

Just as the Aharonov-Bohm effect reflects the $U(1)_{em}$ *gauge invariance* of quantum theory, so does the *quantum Hall (QH) effect for the electric current*, as emphasized by Laughlin (1981; see also Halperin, 1982). In the same vein, the Aharonov-Casher effect and the *quantum Hall effect for the spin current* reflect the $SU(2)_{spin}$ *gauge invariance* of non-relativistic quantum theory, as emphasized by Fröhlich and Studer (1992a, 1992c). In this subsection, we review some basic facts concerning the integer (von Klitzing, Dorda, and Pepper, 1980) and fractional (Tsui, Stormer, and Gossard, 1982) quantum Hall effect; for comprehensive reviews see Chakraborty and Pietiläinen (1988), Morandi (1988), Prange and Gervin (1990), Wilczek (1990), and Stone (1992). The purpose of this review is to set the stage for Sects. 6–8, where we attempt to unravel the *universal* aspects of the quantum Hall effect in two-dimensional, *incompressible* quantum fluids.

Experimentally, the *QH effect* is observed in two-dimensional systems of electrons confined to a planar region Ω and subject to a strong, uniform magnetic field $\mathbf{B}_c$ transversal to Ω. For definiteness, we choose the region Ω to be a rectangle in the (x, y)-plane Ω with dimensions l_x and l_y in the x- and y-directions, respectively. By tuning the y-component, I_y, of the total electric current to some value and then measuring the voltage drop, V_x, in the x-direction – i.e., the difference in the chemical potentials of the electrons at the two edges parallel to the y-axis – one can calculate the *Hall resistance*

$$R_H := \frac{V_x}{I_y} , \tag{4.8}$$

and finds that, for a fixed density, n, of electrons and at temperatures close to $0\,K$, the value of R_H is independent of the current I_y. It depends only on the external magnetic field $\mathbf{B}_c$. If the electrons are treated classically one finds, by equating the electrostatic - to the Lorentz force, that

$$R_H = \frac{B_c}{nec} , \tag{4.9}$$

where B_c is the z-component of $\mathbf{B}_c$ perpendicular to the plane of the system, e is the elementary electric charge, and c is the velocity of light.

By also measuring the voltage drop, V_y, in the y-direction, one can determine the longitudinal resistance, R_L, from the equation

$$R_L := \frac{V_y}{I_y} . \tag{4.10}$$

Neither classical, nor quantum theory make simple predictions about the behaviour of R_L, but $R_L > 0$ means that there are *dissipative processes* in the system.

Two-dimensional systems of electrons (and, similarly, of holes) are realized, in the laboratory, as *inversion layers* which form at the interface between an insulator and a semiconductor when an electric field (gate voltage) perpendicular to the interface, the plane of the system, is applied. An example of such a material is a heterojunction (a "sandwich") made from $GaAs$ and $Al_xGa_{1-x}As$. The quantum-mechanical motion of the electrons in the z-direction perpendicular to the interface is then constrained by a deep potential well with a minimum on the interface. Quantum theory predicts that electrons of sufficiently low energy, i.e., at low enough temperatures, remain bound to the interface and form a very nearly two-dimensional system.

In classical physics, the connection between the electric field $\vec{E} = (E_x, E_y)$ in the plane of the system and the electric current density $\vec{J} = (J_x, J_y)$ is given by the *Ohm-Hall law*

$$\vec{E} = \boldsymbol{\rho} \vec{J} , \qquad \text{with} \qquad \boldsymbol{\rho} := \begin{pmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix} , \tag{4.11}$$

where the components of the resistivity tensor $\boldsymbol{\rho}$ are given as follows: $\rho_{xx} = R_L\, l_y/l_x$, $\rho_{yy} = R_L\, l_x/l_y$, and $\rho_H = R_H$. This is a phenomenological law valid on macroscopic distance scales and at low frequencies.

It is convenient to introduce a dimensionless quantity, the so-called *filling factor* ν, by setting

$$\nu := \frac{n \frac{hc}{e}}{B_c} \tag{4.12}$$

where $\frac{hc}{e}$ is the flux quantum. Then the *classical* Hall law (4.9) says that R_H^{-1} rises *linearly* in ν, $R_H^{-1} = \frac{e^2}{h}\, \nu$, the constant of proportionality being a constant of nature, $\frac{e^2}{h}$. Since, experimentally, B_c can be varied and n can be varied (by varying the gate voltage), this prediction of classical theory can be put to experimental tests. Experiments at very low temperatures and for rather pure inversion layers yield the following very surprizing data some of which are summarized in Fig. 4.1 below:

(D1) $\sigma_H = \frac{h}{e^2}\, R_H^{-1}$ has plateaux at *rational heights*, i.e., $\sigma_H = n_H/d_H$, with n_H, d_H relatively prime integers (see, e.g., Tsui, 1990; Stormer, 1992; Du *et al.*, 1993). Typically d_H is *odd*, but lately plateaux at $\sigma_H = \frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein *et al.*, 1988; Eisenstein *et al.*, 1990) and $\sigma_H = \frac{1}{2}$ (Suen *et al.*, 1992; Eisenstein *et al.*, 1992) have been observed. The plateaux at integer height occur with an astronomical accuracy (measurements are precise to one part in 10^8!).

(D2) When (ν, σ_H) belongs to a plateau the longitudinal resistance, R_L, very nearly *vanishes*, i.e., in plateau regions the system is *dissipationless*. Inverting the resistivity tensor $\boldsymbol{\rho}$ (see (4.11)) to obtain the conductivity tensor, $\boldsymbol{\sigma} = \boldsymbol{\rho}^{-1}$, yields the result that the diagonal part of $\boldsymbol{\sigma}$ *vanishes* on a plateau.

(D3) The precision of the plateau quantization is insensitive to details of sample preparation and geometry, hence this quantization is a "*universal*" phenomenon.

(D4) One has found (Clark *et al.*, 1988; Chang and Cunningham, 1989; Simmons *et al.*, 1989; Clark *et al.*, 1990; Hwang *et al.*, 1992) that, when (ν, σ_H) belongs to a plateau at non-integer height, then the system exhibits *fractionally charged* excitations, (the fractions of e being related to the value of d_H).

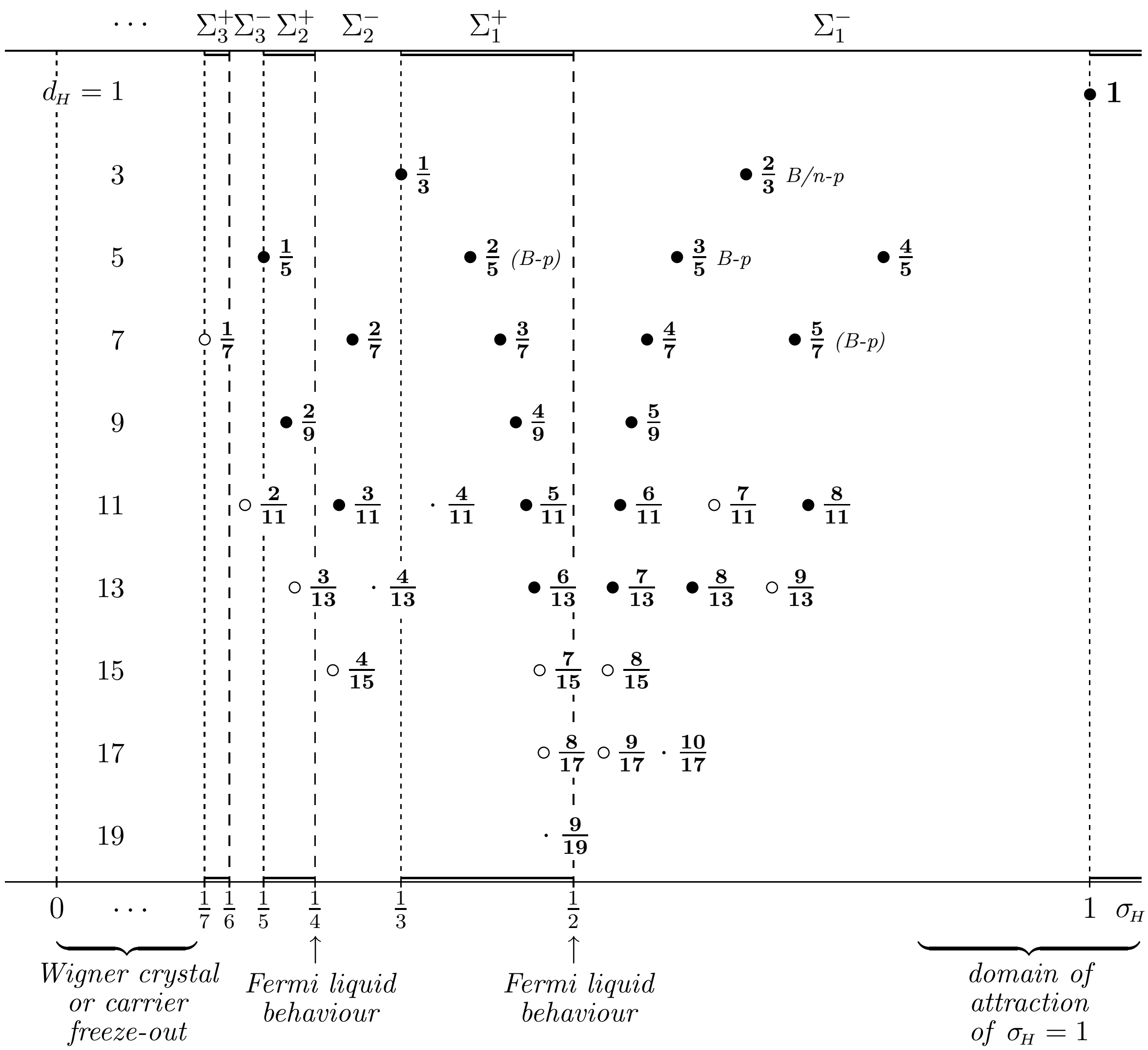

Figure 4.1. *Observed Hall fractions $\sigma_H = n_H/d_H$ in the interval $0 < \sigma_H \leq 1$, and their experimental status in single-layer quantum Hall systems.*

Well established Hall fractions are indicated by "$\bullet$". These are fractions for which an R_{xx}-minimum and a plateau in R_H have been clearly observed, and the quantization accuracy of $\sigma_H = 1/R_H$ is typically better than 0.5%. Fractions for which a minimum in R_{xx} and typically an inflection in R_H (i.e., a minimum in $dR_H/dB_c^\perp$, but no well developed plateau in R_H) have been observed are indicated by "$\circ$". If there are only very weak experimental indications or controversial data for a given Hall fraction, the symbol "$\cdot$" is used. Finally, "*B/n-p*" is appended to fractions at which a magnetic field *(B)* and/or density *(n)* driven phase transition has been observed.

(D5) Studies in "*tilted magnetic fields*" provide evidence that, when (ν, σ_H) belongs to a plateau at height $\sigma_H = \frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein, Willett *et al.*, 1988, 1990), $\frac{4}{3}$ (Clark *et al.*, 1989; Clark *et al.*, 1990), $\frac{8}{5}$ (Eisenstein, Stormer *et al.*, 1989, 1990a), or $\frac{2}{3}$ (Clark *et al.*, 1990; Eisenstein, Stormer *et al.*, 1990b), then the ground state of the system can be *spin-unpolarized*. For certain plateaux it might be a *spin-singlet* state.

(D6) Magnetic field and density driven *phase transitions* have been reported at $\sigma_H = 2/3$ (Clark *et al.*, 1990; Eisenstein, Stormer *et al.*, 1990b; Engel *et al.*, 1992). A magnetic field driven phase transition has been established at $\sigma_H = 3/5$ (Engel *et al.*, 1992), and a possible observation of such a phase transition at $\sigma_H = 5/7$ has been discussed in Sajoto *et al.* (1990).

Next, we propose to study what the Ohm-Hall law (4.11) tells us about a two-dimensional system of electrons in an external magnetic field when (ν, σ_H) belongs to a *plateau*. As noted in **(D2)**, experimentally one finds that, in this situation, the longitudinal resistance R_L vanishes. This signals the *absence of dissipative processes*. The absence of dissipative processes could be explained if one succeeded in showing that the spectrum of the many-electron Hamiltonian of the system exhibits an *energy gap*, $\delta > 0$, above the ground-state energy, (or, at least, that states of very small energy above the ground-state energy are *localized*). To exhibit a positive energy gap for certain values of the filling factor ν, physically interpreted as *incompressibility* of the system, poses difficult analytical problems. Some recent ideas about how to establish incompressibility at particular filling factors can be found in the following papers: studies of Laughlin states are given in Haldane (1983, 1990), Halperin (1983, 1984), Laughlin (1983a, 1983b, 1984, 1990), Arovas, Schrieffer, and Wilczek (1984), and Trugman and Kivelson (1985); off-diagonal long-range order and Chern-Simons-Landau-Ginzburg theory in fractional QH fluids have been studied in Girvin and MacDonald (1987), Read (1989), Zhang, Hansson, and Kivelson (1989), Lee and Zhang (1991), Fröhlich (1992), Fröhlich, Kerler, and Marchetti (1992), and Zhang (1992); finally, for some numerical studies concerning the question of incompressibility see, e.g., Fano, Ortolani, and Colombo (1986), Yoshioka (1986), Chakraborty and Pietiläinen (1987, 1988), Rezayi (1987), and d'Ambrumenil and Morf (1989). What is easier to show is for *which values* of the parameter $\sigma_H = \frac{h}{e^2} R_H^{-1}$ a positive energy gap δ *cannot* occur; more precisely, to prove a "*gap labelling theorem*". Such a theorem is stated at the end of Subsect. 7.2, and will form the main theme of Sect. 8.

Thus, if the system is incompressible, in the sense that $R_L = 0$, then we have the following form for the *Hall law* (we use units such that $e^2/h = 1$):

$$\vec{J} = \boldsymbol{\sigma} \vec{E} , \quad \text{with} \quad \boldsymbol{\sigma} := \boldsymbol{\rho}^{-1} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} , \tag{4.13}$$

where $\sigma_H = R_H^{-1}$. This is a *phenomenological* law valid on macroscopic distance scales and at low frequencies, as mentioned after (4.11). More fundamental are the following two laws: *Charge conservation*

$$\frac{1}{c}\frac{\partial}{\partial t} J^0 + \vec{\nabla} \cdot \vec{J} \;=\; 0 \;, \tag{4.14}$$

(continuity equation), where J^0 is (c times) the electric charge density, and *Faraday's induction law*,

$$\frac{1}{c}\frac{\partial}{\partial t} B + \vec{\nabla} \times \vec{E} \;=\; 0 \;, \tag{4.15}$$

where B denotes the component of the magnetic field perpendicular to the plane of the system, and $\vec{E}$ is the electric field in the plane of the system. We note that the dynamics of charged, spinless particles confined to a plane only depends on the component of the magnetic field perpendicular to the plane of the system and the components of the electric field in that plane. Combining Eqs. (4.13) through (4.15), we find that

$$\frac{\partial}{\partial t} J^0 \;=\; \sigma_H \frac{\partial}{\partial t} B \;. \tag{4.16}$$

Eq. (4.16) can be integrated with respect to time t. By $J^0 = J^0_{tot} - nec$ we denote the difference between the total electric charge density, J^0_{tot}, and the uniform background density, nec, of a system in a uniform background magnetic field B_c. Likewise, B denotes the difference between the total magnetic field, B_{tot}, and the uniform background field B_c. Then Eq. (4.16) implies the *charge-flux relation*

$$J^0 \;=\; \sigma_H B \;. \tag{4.17}$$

It is convenient to introduce the electromagnetic field tensor which is a 2-form, F, given by

$$F \;=\; \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \;, \quad \text{with} \quad (F_{\mu\nu}) \;=\; \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{pmatrix} \;, \tag{4.18}$$

and the 2-form $\mathcal{J}$ dual to the current density $J^\mu = (J^0, \vec{J})$, i.e.,

$$\mathcal{J} = \frac{1}{2}\sum_{\mu,\nu} \mathcal{J}_{\mu\nu}\, dx^\mu \wedge dx^\nu \;, \quad \text{with} \quad \mathcal{J}_{\mu\nu} = \varepsilon_{\mu\nu\rho}\, J^\rho \;. \tag{4.19}$$

Then Eqs. (4.13) and (4.17) can be combined into one equation,

$$\mathcal{J} \,=\, -\sigma_{H}\, F \;, \tag{4.20}$$

while current conservation (4.14) is expressed as

$$\mathrm{d}\mathcal{J} = \frac{1}{2} \sum_{\alpha,\mu,\nu} \partial_\alpha \mathcal{J}_{\mu\nu}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu \;=\; 0 \;, \tag{4.21}$$

and Faraday's induction law (4.15) becomes

$$\mathrm{d}F \;=\; 0 \;. \tag{4.22}$$

Eqs. (4.20) through (4.22) are compatible with each other if, and only if, σ_H is *constant*. If the values of σ_H along the two sides of a curve Γ differ from each other – which happens, for example, at the boundary of the system – then an *additional* current, $\vec{\mathcal{I}}$, *not* described by Eq. (4.13), is observed in the vicinity of Γ, in order to reconcile charge conservation with the induction law. [For time-independent fields one finds that $\vec{\nabla} \cdot \vec{\mathcal{I}} = (\vec{\nabla}\sigma_H) \times \vec{E}$; see also Halperin (1982) and Sect. 7 below.]

Note that Eqs. (4.20) through (4.22) are *generally covariant* and independent of metric properties of the system. Eqs. (4.21) and (4.22) can be integrated by introducing the 1-forms (or "vector potentials") A and b, with

$$\mathcal{J} \,=\, \mathrm{d}b \;, \quad \text{i.e.,} \quad \mathcal{J}_{\mu\nu} \,=\, \partial_\mu b_\nu - \partial_\nu b_\mu \;, \quad \text{and} \quad F \,=\, \mathrm{d}A \;. \tag{4.23}$$

Eq. (4.20) then reads

$$\mathrm{d}b \;=\; -\sigma_H\, \mathrm{d}A \;. \tag{4.24}$$

Eq. (4.24) is the Euler-Lagrange equation derived form an *action principle*! The corresponding action, $S(b;A)$, is given by

$$\begin{aligned} S(b;A) \;&:=\; \frac{1}{2c^2\sigma_H}\int_\Lambda b \wedge \mathrm{d}b \,+\, \frac{1}{c^2}\int_\Lambda A \wedge \mathrm{d}b \,+\, \mathrm{B.T.}(A|_{\partial\Lambda}, b|_{\partial\Lambda}) \\ &=\; \frac{1}{2c^2\sigma_H}\int_\Lambda (\mathrm{d}^{-1}\mathcal{J}) \wedge \mathcal{J} + \frac{1}{c^2}\int_\Lambda A \wedge \mathcal{J} + \mathrm{B.T.}(A|_{\partial\Lambda}, \mathcal{J}|_{\partial\Lambda})\,, \end{aligned} \tag{4.25}$$

where $\Lambda := \mathbb{R} \times \Omega$ is the space-time domain to which the system is confined, and "B.T." stands for *boundary terms* which only depend on the restrictions $A|_{\partial\Lambda}$ and $b|_{\partial\Lambda}$ of A and b to the space-time boundary $\partial\Lambda$, (see the remark after (4.27) below). Moreover, $S(b;A)$

shall be varied with respect to the *dynamical* variable, that is, with respect to b; (the vector potential A of the electromagnetic field is a tunable, external field).

Why is the result (4.25) interesting? It is interesting, because an equation of motion, like (4.24), that can be derived form an action principle can be quantized easily; for example by using *Feynman path integrals*. Clearly, the current density $\mathcal{J}$ of a system of electrons must be interpreted as a quantum-mechanical operator-valued distribution. Hence (4.24) must be *quantized.* We note that, in the present example, Feynman path integral quantization has a mathematically rigorous interpretation. In formal, "physical" notation, Feynman's path space measure is given by

$$dP_A(b) \; = \; Z(A)^{-1} \, \exp\Big[\frac{i}{\hbar} S(b;A)\Big] \, \mathcal{D}b \, , \tag{4.26}$$

where the *partition* - or *generating function* $Z(A)$ is chosen such that (formally) $\int dP_A(b) = 1$. This implies that

$$Z(A) \; = \; Z_o \, \exp\Big[-\frac{i\,\sigma_H}{2\hbar c^2}\int_\Lambda A \wedge \mathrm{d}A \; + \; \mathrm{B.T.}(A|_{\partial\Lambda})\Big] \, , \tag{4.27}$$

where Z_o is a constant *independent* of A. [In (4.26), we have omitted a gauge fixing term for the integration over the field b.] At this point, we emphasize that a non-trivial boundary term, $\mathrm{B.T.}(A|_{\partial\Lambda})$, in Eq. (4.27) is a necessity forced upon us by the $U(1)_{em}$ *gauge invariance of quantum mechanics* (see Sect. 3): Under a $U(1)$ gauge transformation, $A \mapsto A + \mathrm{d}\chi$, the Chern-Simons term in (4.27) transforms according to

$$\int_\Lambda A \wedge \mathrm{d}A \;\; \mapsto \;\; \int_\Lambda A \wedge \mathrm{d}A \; - \; \int_{\partial\Lambda} \mathrm{d}\chi \wedge A \, , \tag{4.28}$$

i.e., there is a *gauge anomaly* localized at the boundary $\partial\Lambda$. Hence, in order for the partition function $Z(A)$ to be $U(1)$ gauge invariant, the presence of a boundary term, $\mathrm{B.T.}(A|_{\partial\Lambda})$, exhibiting a gauge anomaly cancelling the one in (4.28) is indispensable! We shall see that the anomalous part of $\mathrm{B.T.}(A|_{\partial\Lambda})$ turns out to be the generating functional of the connected Green functions of chiral current operators generating a $\hat{u}(1)$ current (Kac-Moody) algebra which, physically, describes charged, chiral waves circulating at the edge of the QH sample. Sect. 7 will be devoted to an investigation of this current algebra. Our analysis will lead to a list of the possible (quantized) values of the response coefficient of the Hall conductivity σ_H; see also remark **(iii)** at the end of this subsection.

Given the expression (4.27) for the partition function $Z(A)$, we may ask whether there is a simple way of recovering the action $S(b;A)$ as a functional of the vector potential b of the conserved current density $\mathcal{J}$, as given in (4.25). The answer is yes. It is provided by the following *functional Fourier transform* identity:

$$\exp\Big[\frac{i}{\hbar}S(b;A)\Big] \;=\; \text{const.} \int \mathcal{D}\alpha\; \exp\Big[-\frac{i}{\hbar c^2}\int \alpha \wedge \mathrm{d}b\Big]\, Z(A+\alpha)\;, \tag{4.29}$$

where we again omit a suitable gauge fixing term for the integral over α; see Fröhlich and Kerler (1991), and the general discussion in Sect. 5 below.

Remarks. **(i)** Defining the *effective action*, $S^{\text{eff}}(A)$, of a two-dimensional electronic system by

$$S^{\text{eff}}(A) \;:=\; \frac{\hbar}{i}\,\ln Z(A)\;, \tag{4.30}$$

our circle of arguments can be closed from Eq. (4.27) back to the starting point of the Hall law (4.13) (and (4.17)) by noting that

$$J^i \;=\; c^2\,\frac{\delta S^{\text{eff}}(A)}{\delta A_i} \;=\; \sigma_H\,\varepsilon^{ij}\,E_j\;, \tag{4.31}$$

where $E_j = \partial_j A_0 - \partial_0 A_j$, for $j = 1, 2$.

Eqs. (4.31), (4.30) and (4.29) make it clear that one approach leading to an understanding of the QH effect is to derive, for a QH fluid at particular values of the filling factor ν, the effective action $S^{\text{eff}}(A)$ corresponding to Eq. (4.27) from "first principles". In Subsect. 6.1, we show that *gauge invariance* of non-relativistic quantum mechanics and the single assumption of *incompressibility* of QH fluids are sufficient to uniquely determine their effective action $S^{\text{eff}}(A)$ in the "scaling limit" and thereby to derive (4.27). Hence the phenomenology of QH systems at low frequencies and on large distance scales, including the quantization of the Hall conductivity σ_H (see remark **(iii)** below), can be derived from gauge invariance and incompressibility. This shows that a proof of incompressibility of electronic systems at particular values of the filling factor ν is really the essential problem in the theory of the QH effect in need of further investigation.

(ii) It can be seen directly from the Ohm-Hall law (4.11) that the incompressibility of QH fluids is a crucial property which allows for a description of these systems in terms of an effective action formalism: Only for *dissipationless* systems, i.e., for systems with an *antisymmetric* conductivity tensor $\boldsymbol{\sigma}$, it is possible to functionally integrate relation (4.31) in order to obtain an effective action. In other words, for electronic systems with *dissipation*, one *cannot* formulate the Ohm-Hall law (coming from transport theory) within an effective action formalism.

(iii) Clearly, we must require that the quantum theory with action $S(b;A)$ given by (4.25), defined by the Feynman path integral (4.26), describe *localized, particle-like excitations* with the quantum numbers of the *electron* or *hole*, i.e., with electric charge $\pm e$ and Fermi statistics. We shall see in Sects. 7.2 and 8.2 that this requirement implies that, for *consistency* of the theory, the dimensionless Hall conductivity (*Hall fraction*) $\sigma = \frac{h}{e^2}\,\sigma_H$ must be a *rational number*.

5 Scaling Limit of the Effective Action of Fermi Systems, and Classification of States of Non-Relativistic Matter

In this section, we generalize the observations made at the end of the last subsection. We review a conceptual framework (Fröhlich, Götschmann, and Marchetti, 1995a, 1995b), based on the bosonization of quantum systems with infinitely many degrees of freedom, which has proven to be useful in attempting to classify states of non-relativistic matter at very low temperatures. In this section, we focus our attention on the analysis of electric charge transport properties. Magnetic properties can be studied in a similarly general manner; for an example see Sect. 6 where two-dimensional, incompressible quantum fluids are treated, *including* their magnetic properties.

The basic ideas underlying our approach are very simple: The starting point is to study the response of a quantum system of charged particles to perturbations by external electromagnetic fields. Thus we couple the electric current density, J^μ, to an arbitrary, smooth, external electromagnetic vector potential (1-form), A, and then attempt to calculate the partition function, $Z(A)$, of the system as a functional of A.

Of course, this is a very complicated task. However, in order to *classify electric properties of non-relativistic matter*, we are really only interested in understanding the behaviour of the *effective action* (see also (4.30))

$$S^{\mathrm{eff}}(A) \;:=\; \frac{\hbar}{i}\,\ln Z(A)\;, \tag{5.1}$$

on very large distance scales and at very low frequencies. We thus study families of systems confined to ever larger cubes, $\Omega^{(\theta)} := \{\vec{x}\,|\,\vec{x}/\theta \in \Omega\}$, in physical space $\mathbb{E}^d$, $d = 1, 2, 3$, where Ω is a fixed compact subset of $\mathbb{E}^d$, and $1 \leq \theta < \infty$ is a scale parameter. We keep the particle density, n, and the temperature $T\,(\approx 0\,K)$ constant. We then couple the electric current density of the system confined to $\Omega^{(\theta)}$ to a vector potential $A^{(\theta)}$ given by

$$A^{(\theta)}(t,\vec{x}\,) \;:=\; \theta^{-1}\,A(\frac{t}{\theta},\frac{\vec{x}}{\theta})\;, \qquad \frac{\vec{x}}{\theta}\in\Omega\;, \tag{5.2}$$

where A is an arbitrary, but θ-independent vector potential on $\mathbb{R}\times\Omega$. We then study the behaviour of $S^{\mathrm{eff}}_{\Omega^{(\theta)}}(A^{(\theta)})$ when θ becomes large.

More precisely, we attempt to expand $S^{\mathrm{eff}}_{\Omega^{(\theta)}}(A^{(\theta)})$ in powers of θ^{-1} around $\theta = \infty$ (to a finite order) and define the *scaling limit*, $S^*(A)$, *of the effective acction* to be the coefficient of the leading power of θ in that expansion; more details are given in Subsect. 6.1.

Remarkably, all many-body systems of non-relativistic electrons that can be controled analytically have the property that $S^*(A)$ is *quadratic* in A; this holds, in particular, for insulators, Landau-Fermi (non-interacting electron) liquids, metals, incompressible quantum

Hall fluids (e.g., Laughlin fluids), and superconductors. Since we do not know of a proof of this fact for any imagineable system of non-relativistic electrons at positive density, we call this fact a *"quasi-theorem"* and summarize below the explicit results of a case-by-case analysis that has been presented in Fröhlich, Götschmann, and Marchetti (1995a, 1995b). It should be emphasized at this point, that, except in the case of insulators and incompressible quantum Hall fluids (see Subsect. 6.1), the proof of the quasi-theorem is never trivial and does *not* simply follow from dimensional analysis (and gauge-invariance).

It is well known that $S^{\mathrm{eff}}(A)$ is the generating functional of connected Green functions of the electric current density J^μ and that it is *gauge-invariant*, i.e.,

$$S^{\mathrm{eff}}(A + \mathrm{d}\chi) \;=\; S^{\mathrm{eff}}(A) \;, \tag{5.3}$$

where χ is an arbitrary, real function on space-time, and $\mathrm{d}\chi$ is its exterior derivative. Clearly, gauge-invariance persists upon passing to the scaling limit. Using our quasi-theorem, we conclude that

$$S^*(A) \;=\; \frac{1}{2}(A, \Pi^*\, A) \;=\; \frac{1}{2}\int d^{d+1}x \int d^{d+1}y\; A_\mu(x)\Pi^{*\mu\nu}(x-y)A_\nu(y) \;, \tag{5.4}$$

where, for $x \neq y$, $\Pi^{*\mu\nu}(x-y)$ is given by the scaling limit of the connected two-current Green function $\langle\, T(\, J^\mu(x) J^\nu(y))\,\rangle^{\mathrm{con}}$.

By gauge-invariance, or, equivalently, current conservation,

$$\partial_\mu \Pi^{*\mu\nu} \;=\; \partial_\nu \Pi^{*\mu\nu} \;=\; 0 \;. \tag{5.5}$$

Furthermore, $\Pi^{*\mu\nu}$ inherits all the symmetries of the system; (in (5.4), we have assumed translation invariance, in the limit when Ω increases to $\mathbb{E}^d$). Thus classifying electric properties of a system reduces, in the scaling limit and under the assumption that the quasi-theorem holds, to a classification of "vacuum polarisation tensors", $\Pi^{*\mu\nu}$, satisfying (5.5) and having certain symmetries.

In principle, essentially all information concerning electric properties of a system can be retrieved from its effective action $S^{\mathrm{eff}}(A)$ by fairly straightforward calculations. However, in order to evoke and then apply analogies with other physical systems, in particular with gauge theories of elementary particle physics, it is useful to embark on a detour. The detour chosen here is *bosonization*. The idea (exemplified by our discussion in Subsect. 4.5) is as follows: Since the electric current density J^μ $(\mu = 0, \ldots, d)$ is conserved, it can be derived from a "potential",

$$J^\mu \;=\; \varepsilon^{\mu\mu_1\cdots\mu_d}\, \partial_{\mu_1} b_{\mu_2\cdots\mu_d} \;, \tag{5.6}$$

where ε is the totally antisymmetric tensor in $d+1$ dimensions, and $b_{\mu_2 \cdots \mu_d}$ is an antisymmetric tensor field of rank $d-1$. In the language of differential forms, Eq. (5.6) is expressed as

$$J \;=\; J_\mu \, dx^\mu \;=\; * \, \mathrm{d} b \; , \tag{5.7}$$

where $*$ is the Hodge $*$-operation and d denotes exterior differentiation; see Subsect. 3.1. The "potential" b of the current density J is determined by (5.7) only up to the exterior derivative of an antisymmetric tensor field of rank $d-2$, (a (d–2)-form). Thus b is what one calls a "*gauge form*". For one-dimensional systems, b is a scalar and is determined by (5.7) up to a constant. In two dimensions, b is a 1-form determined by J up to the exterior derivative of an arbitrary function.

The basic idea is then to derive the effective field theory for the field b from $S^{\mathrm{eff}}(A)$. This field theory is determined by an action $\tilde{S}^{\mathrm{eff}}(b)$. Choosing units in which $\hbar = c = 1$ and using an *imaginary-time (euclidean)* formulation, $\tilde{S}^{\mathrm{eff}}(b)$ is obtained from $S^{\mathrm{eff}}(A)$ by functional Fourier transformation

$$e^{-\tilde{S}^{\mathrm{eff}}(b)} \;=\; N^{-1} \int \mathcal{D}A \, e^{-S^{\mathrm{eff}}(A)} \, e^{i \int A \wedge db} \; , \tag{5.8}$$

where N is a (divergent) normalization factor (proportional to the volume of the Lie algebra of $U(1)$ gauge transformations); see also (4.29) in the previous subsection. Gauge invariance implies that

$$\tilde{S}^{\mathrm{eff}}(b + \mathrm{d}\Lambda) \;=\; \tilde{S}^{\mathrm{eff}}(b) \; , \tag{5.9}$$

for an arbitrary (d–2)-form Λ ($d \geq 2$). Our quasi-theorem then implies that the low-wave-vector, low-energy modes of the *bosonic* field b are *non-interacting*, i.e., the scaling limit of the system described in terms of the b-field has a *quadratic action*, $\tilde{S}^*$, whose form is constrained by its gauge invariance, Eq. (5.9), and by the symmetries of the system.

Many quantities of interest in the original system can be expressed in terms of quantities referring to the b-field. For example, current Green functions (at imaginary time) are given by expectations of products of the "field strength" db in the functional measure

$$Z^{-1} \, e^{-\tilde{S}^{\mathrm{eff}}(b)} \, \mathcal{D}b \; . \tag{5.10}$$

Green functions of electron creation - and annihilation operators turn out to be proportional to expectations of *disorder operators* of the "dual" theory formulated in terms of the b-field (Fröhlich, Götschmann, and Marchetti, 1995a).

If Σ is an open subset of physical space, and $\mathcal{Q}_\Sigma$ denotes the operator measuring the total electric charge inside Σ then

$$e^{i\alpha\mathcal{Q}_\Sigma} \propto e^{i\alpha\int_{\partial\Sigma} b} , \quad \alpha \in \mathbb{R} , \tag{5.11}$$

i.e., the charge operator can be reconstructed from operators analogous to the *Wilson loops* of gauge theory. The operators in (5.11) are *dual* to the disorder operators describing electron creation and annihilation, in the sense of Wegner - 't Hooft duality (Wegner, 1971; 't Hooft, 1978). This suggests that we can carry over the consequences of Wegner - 't Hooft duality form gauge theory to condensed matter physics. As a consequence, we mention that if, at zero temperature, the total electric charge operator, $\mathcal{Q} = \lim_{\Sigma \nearrow \mathbb{R}^d} \mathcal{Q}_\Sigma$, is well defined on the Hilbert space of all physical states of the system then disorder Green functions – e.g., the Green function of an electron creation - and an annihilation operator – exhibit strong spatial cluster decomposition properties, and conversely. This applies to *insulators, incompressible quantum Hall fluids, and superconductors with (unscreened) Coulomb repulsion.* In contrast, if the field b couples the ground state of the system to a massless quasi-particle then the total charge operator does *not* exist, because charge fluctuations are divergent in the thermodynamic limit, as in metals and massless superconductors.

For *two-dimensional* systems, both fields, A and b, are 1-forms defined up to gradients of scalar functions. In this case, Eq. (5.8) enables us to define a notion of *duality*: a system 1 and a system 2 are *dual* to each other if, and only if,

$$\tilde{S}_1^* \propto S_2^* . \tag{5.12}$$

It turns out that, in the sense of Eq. (5.12), a two-dimensional *insulator* is *dual* to a two-dimensional *London superconductor*, a metal to a "semi-conductor", and an incompressible quantum Hall (e.g. a Laughlin) fluid is *self-dual.*

Besides the duality expressed in Eqs. (5.8) and (5.12), there is also a notion of Kramers -Wannier duality: in *any dimension* d, a $U(1)$ gauge theory of an antisymmetric tensor field b of rank $d-1$ is "Kramers-Wannier dual" to a *scalar* field theory. Thus, for example, a London superconductor, corrected by dynamical Abrikosov vortices, is Kramers-Wannier dual to a Landau-Ginzburg superconductor; see Subsect. IV.D in Fröhlich and Studer (1993b).

The two notions of duality sketched here are conceptually quite clarifying and useful in a classification of electric properties of non-relativistic matter. One of the principal advantages of reformulating the theory of a system of electrons in terms of the tensor field b (bosonization) is that this formulation is convenient to explore properties of systems obtained by perturbing a given one by *two-body interactions.* A translation-invariant two-body interaction, I_{pert}, has the form

$$I_{\text{pert}} = \frac{1}{2}\int d^{d+1}x \int d^{d+1}y\, J^\mu(x) V_{\mu\nu}(x-y) J^\nu(y) . \tag{5.13}$$

After bosonization, the effective action of the perturbed system is given by

$$\tilde{S}^{\mathrm{eff}}_{\mathrm{tot}}(b) \;=\; \tilde{S}^{\mathrm{eff}}(b) + \frac{1}{2}\int d^{d+1}x \int d^{d+1}y \, (*\mathrm{d}b)^{\mu}(x) V_{\mu\nu}(x-y)(*\mathrm{d}b)^{\nu}(y) \;, \tag{5.14}$$

where $\tilde{S}^{\mathrm{eff}}(b)$ is the effective action of the unperturbed system. Note that, expressed in terms of the field b, the two-body interaction I_{pert} is *quadratic* (rather than quartic)! A conventional two-body interaction described by an instantaneous two-body potential corresponds to a kernel $V_{\mu\nu}$ given by

$$V_{\mu\nu}(x-y) \;=\; \delta_{\mu 0}\, \delta_{\nu 0}\, V(\vec{x}-\vec{y}\,)\, \delta(x^0 - y^0) \;, \tag{5.15}$$

with $x := (x^0, \vec{x}\,)$ and $y := (y^0, \vec{y}\,)$.

Suppose now that the action of the perturbed system in the scaling limit (scale parameter $\theta \to \infty$), $\tilde{S}^*_{\mathrm{tot}}$, is given by the scaling limit, $\tilde{S}^*$, of the action of the unperturbed system, perturbed by the long-range tail, I^*_{pert}, of the two-body interaction I_{pert}. From our quasi-theorem we infer that $\tilde{S}^*$ is quadratic in b, and hence, since I^*_{pert} is quadratic in b, $\tilde{S}^*_{\mathrm{tot}}$ is *quadratic* in b, too, and, by (5.9) and (5.14), gauge-invariant. It is given by

$$\tilde{S}^*_{\mathrm{tot}} \;=\; \tilde{S}^* + \theta^{\kappa}\, I^*_{\mathrm{pert}} \;, \qquad (\theta \to \infty) \;, \tag{5.16}$$

for some exponent κ. It is plausible that the assumption that perturbation by I_{pert} and passage to the scaling limit are commuting operations is justified if $V_{\mu\nu}$ is *positive-definite* and of very *long range* (e.g., Coulomb repulsion). In this case, the analysis sketched above yields a variant of the "random phase approximation" (RPA). These ideas have been applied in Fröhlich, Götschmann, and Marchetti (1995a, 1995b) to the following systems:

(i) A metal pertubed by repulsive two-body Coulomb interactions. In this case, one obtains the exact formula for the *plasmon gap*.

(ii) A massless London superconductor perturbed by repulsive two-body Coulomb interactions. In this example, one recovers a precise formulation of the *Anderson-Higgs mechanism*; (the $U(1)$-Goldstone boson becomes massive).

(iii) A Landau-Fermi liquid perturbed by repulsive two-body interactions, in the presence of a static source. For a two-dimensional system of this type with Coulomb-Ampère interaction, a possible crossover to a non-Fermi-liquid behaviour ("*Luttinger liquid*") has been observed.

Remark: An effective field theory similar to the one described here for the electric current density can also be derived for the *spin* ($SU(2)-$) *current density* by using tools developed in the study of the so-called BF-theories (see Cattaneo et al. (1995)). It is useful in the analysis of *magnetic properties* of electron gases (and will be described elsewhere).

Examples. In the second part of this section, we provide some explicit examples illustrating the ideas presented above.

(F) We start with the *free non-relativistic fermion gas*. The euclidean action of the system is given by

$$S_E(\Psi, \Psi^*) \;=\; \int d^{d+1}x \Big[\Psi^* \partial_0 \Psi + \frac{1}{2m} |\vec{\nabla}\Psi|^2 + \mu\, \Psi^* \Psi\Big](x) \;, \tag{5.17}$$

where m is the mass of the fermions and μ the chemical potential.

In $d = 1$, the Fermi surface consists of only two points, $\pm k_F$. Since physics in the scaling limit is dominated by excitations with momenta close to the Fermi surface, one expands the momentum-space fermionic two-point Green function around the points $\pm k_F$ on the Fermi surface, in order to calculate its scaling limit.

The result of this analysis is that, in the scaling limit, one can introduce quasi-particle fields, Ψ_L, Ψ_L^* and Ψ_R, Ψ_R^*, corresponding to the points $-k_F$ and k_F, respectively. The fields Ψ_L, Ψ_L^* describe left-moving excitations, while Ψ_R, Ψ_R^* describe right movers. These excitations approximately obey a "relativistic" dispersion relation, $\omega \approx |p|$, where $\omega = E - E_F$ and $p = k - k_F$. The original field variable Ψ is related to the relativistic field variables Ψ_L and Ψ_R by

$$\Psi(x) \;\approx\; e^{-ik_F x^1} \Psi_R(x) + e^{ik_F x^1} \Psi_L(x) \;. \tag{5.18}$$

The euclidean action of Ψ_L and Ψ_R is given by

$$S_E(\Psi_R, \Psi_R^*, \Psi_L, \Psi_L^*) \;=\; \int d^2x \Big[\Psi_R^* (\partial_0 + i v_F \partial_1) \Psi_R + \Psi_L^* (\partial_0 - i v_F \partial_1) \Psi_L\Big](x) \;. \tag{5.19}$$

The action (5.19) is the euclidean action of *free, massless Dirac fermions* (in two dimensions), with the Fermi velocity v_F playing the role of the velocity of light. As a consequence, in the scaling limit, the effective action $S^*(A)$ obtained by minimal coupling of Ψ and Ψ^* to the gauge field A is *quadratic* in A; for more details, see Subsect. 7.1. Using the bosonization method sketched above, one obtains a quadratic action for the potential b of the conserved $u(1)$ current. These features of one-dimensional systems have been known for many years; see Lieb and Mattis (1965), and Luther and Peshel (1975).

More recently, it has been realized (see Luther, 1979, for the original observations) that similar ideas on the scaling limit can be used in any dimension d. The sum over the two points of the Fermi surface in (5.18) must be replaced by an integration over a $(d-1)$-dimensional

Fermi surface; see Feldman and Trubowitz (1990), Benfatto and Gallavotti (1990), Anderson (1990), Shankar (1991), and Feldman, Magnen *et al.* (1992).

Let Ω be the $(d-1)$-dimensional unit sphere in momentum space. Elements of Ω are denoted by $\vec{\omega}$. The extension of formula (5.18) to higher dimensions is given by

$$\Psi(x) \;\approx\; \int_\Omega d\vec{\omega}\, e^{-ik_F\vec{x}\cdot\vec{\omega}}\,\Psi_{\vec{\omega}}(x)\ , \tag{5.20}$$

and the euclidean action for the fields $\Psi_{\vec{\omega}}$ describing the scaling limit of the free Fermi gas has the form

$$S_E(\{\Psi_{\vec{\omega}},\Psi^*_{\vec{\omega}}\}) \;=\; \text{const.}\,(k_F\alpha)^{d-1}\ \int_\Omega d\vec{\omega}\int d^{d+1}x\Big[\Psi^*_{\vec{\omega}}(\partial_0+iv_F\vec{\omega}\cdot\vec{\nabla})e^{-\alpha(\vec{\omega}\wedge\vec{\nabla})^2}\Psi_{\vec{\omega}}\Big](x)\ , \tag{5.21}$$

where α is a large constant.

Coupling Ψ and Ψ^* minimally to the gauge field A, one can show (see Fröhlich, Götschmann, and Marchetti (1995b)) that $S^*(A)$ is given by an integral over the set of directions, $\pm\vec{\omega}$, in momentum space of contributions coming from the degrees of freedom described by the fields $\Psi^{\#}_{\vec{\omega}}$ and $\Psi^{\#}_{-\vec{\omega}}$, corresponding to antipodal points, $\pm\vec{\omega}$, on the Fermi surface. Every such contribution corresponds to the one of a one-dimensional free fermion system and is quadratic in $A_{\vec{\omega}} = (A_0, \vec{\omega}\cdot\vec{A})$. Since $S^*(A)$ is an integral of the effective actions $S^*(A_{\vec{\omega}})$ of one-dimensional systems over all pairs of points $\pm\vec{\omega}$ in Ω, it, too, is quadratic in A. These considerations yield an explicit expression for $S^*(A)$, in particular for the polarization tensor Π^*.

Let $\Pi^{\mu\nu}(x,y)$ denote the vacuum polarization tensor of a system of electrons at zero temperature, and let $n^{(1)} = n\,dx^0$, where n is the density of the system. Then (assuming that the quasi-theorem holds)

$$S^*(A) \;=\; \frac{1}{2}(A,\Pi^*A) + i(n^{(1)},A)\ . \tag{5.22}$$

By invariance under $U(1)$ gauge transformations, translations, rotations and parity, the expression for $\Pi^{\mu\nu}$ is given, in momentum space, by

$$\begin{aligned}
\Pi^{ij}(k) &= \Pi_\perp(k)\Big(\delta^{ij}-\frac{k^ik^j}{\vec{k}^2}\Big)+\Pi_\parallel(k)\,\frac{k^ik^j}{\vec{k}^2}\ ,\\
\Pi^{i0}(k) &= -\Pi_\parallel(k)\,\frac{k^i}{k_0}\ ,\\
\Pi^{00}(k) &= \Pi_\parallel(k)\,\frac{\vec{k}^2}{k_0^2} \;=:\; \Pi_0(k)\ ,
\end{aligned} \tag{5.23}$$

for $i, j = 1, \ldots, d$. For non-interacting electrons the explicit form of Π_0^* in the scaling limit, in $d = 1$, is given by

$$\Pi_0^*(k) \;=\; \chi_0 \, \frac{v_F^2 \, k_1^2}{k_0^2 + v_F^2 k_1^2} \;, \tag{5.24}$$

and, in higher dimensions, to leading order in $|\frac{v_F \vec{k}}{k_0}|$ and $|\frac{k_0}{v_F \vec{k}}|$, we have that

$$\begin{aligned} \Pi_0^*(k) \;&=\; (\chi_0 + \lambda_0 \, |\frac{k_0}{v_F \vec{k}}|) \, \Theta\Big(1 - |\frac{k_0}{v_F \vec{k}}|\Big) + \tilde{\chi}_0 \, |\frac{v_F \vec{k}}{k_0}|^2 \, \Theta\Big(|\frac{k_0}{v_F \vec{k}}| - 1\Big) \;, \\ \Pi_\perp^*(k) \;&=\; (\chi_\perp \vec{k}^2 + \lambda_\perp |\frac{k_0}{v_F \vec{k}}|) \, \Theta\Big(1 - |\frac{k_0}{v_F \vec{k}}|\Big) + \tilde{\chi}_0 \, \Theta\Big(|\frac{k_0}{v_F \vec{k}}| - 1\Big) \;, \end{aligned} \tag{5.25}$$

where χ_0, λ_0, $\tilde{\chi}_0$, $\chi_\perp$, and $\lambda_\perp$ are constants depending on d, m, and v_F; see Pines and Nozière (1989), and Fetter and Walecka (1980).

For $d > 1$, according to the argument given before, the expression for Π^* can be derived from an integration over $\pm\vec{\varpi}$ of the one-dimensional vacuum polarizations, $\Pi_{\vec{\varpi}}$, referring to the quasi-particle fields $\Psi_{\vec{\varpi}}$, $\Psi_{-\vec{\varpi}}$:

$$(A, \Pi^* \, A) \;=\; \frac{1}{2} \int_\Omega d\vec{\varpi} \, (A_{\vec{\varpi}}, \Pi_{\vec{\varpi}}^* \, A_{\vec{\varpi}}) \;. \tag{5.26}$$

It is important to note that the quadratic nature of $S^*(A)$ does not just follow from a "small A" approximation and dimensional analysis, but it is the result of explicit cancellations arising from the structure of the fermion two-point function in the scaling limit.

Remark. One can bosonize directly the scaling limit action of the free Fermi gas expressed in terms of the quasi-particle fields $\{\Psi_{\vec{\varpi}}, \Psi_{-\vec{\varpi}}\}$, $\vec{\varpi} \in \Omega$, by introducing real scalar fields $\{\Phi_{\vec{\varpi}}\}$, $\vec{\varpi} \in \Omega$, identifying $\Phi_{\vec{\varpi}}$ with $\Phi_{-\vec{\varpi}}$. The result is Luther-Haldane bosonization (Luther, 1979; Haldane, 1992) in the euclidean path-integral formalism; see also Fröhlich, Götschmann, and Marchetti (1995a, 1995b).

From now on we omit the trivial term $i(n^{(1)}, A)$ in the effective actions. This corresponds to redefining the density $J^0(\Psi, \Psi^*)$ by subtracting the background density n. Furthermore, we set $v_F = 1$.

(I) *Insulators and incompressible quantum fluids* form another class of fermionic systems whose (bulk) effective action in the scaling limit, $S^*(A)$, is quadratic in A. Here incompressibility means that the connected correlation functions of the current density $J^\mu(\Psi, \Psi^*)$ have cluster properties better than those encountered in systems whose large-scale physics is dominated by Goldstone bosons; see Subsects. 4.5 and 6.1. From incompressibility it follows, as we will review in Subsect. 6.1, that S^* is *local*, and, for systems with translation, rotation and parity invariance, it is given by

$$S^*(A) \;=\; \frac{1}{2}\int d^{d+1}x \left[g_{\|}\,(\mathrm{d}A)^{0i}(\mathrm{d}A)_{0i} + g_{\perp}(\mathrm{d}A)^{ij}(\mathrm{d}A)_{ij}\right](x)\ , \tag{5.27}$$

where $g_{\|}$, $g_{\perp}$ are constants.

(H) The *quantum Hall fluids* are parity-breaking, two-dimensional, incompressible electron systems. For such fluids (Laughlin, 1983a, 1983b) with translation and rotation invariance, it has been shown in Fröhlich and Kerler (1991), and in Fröhlich and Studer (1992b, 1992c, 1993b) – see also Sect. 6 below – that $S^*(A)$ is the abelian Chern-Simons action

$$S^*(A) \;=\; \frac{i\sigma_H}{4\pi}\int \mathrm{d}A\wedge A\ , \tag{5.28}$$

where σ_H is the dimensionless Hall conductivity.

(S) London theory and computations based on perturbation theory suggest that the effective action $S^*(A)$ of *BCS superconductors* is quadratic and is given by

$$\begin{aligned} S^*(A) \;=\; & \frac{1}{2}\int d^{d+1}x \left[\frac{1}{\lambda_L^2}\,(\vec{A}^T)^2(x)\right. \\ & \left. + \int d^{d+1}y\,(A_0-\partial_0\Delta_d^{-1}\partial_jA^j)(x)\Pi_s^*(x-y)(A_0-\partial_0\Delta_d^{-1}\partial_jA^j)(y)\right]\ , \end{aligned} \tag{5.29}$$

where λ_L is a constant (the London penetration depth),

$$A_i^T \;:=\; A_i - \partial_i\Delta_d^{-1}\partial_jA^j\ , \tag{5.30}$$

with $i,j=1,\ldots,d$, Δ_d denotes the d-dimensional Laplacian, and Π_s^* is the scaling limit of the scalar component of the vacuum polarization tensor in a superconductor. To leading order in $|\frac{\vec{k}}{k_0}|$ and $|\frac{k_0}{\vec{k}}|$, Π_s is given, in $d>1$, by

$$\Pi_s^*(k) \;=\; \frac{1}{\lambda_L^2}\,|\frac{\vec{k}}{k_0}|^2\,\Theta\Big(|\frac{k_0}{\vec{k}}|-1\Big)+\chi_s\,\Theta\Big(1-|\frac{k_0}{\vec{k}}|\Big)\ , \tag{5.31}$$

where χ_s is a constant depending on d and m; see Schrieffer (1964).

The scaling limits of the effective actions of systems **(F)**, **(I)**, **(H)**, and **(S)**, given by Eqs. (5.22) through (5.25), (5.27), (5.28), and (5.29) through (5.31), respectively, yield b-field (dual) actions of the form

$$\tilde{S}^*(\mathrm{d}b) \;=\; \frac{1}{8\pi^2}\,(*\mathrm{d}b,(\Pi^*)^{-1}{*\mathrm{d}b})\ , \tag{5.32}$$

where, in the notation of (5.23), Π^* is given by

$$\textbf{(F)} \qquad \text{Eqs. (5.24) and (5.25)} \; , \tag{5.33}$$

$$\textbf{(I)} \qquad \Pi_0^*(k) \;=\; g_{\|}\, \vec{k}^2 \; , \qquad \Pi_\perp^*(k) \;=\; g_\perp \vec{k}^2 + g_{\|}\, k_0^2 \; , \tag{5.34}$$

$$\textbf{(H)} \qquad \Pi^{*\mu\nu}(k) \;=\; \frac{i}{2\pi\, \sigma_H}\, \varepsilon^{\mu\nu\rho} k_\rho \; , \tag{5.35}$$

$$\textbf{(S)} \qquad \Pi_0^*(k) \;=\; \Pi_s^*(k) \; , \qquad \Pi_\perp^*(k) \;=\; \frac{1}{\lambda_L^2} \; . \tag{5.36}$$

For Π^* given by Eq. (5.23), one finds

$$\begin{aligned} \tilde{S}^*(\mathrm{d}b) \;=\; \frac{1}{8\pi^2} \int d^{d+1}x\, d^{d+1}y \Big[& (*\mathrm{d}b)^i(x)\, [(\Pi_\perp^*)^{-\frac{1}{2}} (\delta_{ij} - \partial_i \Delta_d^{-1} \partial_j)(\Pi_\perp^*)^{-\frac{1}{2}}](x,y)\, (*\mathrm{d}b)^j(y) \\ & + (*\mathrm{d}b)^0(x)\, (\Pi_0^*)^{-1}(x,y)\, (*\mathrm{d}b)^0(y) \Big] \; . \end{aligned} \tag{5.37}$$

In particular, in two space dimensions, b is a 1-form, and (5.37) simplifies to

$$\begin{aligned} \tilde{S}^*(\mathrm{d}b) \;&=\; \frac{1}{8\pi^2} \int d^{2+1}k \Big[(\Pi_\perp^*)^{-1}(k)\, \vec{k}^2\, (b_0 - k_0 \frac{\vec{k}\cdot\vec{b}}{\vec{k}^2})^2 \\ & \qquad + (\Pi_0^*)^{-1}(k)\, \vec{k}^2\, (\vec{b}^T)^2 \Big] \; . \end{aligned} \tag{5.38}$$

For incompressible Hall fluids, the dual action is given by

$$\tilde{S}^*(\mathrm{d}b) \;=\; -\frac{i}{4\pi\, \sigma_H} \int b \wedge \mathrm{d}b \; . \tag{5.39}$$

Duality in two-dimensional systems. Note that, in two space dimensions, A and b are 1-forms, and from formulae (5.22), (5.25), (5.27), (5.29), (5.32), (5.36), (5.38), and (5.39) it follows that the actions $S^*(A)$ and $\tilde{S}^*(\mathrm{d}b)$ are related by the remarkable "duality":

$$\begin{aligned} S^*|_I \;&\propto\; \tilde{S}^*|_S \; , \\ S^*|_S \;&\propto\; \tilde{S}^*|_I \; , \\ S^*|_H \;&\propto\; \tilde{S}^*|_H \; , \\ &\text{(etc.)} \end{aligned} \tag{5.40}$$

with $(g_\|, g_\perp)$ corresponding to $(\lambda_L^2, \chi_s^{-1})$, and σ_H to σ_H^{-1}.

In particular, since $S^*(A)|_I$ is the Maxwell action, it follows that $\tilde{S}^*(\mathrm{d}b)|_S$ describes a massless mode, the Goldstone boson of the superconducting state with broken $U(1)$-gauge symmetry, known as the Anderson-Bogoliubov mode. By Kramers-Wannier duality, this mode can also be described by an angular variable which is a free field.

6 Scaling Limit of the Effective Action of a Two - Dimensional, Incompressible Quantum Fluid

In this section, we study the *partition* - or *generating function* (at $T = 0$ and for *real* time) of a two-dimensional, non-relativistic quantum system confined to a space-time region $\Lambda = \mathbb{R} \times \Omega$ and coupled to external electromagnetic, "tidal", and geometric fields:

$$Z_\Lambda(a, w) := \int \mathcal{D}\psi^* \mathcal{D}\psi \, \exp\Big[\frac{i}{\hbar} S_\Lambda(\psi^*, \psi\,; a, w)\Big] \,, \tag{6.1}$$

where the gauge potentials a and w have been introduced in (3.26)–(3.31), and $S_\Lambda(\psi^*, \psi\,; a, w)$ is the action of the system given in (3.34) and (3.35); see also (3.55)–(3.58). The integration variables ψ^* and ψ are Grassmann variables (i.e., anticommuting c-numbers) for Fermi statistics, and complex c-number fields for Bose statistics.

We have not displayed the metric, g_{ij}, on the background space, M, explicitly, since it will be kept fixed, and usually $M = \mathbb{E}^2$ with $g_{ij} = \delta_{ij}$, for simplicity. We note, however, that for the study of the *stress tensor*, pressure - and density fluctuations, and curvature - and torsion effects, we would have to choose a variable external metric (or, at least, a variable conformal factor in g_{ij}). This is important in the study of density waves, in particular of surface density waves (which are interesting in two-dimensional quantum fluids), and of *critical phenomena* in the theory of phase transitions. We note, however, that curvature - and torsion effects can be studied by analyzing the dependence of $Z_\Lambda(a, w)$ on w which contains the affine spin connection, ${\omega^A}_B$; see (3.28) and (3.29), and the remarks at the end of Subsect. 3.1 and after (3.39).

Calculating the partition function (6.1) for an arbitrary (two-dimensional) non-relativistic quantum system is surely a major task. In the first part of this section, we show how the calculation can be carried out for *incompressible* systems, provided one passes to the *scaling limit*. Once we have an explicit expression for the partition function of a system, many of its physical properties can be derived. For incompressible systems, this is the topic of the rest of this section and of Sects. 7 and 8 where we review, in partucular, a classification of incompressible quantum Hall fluids. Since we are working in the scaling limit, we can only analyze *universal* properties of such systems, (i.e., properties independent of the small-scale structure of the system).

6.1 Scaling Limit of the Effective Action

In this subsection, we sketch how one calculates, for *incompressible* systems, the scaling limit of the effective action (see (6.5) below) associated with the partition function (6.1).

One of our main motivations for studying two-dimensional, incompressible quantum fluids comes from the phenomenology of the quantum Hall effect; see Subsect. 4.5. In discussing quantum Hall fluids, it is often assumed that the magnetic field transversal to the samples is so strong that the Zeeman energies are large enough for the systems to be totally spin-polarized. Moreover, spin-orbit interaction terms are expected to be negligible in quantum Hall fluids. One might therefore ask why, when studying quantum Hall fluids, one should worry about the dependence of the partition function $Z_\Lambda(a,w)$ on the $SU(2)$ connection w, thereby taking into account Zeeman and spin-orbit interaction effects?

To answer this question, we first argue that it is of principal, theoretical interest to know how to incorporate the *spin degrees of freedom* in a consistent way into the description of two-dimensional electronic systems.

Second, as first pointed out by Halperin (1983), in $GaAs$ (for example) the g-factor of the electron is $\frac{1}{4}$ of the value in empty space, and the effective mass, m, of the electron is about $\frac{7}{100}$ of the mass, m_o, in the vacuum. Thus, in $GaAs$, the Zeeman energies are only approximately $\frac{1}{60}$ of the cyclotron energy (i.e., the splitting between Landau levels). Furthermore, they are of the same magnitude as the quasi-particle energies of the fractional quantum Hall states in magnetic fields of the order of $10\,\mathrm{T}$. One expects therefore that, at some values of the filling factor ν, the ground state of the system will contain electrons with reversed spins. Experimental evidence that *spin-unpolarized* quantum Hall fluids exist has been given in the works cited in **(D5)** in Subsect. 4.5. We emphasize that unpolarized (or partially polarized) quantum Hall fluids can arise in two fundamentally different ways: either through the presence of two (or more) *independent*, but oppositely polarized bands, or through the formation of *spin-singlet* bands; see Subsect. VI.C in Fröhlich and Studer (1993b) and Fröhlich and Thiran (1994).

Third, we will show in Sects. 7 and 8 how one can infer, from the form of $Z_\Lambda(a,w)$, *universal* properties of quantum Hall fluids that will lead to a classification of such systems in terms of "universality classes".

Fourth, in Subsect. 6.2, we sketch how one can derive the linear response theory of quantum Hall fluids from $Z_\Lambda(a,w)$ describing, among other effects, a *quantum Hall effect for spin currents*. For a discussion of possible Hall systems where this effect might be tested experimentally, see Subsect. VII.A in Fröhlich and Studer (1993b).

We define the *electric charge* - and *current densities*, $j^0(x)$ and $\vec{j}(x)$, by

$$\begin{aligned} j^0(x) &:= \psi^*(x)\psi(x) \,, \\ j^k(x) &:= -\frac{i\hbar}{2mc}\, g^{kl}(x)\,[(\mathcal{D}_l\psi)^*(x)\psi(x) - \psi^*(x)(\mathcal{D}_l\psi)(x)] \,, \end{aligned} \tag{6.2}$$

and the *spin* - and *spin current densities*, $s_A^0(x)$ and $\vec{s}_A(x)$, by

$$\begin{aligned} s_A^0(x) &:= \psi^*(x)\, L_A^{(s)}\, \psi(x) \,, \\ s_A^k(x) &:= -\frac{i\hbar}{2mc}\, g^{kl}(x) \left[(\mathcal{D}_l\psi)^*(x)\, L_A^{(s)}\, \psi(x) - \psi^*(x)\, L_A^{(s)}\, (\mathcal{D}_l\psi)(x)\right] \,, \end{aligned} \tag{6.3}$$

where $(L_1^{(s)}, L_2^{(s)}, L_3^{(s)})$ are the three generators of the spin-s representation of $su(2)$; see (3.27). Similarly, one defines currents associated with internal symmetries. The electric current is conserved (i.e., the continuity equation holds; see (4.14)), but the spin current is, in general, *not* conserved, because it couples to a non-abelian gauge potential. It is, however, *covariantly conserved*; see (6.12) below.

It is straightforward to infer from (6.1), (3.34), (6.2) and (6.3) that the *connected, time-ordered current Green functions* of the system are given, *at non-coinciding arguments*, by

$$\Big\langle T\Big[\prod_{i=1}^{n} j^{\mu_i}(x_i)\prod_{j=1}^{m} s^{\nu_j}_{A_j}(y_j)\Big]\Big\rangle^{\text{con}}_{a,\,w} = i^{\,n+m}\prod_{i=1}^{n}\frac{\delta}{\delta a_{\mu_i}(x_i)}\prod_{j=1}^{m}\frac{\delta}{\delta w_{\nu_j A_j}(y_j)}\ \ln Z_\Lambda(a,w)\ , \tag{6.4}$$

where $\langle\,(\,\cdot\,)\,\rangle^{\text{con}}_{a,\,w}$ denotes the connected expectation functional of the system in an external gauge field configuration, (a,w), (with "ground state asymptotic conditions", as $t\to\pm\infty$, to be specified), and T indicates time-ordering. At *coinciding arguments*, Eq. (6.4) is modified by *Schwinger terms*, (but their precise form will not be of importance in our analysis, and therefore we do not display them).

As in Sect. 5, the *effective action* of the system is defined by

$$S^{\text{eff}}_\Lambda(a,w)\ :=\ \frac{\hbar}{i}\ \ln Z_\Lambda(a,w)\ . \tag{6.5}$$

The idea is to try to calculate the "leading terms" in $S^{\text{eff}}_\Lambda(a,w)$ which, via (6.4), will provide us with information on the current Green functions. By leading terms we mean those terms which dominate at *large-distance scales* and *low frequencies*. The calculation of the leading terms in $S^{\text{eff}}_\Lambda(a,w)$ might appear to be an intractable problem. Actually, making a single assumption on the excitation spectrum of the system, *incompressibility*, and using the $U(1)\times SU(2)$ *gauge-invariance* of non-relativistic quantum mechanics, they can be found explicitly.

Let χ be a real-valued function and R an $SU(2)$-valued function on space-time. Consider the gauge transformations in Eq. (3.72), i.e.,

$$a\ \mapsto\ {}^{\chi}a\ ,\qquad\text{with}\qquad {}^{\chi}a_\mu(x)\ :=\ a_\mu(x)+\partial_\mu\chi(x)\ , \tag{6.6}$$

and

$$\begin{aligned} w\ &\mapsto\ {}^{R}w\ ,\qquad\text{with}\\ {}^{R}w_\mu(x)\ &:=\ U^{(s)}(R(x))\,w_\mu(x)\,U^{(s)}(R(x))^* + U^{(s)}(R(x))\,\partial_\mu U^{(s)}(R(x))^*\ . \end{aligned} \tag{6.7}$$

Changing integration variables,

$$\psi(x) \;\mapsto\; {}^{\chi,\,R}\psi(x) \;:=\; e^{-i\chi(x)}\, U^{(s)}(R(x))\, \psi(x) \;, \tag{6.8}$$

in the functional integral (6.1), and using the gauge invariance of $S_\Lambda(\psi^*, \psi; a, w)$ under the transformations (6.6)–(6.8) and the fact that the Jacobian of (6.8) is unity, we find the *Ward identity*

$$S_\Lambda^{\rm eff}({}^{\chi}a, {}^{R}w) \;=\; S_\Lambda^{\rm eff}(a, w) \;, \tag{6.9}$$

for all χ and R. For a system of finitely many particles in a bounded region, Ω, of space, (6.9) can be proven rigorously (Fröhlich and Studer, 1992b). The identity is stable under passing to limits, for χ's and $\partial_\mu R$'s of compact support.

By differentiating (6.9) in χ or R and setting $\chi = 0$, $R = \mathbf{1}$, we find, using (6.4) (for $n + m = 1$), that

$$\frac{1}{\sqrt{g(x)}}\, \partial_\mu \Big(\sqrt{g(x)} \langle j^\mu(x) \rangle_{a,\,w} \Big) \;=\; 0 \;, \tag{6.10}$$

and

$$\frac{1}{\sqrt{g(x)}}\, \Big(\mathcal{D}_\mu \Big(\sqrt{g(x)}\, \langle s^\mu(x) \rangle_{a,\,w} \Big) \Big)_A \;=\; 0 \;, \quad A = 1, 2, 3 \;, \tag{6.11}$$

i.e.,

$$\frac{1}{\sqrt{g(x)}}\, \partial_\mu \Big(\sqrt{g(x)}\, \langle s_A^\mu(x) \rangle_{a,\,w} \Big) - 2\, \varepsilon_{ABC}\, w_{\mu B}(x)\, \langle s_C^\mu(x) \rangle_{a,\,w} \;=\; 0 \;, \tag{6.12}$$

for arbitrary a and w. These infinitesimal Ward identities play an important role in determining the general form of $S_\Lambda^{\rm eff}(a, w)$. They can be generalized, in an obvious way, to systems with internal symmetries.

We now proceed to determine the form of $S_\Lambda^{\rm eff}(a, w)$ in the *scaling limit*. We need to consider *ever larger* systems and *ever slower* variations in time. Let $1 \le \theta < \infty$ be a scale parameter as in Sect. 5. We set

$$\begin{aligned} g_{ij}(x) \;&:=\; g_{ij}^{(\theta)}(x) \;=\; \gamma_{ij}\Big(\frac{x}{\theta}\Big) \;, \quad \text{and} \\ \Lambda \;&:=\; \Lambda^{(\theta)} \;=\; \theta\, \Lambda_o \;, \end{aligned} \tag{6.13}$$

where $\gamma_{ij}(x)$ is a fixed metric on M (e.g., $\gamma_{ij}(x) = \delta_{ij}$), and $\Lambda_o \subset N = \mathbb{R} \times M$ is a fixed space-time cylinder;

$$x := (x^0 = ct, \vec{x}\,) = \theta\,(\xi^0, \vec{\xi}\,) = \theta\,\xi\ , \quad \xi \in \Lambda_o\ ; \tag{6.14}$$

hence

$$\frac{\partial}{\partial x^\mu} = \theta^{-1}\,\frac{\partial}{\partial \xi^\mu}\ . \tag{6.15}$$

We propose to study the reaction of the system to a small change in the external gauge potentials a and w. We choose *fixed background potentials*, a_c and w_c, defined on all of space-time N, and set

$$a_\mu^{(\theta)}(x) := a_{c,\mu}(x) + \theta^{-1}\,\tilde{a}_\mu\Big(\frac{x}{\theta}\Big)\ , \tag{6.16}$$

and

$$w_\mu^{(\theta)}(x) := w_{c,\mu}(x) + \theta^{-1}\,\tilde{w}_\mu\Big(\frac{x}{\theta}\Big)\ , \tag{6.17}$$

where $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)$ are *fixed* functions defined on Λ_o. If m is the effective mass of the particles and μ is the quantity determining their magnetic moment (see (2.4)) in physical $(t, \vec{x})$-coordinates, then the mass $m^{(\theta)}$ and the strength of the magnetic moment $\mu^{(\theta)}$ in *rescaled coordinates*, $(\tau = \frac{\xi^0}{c}, \vec{\xi}\,)$, are given by

$$m^{(\theta)} = \theta\,m\ , \quad \text{and} \quad \mu^{(\theta)} = \theta^{-1}\mu\ , \tag{6.18}$$

as follows from Eqs. (3.34), (3.37), and (3.38), (i.e., in the *rescaled* systems, the particles are heavy and have small magnetic moment. Moreover, the range of the two-body potential, in the *rescaled* systems, becomes shorter and shorter, as the scale parameter θ becomes large).

One *basic assumption* underlying our analysis is that $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ is *four times continuously (Fréchet) differentiable* in $\tilde{a}_\mu^{(\theta)}(x) := \theta^{-1}\tilde{a}_\mu\big(\frac{x}{\theta}\big)$ and $\tilde{w}_\mu^{(\theta)}(x) := \theta^{-1}\tilde{w}_\mu\big(\frac{x}{\theta}\big)$ at $\tilde{a}_\mu^{(\theta)}(x) = 0 = \tilde{w}_\mu^{(\theta)}(x)$, for a *suitable choice of background potentials*, $a_{c,\mu}(x)$ and $w_{c,\mu}(x) = \omega_{c,\mu}(x) + \rho_{c,\mu}(x)$, and for $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi) = \tilde{\omega}_\mu(\xi) + \tilde{\rho}_\mu(\xi)$ constrained to *suitable function spaces*, $\mathcal{A}$ and $\mathcal{W}$, *of perturbation potentials*, to be specified later. We may then (functionally) expand $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ to third order in $\tilde{a}^{(\theta)}(x)$ and $\tilde{w}^{(\theta)}(x)$, with a fourth order remainder term. Among the terms thus generated we shall retain only the *leading terms* in θ, namely those *scaling with a non-negative power of* θ which are commonly called *relevant* and *marginal* terms. The coefficients of these terms will be denoted by $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, a functional that we call the *scaling limit of the effective action*.

Using identity (6.4) and Eq. (6.5) to find the Taylor coefficients of $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$, plugging (6.16) and (6.17) into the resulting expression, and passing to $(\xi^0, \vec{\xi}\,)$-coordinates, we find that the coefficient of the term of n^{th} order in $\tilde{a}(\xi)$ and of m^{th} order in $\tilde{w}(\xi)$ in $S^{\mathrm{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ is given by a distribution

$$\varphi^{(\theta)\,\mu_1\ldots\mu_n,\,\nu_1\ldots\nu_m}_{\qquad\qquad A_1\ldots A_m}(a_c, w_c; \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \;=:\; \varphi^{(\theta)\,\underline{\mu},\underline{\nu}}_{\qquad\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta}) \tag{6.19}$$

which, at non-coinciding arguments, is given by

$$\frac{(-i)^{n+m+1}\,\hbar}{n!\,m!}\,\left\langle T\left[\prod_{i=1}^{n}\left(\theta^2\, j^{\mu_i}(\theta\xi_i)\right)\,\prod_{j=1}^{m}\left(\theta^2\, s^{\nu_j}_{A_j}(\theta\eta_j)\right)\right]\right\rangle^{\mathrm{con}}_{a_c,\,w_c}\,, \tag{6.20}$$

in accordance with the circumstance that, in $2+1$ space-time dimensions, the scaling dimension of currents is 2! [Note, e.g., that $\int_{t=\mathrm{const.}} j^0(ct, \vec{x}\,)\, d^2\vec{x}$ is a *dimensionless* number, a charge.]

We now formulate our *basic assumption of incompressibility*: We assume that, for *certain choices of the background potentials* a_c *and* w_c, the excitation spectrum of the system above its ground-state energy is such that (in the bulk) the *connected Green functions of its currents* have "*good*" *cluster properties* (better than in a metal or in a system with Goldstone bosons). More precisely, we assume that the distributions in (6.19) exhibit the following behaviour, for $\theta \to \infty$:

$$\begin{aligned}\varphi^{(\theta)\,\underline{\mu},\underline{\nu}}_{\qquad\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta}) \;&=\; \sum_{\Delta=0}^{N}\theta^{-D_\Delta}\,\varphi_{\Delta\;\underline{A}}^{\underline{\mu},\underline{\nu}}(a_c, w_c; \underline{\xi}, \underline{\eta}) \\ &+\;\; \mathrm{B.T.}^{\underline{\mu},\underline{\nu}}_{\;\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta}) \;+\; o\!\left(\theta^{-D_N}\right)\,,\end{aligned} \tag{6.21}$$

where $\varphi_{\Delta\;\underline{A}}^{\underline{\mu},\underline{\nu}}(a_c, w_c; \underline{\xi}, \underline{\eta})$ are *local distributions*, i.e., sums of products of derivatives of δ-functions, and the *scaling dimensions*, D_Δ, are given by

$$D_\Delta \;:=\; -2(n+m) + 3(n+m-1) + \Delta \;=\; n+m-3+\Delta\; \in \mathbb{Z}\,, \tag{6.22}$$

with Δ the number of derivatives present in the corresponding local distribution. The upper limit, N, in the sum on the r.h.s. of Eq. (6.21) is chosen such that $D_N \geq 0$. Finally, $\mathrm{B.T.}^{\underline{\mu},\underline{\nu}}_{\;\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta})$ are distributions (*not* necessarily local ones) that are completely localized on the space-time boundary $\partial\Lambda_o$ of the rescaled system in Λ_o. These terms will not be discussed in this section; they form the subject matter of Sect. 7. [For a different way of formulating the incompressibility of a system, see the discussion on the quantum Hall effect in Subsect. 4.5.]

Our incompressibility assumption is by no means a mild or minor assumption. It tends to be a hard analytical problem of many-body theory to show that, for a concrete system, it is satisfied. [For some recent ideas about how to establish it for quantum Hall fluids at certain filling factors, see the references given after **(D5)** in Subsect. 4.5.] What we propose to do here is to use it to calculate the general form of $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, the scaling limit of the effective action of the system, thereby elucidating the *universal* properties of two-dimensional, incompressible quantum fluids. We only sketch some ideas; for details, see Fröhlich and Studer (1992b), and Appendix A in Fröhlich and Studer (1993b).

The calculation is based on the following four principles:

(P1) *Incompressibility*: For all n and m, with $2 \leq n + m \leq 4$, the distributions $\varphi^{(\theta)\,\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta})$ "converge" (in the bulk), as $\theta \to \infty$, to local distributions, as specified in Eq. (6.21).

(P2) $U(1) \times SU(2)$ *gauge invariance*: Ward identities (6.9) to (6.12).

(P3) *Only relevant and marginal terms are kept in* $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$.

(P4) *Extra symmetries of the system*, e.g., for $w_{c,\mu A}(x) = \delta_{A3}\, w_{c,\mu 3}(x)$ (and hence, by (3.38), for $a_{c,\mu}(x)$ such that $E_{c,3}(x) = 0$), global rotations around the 3-axis in spin space are a continuous, global symmetry of the system with an associated *conserved* Noether current, $s^\mu_3(x)$; or translation invariance in the scaling limit $(\theta \to \infty)$, ..., *are exploited to reduce the number of terms.*

From **(P1)** and Eqs. (6.16) and (6.17) it immediately follows that all terms contributing to $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ of order 4 or higher in $\tilde{a}(\xi)$ and $\tilde{w}(\xi)$ are *irrelevant*, i.e., they scale like θ^{-D}, with $D > 0$. In particular, a fourth-order remainder term does *not* contribute to $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, (principle **(P3)**). We present the final result of our analysis in the special case of a system which is incompressible for a choice of w_c satisfying

$$w_{c,\mu A}(x) \;=\; \delta_{A3}\, w_{c,\mu 3}(x) \;, \tag{6.23}$$

or, in view of Eqs. (3.28)–(3.30), (3.37) and (3.38), for a background electromagnetic field $(\mathbf{E}_c(x), \mathbf{B}_c(x))$ with

$$\mathbf{E}_c(x) \;=\; (E_{c,1}(x), E_{c,2}(x), 0) \;, \quad \mathbf{B}_c(x) \;=\; (0, 0, B_c(x)) \;, \tag{6.24}$$

and possibly for some affine spin connection, ω, of the following form (see (3.29), (3.11), (3.12), and (3.31), as well as the remarks about the physical relevance of ω at the end of Subsect. 3.1 and after (3.39)):

$$\left(\omega^{A}_{\ B\mu}(x)\right) = \begin{pmatrix} 0 & \omega_\mu(x) & 0 \\ -\,\omega_\mu(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{6.25}$$

relative to some orthonormal frames $(e^1(x),\, e^2(x),\, e^3(x))$. [It is natural to work in an $SU(2)$ gauge which respects our convention of choosing $e^3(x)$ perpendicular to the cotangent plane $T^*_{\vec{x}}(M)$ at $\vec{x} \in M$, for all times t; see the discussion preceding (3.1). E.g., if M were the (x, y)-plane in $\mathbb{E}^3$, we would choose $e^3(x)$ to coincide with dz. Hence the choice of the electromagnetic *background* field specified by (6.24) corresponds to an electric field, $\mathbf{E}_c$, which is tangential to the sample and to a magnetic field, $\mathbf{B}_c$, which is perpendicular to it.] In this situation, the *scaling limit of the effective action* is given by

$$\begin{aligned} -\frac{1}{\hbar}\, S^*_{\Lambda_o}(\tilde{a}, \tilde{w}) \;\; = \;\; & \int_{\Lambda_o} j^\mu_c\, \tilde{a}_\mu\, dv \;+\; \int_{\Lambda_o} m^\mu_3\, \tilde{w}_{\mu 3}\, dv \\ & +\; \sum_{A=1}^{2} \int_{\Lambda_o} \tau^{\mu\nu}_1\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu A}\, dv \;+ \sum_{A,B=1}^{2} \int_{\Lambda_o} \tau^{\mu\nu}_2\, \varepsilon_{AB}\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu B}\, dv \\ & +\; \frac{k}{4\pi} \int_{\Lambda_o} \mathrm{tr}\Big(w \wedge \mathrm{d}w + \frac{2}{3}\, w \wedge w \wedge w \Big) \\ & +\; \frac{\sigma}{4\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{a} \;+\; \frac{\chi_s}{2\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{w}_3 \;+\; \frac{\sigma_s}{4\pi} \int_{\Lambda_o} \tilde{w}_3 \wedge \mathrm{d}\tilde{w}_3 \\ & +\; \sum_{A,B,C=1}^{3} \int_{\Lambda_o} \eta^{\mu\nu\rho}_{ABC}\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu B}\, \tilde{w}_{\rho C}\, dv \;+\; \mathrm{B.T.}(a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o}) , \end{aligned} \tag{6.26}$$

where $j^\mu_c(\xi)$ is an electric - and $m^\mu_3(\xi)$ a magnetic current circulating in the system when $\tilde{a}(\xi) = 0 = \tilde{w}(\xi)$; $\tau^{\mu\nu}_1(\xi)$ is a function symmetric in μ and ν, while $\tau^{\mu\nu}_2(\xi)$ is antisymmetric in μ and ν; the function $\eta^{\mu\nu\rho}_{ABC}(\xi)$ is symmetric under interchanges of (μA), (νB) and (ρC) and vanishes if two or more of the indices A, B, C are equal to 3; $dv := \sqrt{\gamma(\xi)}\, d^3\xi$, where $\gamma(\xi) := \det(\gamma_{ij}(\xi))$ is the volume element on the space-time cylinder Λ_o, see (6.13); σ, χ_s, σ_s, and k are real constants; $w(\xi)$ is the total $SU(2)$ connection given by

$$w(\xi) \;:=\; w^{(\theta)}_c(\xi) + \tilde{w}(\xi) \;, \quad \text{with} \quad w^{(\theta)}_c(\xi) \;:=\; \theta\, w_c(\theta\xi) \;, \tag{6.27}$$

see (6.17); and $\mathrm{B.T.}(a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o})$ denotes boundary terms only depending on the gauge potentials $a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o}$ restricted to the boundary, $\partial\Lambda_o$, of the space-time cylinder Λ_o. They will be discussed in Sect. 7. Moreover, on the r.h.s. of (6.26), we are using the notation:

$$
\begin{aligned}
\tilde{a} &= \sum_{\mu=0}^{2} \tilde{a}_\mu(\xi)\, d\xi^\mu \,, & \mathrm{d}\tilde{a} &= \sum_{\mu,\nu=0}^{2} (\partial_\mu \tilde{a}_\nu)(\xi)\, d\xi^\mu \wedge d\xi^\nu \,, \\
\tilde{w}_3 &= \sum_{\mu=0}^{2} \tilde{w}_{\mu 3}(\xi)\, d\xi^\mu \,, & \tilde{w} &= i \sum_{\mu=0}^{2} \sum_{A=1}^{3} \tilde{w}_{\mu A}(\xi)\, L_A^{(s)}\, d\xi^\mu \,,
\end{aligned} \tag{6.28}
$$

where $\partial_\mu = \frac{\partial}{\partial \xi^\mu}$ whenever we are working in rescaled ξ-coordinates. Finally, we note that the terms in (6.26) are ordered according to their scaling dimensions.

In Fröhlich and Studer (1993b), we have used results on chiral $\hat{u}(1)$ - and $\hat{su}(2)$ current algebras to determine the possible values of the constants σ, χ_s, σ_s, and k; see Sect. 7 below.

Here we wish to point out that the functions j_c^μ, m_3^μ, $\tau_1^{\mu\nu}$, $\tau_2^{\mu\nu}$, and $\eta_{ABC}^{\mu\nu\rho}$ are *not* all independent, but are constrained by the infinitesimal Ward identities (6.10) and (6.12): By (6.4) and (6.5)

$$
\langle j^\mu(\xi) \rangle_{a^{(\theta)},\, w^{(\theta)}} \;=\; -\frac{1}{\hbar}\, \frac{\delta S^*_{\Lambda_o}(\tilde{a}, \tilde{w})}{\delta \tilde{a}_\mu(\xi)} + \cdots \,, \tag{6.29}
$$

and

$$
\langle s_A^\mu(\xi) \rangle_{a^{(\theta)},\, w^{(\theta)}} \;=\; -\frac{1}{\hbar}\, \frac{\delta S^*_{\Lambda_o}(\tilde{a}, \tilde{w})}{\delta \tilde{w}_{\mu A}(\xi)} + \cdots \,. \tag{6.30}
$$

The dots stand for contributions from irrelevant terms in the effective action. We calculate the r.h.s. of these equations by using (6.26) and plug the result into Eqs. (6.10) and (6.12). As a result we obtain the following *constraints* (Fröhlich and Studer, 1992b):

(i)
$$
\frac{1}{\sqrt{\gamma(\xi)}}\, \partial_\mu (\sqrt{\gamma}\, j_c^\mu)(\xi) \;=\; 0 \,.
$$

(ii)
$$
\frac{1}{\sqrt{\gamma(\xi)}}\, \partial_\mu (\sqrt{\gamma}\, m_3^\mu)(\xi) \;=\; 0 \,.
$$

(iii)
$$
\begin{aligned}
&\sum_{B=1}^{2} \varepsilon_{AB} \left[m_3^\mu(\xi) - 2\, \tau_1^{0\mu}(\xi)\, w_{c,03}^{(\theta)}(\xi) \right] \tilde{w}_{\mu B}(\xi) \\
&\qquad + 2\, w_{c,03}^{(\theta)}(\xi) \sum_{j=1}^{2} \tau_2^{0j}(\xi)\, \tilde{w}_{jA}(\xi) \;=\; 0 \,, \qquad A = 1,\, 2 \,.
\end{aligned}
$$

$$
\textbf{(iv)} \qquad \frac{1}{\sqrt{\gamma(\xi)}}\,\partial_\mu\left[\sqrt{\gamma}\,\tau_1^{\mu\nu}\,\tilde{w}_{\nu A} + \sqrt{\gamma}\sum_{B=1}^{2}\varepsilon_{AB}\,\tau_2^{\mu\nu}\,\tilde{w}_{\nu B}\right](\xi) \;=\;
$$
$$
-\,2\left[\sum_{B=1}^{2}\varepsilon_{AB}\,\tau_1^{\mu\nu}(\xi)\,\tilde{w}_{\nu B}(\xi) - \tau_2^{\mu\nu}(\xi)\,\tilde{w}_{\nu A}(\xi)\right]\tilde{w}_{\mu 3}(\xi)
$$
$$
-\,3\sum_{B=1}^{2}\varepsilon_{AB}\sum_{C,D=1}^{3}\eta_{BCD}^{0\nu\rho}(\xi)\,w_{c,03}^{(\theta)}(\xi)\,\tilde{w}_{\nu C}(\xi)\,\tilde{w}_{\rho D}(\xi)\;, \qquad A = 1,\,2\;. \tag{6.31}
$$

Constraints **(i)** and **(ii)** just express the conservation of the currents j_c^μ and m_3^μ when $\tilde{a} = 0 = \tilde{w}$.

If we impose constraints **(iii)** and **(iv)**, for *arbitrary* smooth perturbation potentials $\tilde{w}$, then it follows that

$$
m_3^\mu(\xi) \;=\; \tau_1^{\mu\nu}(\xi) \;=\; \tau_2^{\mu\nu}(\xi) \;=\; 0\;, \qquad \text{for all } \mu,\,\nu = 0,1,2\;; \tag{6.32}
$$

in particular, the system *cannot* be *magnetized* $(m_3^0 = 0)$ and cannot support persistent spin currents. This may seem rather strange, because we would expect that if $\rho_{c,03} = -\frac{g\mu}{2\hbar c}B_c$ (see (3.37)), for some large background magnetic field $\mathbf{B}_c = (0,0,B_c)$, then the system would be magnetized in the 3-direction. What has gone wrong? The point is that the assumed properties, that $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)},\,w^{(\theta)})$ is four times continuously differentiable in $\tilde{a}$ and $\tilde{w}$ and that the system remains incompressible in an *arbitrary* function-space neighbourhood of $(a_c,\,w_c)$ of sufficiently small diameter, must fail for magnetized systems! The reason is that an *arbitrarily small* perturbation field $\tilde{w}$ which oscillates rapidly in time can destroy the incompressibility of the system, and hence our estimate on the fourth order remainder term in the Taylor expansion of $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)},\,w^{(\theta)})$ breaks down!

We thus assume, for example, that, for a *time-independent* background field w_c satisfying (6.23), the system remains incompressible and $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)},\,w^{(\theta)})$ is four times continuously differentiable in $(\tilde{a},\,\tilde{w})$, provided $(\tilde{a}\,,\,\tilde{w})$ belongs to a set of $\mathcal{A}\times\mathcal{W}$ given by

$$
\begin{aligned}
\mathcal{A} \;&:=\; \{\,\tilde{a}\;|\;\tilde{a}_\mu(\tau,\vec{\xi}\,) = \eta(\tau)\,f_\mu(\vec{\xi}\,)\;,\; f_\mu(\vec{\xi}\,)\in\mathcal{S}\,\}\;,\\
\mathcal{W} \;&:=\; \{\,\tilde{w}\;|\;\tilde{w}_{\mu A}(\tau,\vec{\xi}\,) = \eta(\tau)\,g_{\mu A}(\vec{\xi}\,)\;,\; g_{\mu A}(\vec{\xi}\,)\in\mathcal{S}\,\}\;,
\end{aligned} \tag{6.33}
$$

where $\eta(\tau)$ describes an adiabatic process of turning on and off the perturbations: $\eta(\tau) = 1$, for $\tau\in[\tau_1,\tau_2]$, some finite interval in (rescaled) time, and $\eta(\tau) = 0$, for $\tau \ll \tau_1$ or $\tau \gg \tau_2$, while smoothly interpolating in between; and $\mathcal{S}$ is some Schwartz space neighbourhood of 0. Then constraints **(iii)** and **(iv)** imply that

$$\tau_1^{00}(\xi) \;=\; \frac{m_3^0(\xi)}{2\,w_{c,03}^{(\theta)}(\xi)}\;, \quad \tau_1^{0i}(\xi) \;=\; \tau_1^{ij}(\xi) \;=\; 0\;, \quad \tau_2^{\mu\nu}(\xi) \;=\; 0\;, \tag{6.34}$$

and

$$\eta_{AA3}^{000}(\xi) \;=\; -\frac{\tau_1^{00}(\xi)}{3\,w_{c,03}^{(\theta)}(\xi)}\;, \quad \text{for } A=1,2\;, \quad \text{all other } \eta_{ABC}^{\mu\nu\rho}(\xi) \text{ vanish.} \tag{6.35}$$

Hence, $(m_3^\mu) = (m_3^0, \mathbf{0})$. Under somewhat more restrictive assumptions on $\mathcal{W}$, imposing, for example, relations of the form (3.37) and (3.38) on $\tilde{w} = \tilde{\omega} + \tilde{\rho}$ which couple $\tilde{w}$ and $\tilde{a}$, a non-zero spin current $\mathbf{m}_3 = (m_3^1, m_3^2)$ is possible, too. For a more detailed discussion, see Fröhlich and Studer (1992b).

A corollary of our derivation of $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, using gauge invariance and incompressibility, is the *Goldstone theorem* (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962): Recalling that $w_{c,03} = -\frac{g\mu}{2\hbar c}\, B_c$, if $\omega_0 = 0$ (see (3.37) and (6.25)), and denoting by $\mathcal{M} := \frac{g\mu}{2}\, m_3^0$ the *magnetization* in the background field B_c (see (6.52) below), one finds, by (6.34), that, for an incompressible system, the following identity must hold:

$$\mathcal{M}(\xi) \;=\; -\frac{2\hbar c}{g^2\mu^2}\, \tau_1^{00}(\xi)\, B_c(\xi)\;.$$

Hence, with $|\tau_1^{00}| < \infty$, one finds that, if $\mathcal{M}$ does *not* tend to 0, as the background magnetic field, B_c, tends to 0, then the system *cannot* be incompressible at $B_c = 0$, i.e., there are gapless extended modes, the *Goldstone bosons*, coupled to the ground state by the spin current (Fröhlich and Studer, 1992b). We note that our proof also works for systems with continuous non-abelian internal symmetries. An analysis of the form of the effective action and of the spectrum of low-energy modes in systems with broken continuous symmetries and Goldstone bosons has been presented in Leutwyler (1993), using ideas very similar to those described in this section.

Remark. In Eq. (6.26) for the scaling limit of the effective action $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ of a two-dimensional, incompressible quantum fluid we have, as explained above, retained only *relevant* and *marginal terms*, i.e., terms scaling as θ^{-D}, for $\theta \to \infty$, with $D \leq -1$ and $D = 0$, respectively. Although we are mainly interested in the physics corresponding to the relevant and marginal terms in the effective action $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ we display, here, the most important *subleading order terms*. These are the unique terms that are of second order in the perturbation potentials $\tilde{a}$ and $\tilde{w}$ and of scaling dimension $D = 1$, the so-called "Maxwell terms". Added to the r.h.s. of Eq. (6.26) they take the form

$$- \frac{1}{4} \Big[\sum_{i=1}^{2} g_{\|}^{(i)} \int_{\Lambda_o} \tilde{f}_{0i}^{\,2}\, dv + g_{\perp} \int_{\Lambda_o} \tilde{f}_{12}^{\,2}\, dv$$

$$+ \sum_{i=1}^{2} l_{\|}^{(i)} \int_{\Lambda_o} \mathrm{tr}\, [\, h_{0i}^{2}\,]\, dv + l_{\perp} \int_{\Lambda_o} \mathrm{tr}\, [\, h_{12}^{2}\,]\, dv \Big] \,, \tag{6.36}$$

where $\tilde{f}_{\mu\nu} := \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$ is the $U(1)$ curvature (or field strength), and likewise $h_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu + [w_\mu,\, w_\nu]$, where w is given by (6.27), is the $SU(2)$ curvature. Moreover, $g_{\|}^{(i)}$, $g_\perp$, $l_{\|}^{(i)}$ and $l_\perp$ are constants of dimension of a length. Rotation invariance in the scaling limit would imply that $g_{\|}^{(1)} = g_{\|}^{(2)} =: g_{\|}$ and $l_{\|}^{(1)} = l_{\|}^{(2)} =: l_{\|}$. A discussion of the consequences of the $U(1)$ curvature terms can be found in Fröhlich and Studer (1992b). For an application of the $SU(2)$ curvature terms to a spin-pairing mechanism, see the end of Subsect. 6.3.

6.2 Linear Response Theory and Current Sum Rules

Next, we discuss the *linear response equations* (6.29) and (6.30) that follow from our (universal) expression (6.26) for the scaling limit $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ of the effective action of systems characterized by the conditions (6.23) and (6.33)–(6.35). It is a simple exercise to verify that

$$\begin{aligned} \sqrt{\gamma(\xi)}\, \langle j^\mu(\xi) \rangle_{a,\, w} \;&=\; \sqrt{\gamma(\xi)}\, j_c^\mu(\xi) + \frac{\sigma}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{a}_\rho)(\xi) \\ &\quad + \frac{\chi_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{w}_{\rho 3})(\xi) + \cdots \,, \end{aligned} \tag{6.37}$$

and

$$\begin{aligned} \sqrt{\gamma(\xi)}\, \langle s_A^\mu(\xi) \rangle_{a,\, w} \;&=\; \sqrt{\gamma(\xi)}\, \delta_{A3}\, \delta_0^\mu\, m_3^0(\xi) + \delta_{A3}\, \frac{\chi_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{a}_\rho)(\xi) \\ &\quad + \delta_{A3}\, \frac{\sigma_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{w}_{\rho 3})(\xi) \\ &\quad - \frac{k}{\pi}\, \varepsilon^{\mu\nu\rho} \Big[(\partial_\nu w_{\rho A})(\xi) - \varepsilon_{ABC}\, w_{\nu B}(\xi) w_{\rho C}(\xi) \Big] \\ &\quad + \sqrt{\gamma(\xi)}\, 2(1 - \delta_{A3})\, \delta_0^\mu\, \tau_1^{00}(\xi)\, \tilde{w}_{0A}(\xi) + \cdots \,, \end{aligned} \tag{6.38}$$

where the dots stand for terms coming from irrelevant terms in the effective action, or from terms of second order in $\tilde{w}$ (e.g. a term proportional to η^{000}_{ABC}) which are of little interest in linear response theory. Furthermore, we recall that $w_{\mu A} = w^{(\theta)}_{c,\mu A} + \tilde{w}_{\mu A}$; see (6.27).

In order to understand the physical contents of these equtions, we should recall the physical meaning of the connections a and w elucidated in Subsect. 3.2: From Eqs. (3.36), (3.55), and (3.59) we know that

$$a_j(x) \;=\; -\frac{q}{\hbar c} A_j(x) - \frac{m}{\hbar} f_j(x) \;, \tag{6.39}$$

where $\mathbf{A}$ is the electromagnetic vector potential, q is the charge and m the effective mass of the particles in the quantum fluid, (for electrons, we have $q = -e$), and $\mathbf{f}$ is a divergence free velocity field generating some incompressible superfluid flow. Furthermore, by (3.36),

$$a_0(x) \;=\; \frac{q}{\hbar c} \Phi(x) \;, \tag{6.40}$$

where Φ is the electrostatic potential.

In our study of two-dimensional, incompressible quantum fluids on a surface M embedded in $\mathbb{E}^3$, it is natural to choose an $SU(2)$ gauge with the property that $e^3(t, \vec{x})$ is orthogonal to the cotangent space of M at $\vec{x}$, for all times t; see (6.23)–(6.25) and the discussion at the beginning of Subsect. 3.1. Then, by (6.25), a possible affine $SU(2)$ connection, $\omega_\mu^{(s)}$, has the form

$$\omega_\mu^{(s)}(x) \;=\; i\, \omega_\mu(x)\, L_3^{(s)} \;. \tag{6.41}$$

It then follows from (3.28)–(3.30), (3.37), (3.55), and (3.60) that

$$w_{0A}(x) \;=\; -\frac{g\mu}{2\hbar c} B_A(x) + \delta_{A3} \left[\Omega(x) + \omega_0(x) \right] \;, \tag{6.42}$$

where, by (3.44), $\mathbf{\Omega}(x) = (0, 0, \Omega(x)) = \frac{1}{2}\, curl\, \mathbf{f}(x)$ with respect to the orthonormal frame $(e^A(x))_{A=1}^3$ at x, and the magnetic moment of the particles is given by $g\mu\vec{L}^{(s)}$, (for electrons, we have $\mu = -\mu_B$; see (2.4)). Moreover, by (3.28)–(3.30), (3.38), (3.50), and (6.41),

$$w_{jA}(x) \;=\; \left(-\frac{g\mu}{2\hbar c} + \frac{q}{4mc^2} \right) \sum_{B=1}^{3} \varepsilon_{kAB}(x)\, E_B(x) + \delta_{A3} \left[\omega_j(x) + \cdots \right] \;, \tag{6.43}$$

where the dots correspond to terms proportional to derivatives of $\Omega(t', \vec{x})$, $t' \le t$, (and are generated by the $SU(2)$ gauge transformation $U^{(s)}(R)$ with R defined in (3.45)).

Finally, we define, *in physical units*, the *charge density (operator)* by

$$\rho(\xi) \;:=\; q\, \sqrt{\gamma(\xi)}\, j^0(\xi) \;, \tag{6.44}$$

the *electric current density* by

$$\mathcal{J}^i(\xi) \;:=\; qc\, \sqrt{\gamma(\xi)}\, j^i(\xi) \;, \tag{6.45}$$

the *spin density* by

$$S^{\,o}_{A}\,(\xi) \;:=\; \frac{\hbar}{2}\,\sqrt{\gamma(\xi)}\,s^{0}_{A}(\xi)\ , \tag{6.46}$$

and the *spin current density* by

$$S^{\,i}_{A}\,(\xi) \;:=\; \frac{\hbar c}{2}\,\sqrt{\gamma(\xi)}\,s^{i}_{A}(\xi)\ . \tag{6.47}$$

Then, for the $(\mu = 0)$-component, Eq. (6.37) reads

$$\begin{aligned} \langle\rho(\xi)\rangle_{a,\,w} \;&=\; \rho_c(\xi) - \frac{\sigma_{H}}{c}\,\tilde{B}_3(\xi) - \sigma\,\frac{2qm}{h}\,\tilde{\Omega}(\xi) \\ &\quad - \chi_s\left[\,\frac{qg\mu}{4hc}\,\vec{\nabla}\cdot\vec{\tilde{E}}(\xi) - \frac{q}{2\pi}\,\mathcal{R}(\xi)\right] + \cdots\ , \end{aligned} \tag{6.48}$$

where the *Hall conductivity (for the electric current)*, σ_{H}, is defined by

$$\sigma_{H} \;:=\; \frac{q^2}{h}\,\sigma\ , \tag{6.49}$$

$\vec{\tilde{E}}(\xi) := (\tilde{E}_1(\xi), \tilde{E}_2(\xi))\,,\ \vec{\nabla} = (\partial_1, \partial_2) := (\frac{\partial}{\partial\xi^1}, \frac{\partial}{\partial\xi^2}),\ \mathcal{R}(\xi) := curl\ \vec{\omega}(\xi)$ is the scalar curvature of M at ξ, and the dots stand for contributions from irrelevant terms. It will turn out that

$$\chi_{\perp} \;:=\; -\frac{qg\mu}{2hc}\,\chi_s \tag{6.50}$$

is the *magnetic susceptibility* of the system in the 3-direction normal to the surface. In Eq. (6.48) and the following formulas the tildes $\tilde{}$ indicate contributions from the perturbation potentials $\tilde{a}$ and $\tilde{w}$; (we have absorbed the affine spin connection ω into $\tilde{w}$, but without decorating it with a $\tilde{}$). Next, one verifies that

$$\begin{aligned} \langle\mathcal{J}^i(\xi)\rangle_{a,\,w} \;&=\; \mathcal{J}^i_c(\xi) - \sigma_{H}\,\varepsilon^{ij}\tilde{E}_j(\xi) + \sigma\,\frac{qm}{h}\,\varepsilon^{ij}\frac{\partial}{\partial\tau}\tilde{f}_j(\xi) \\ &\quad - \chi_s\left[\,\frac{qg\mu}{2h}\,\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - \frac{qc}{2\pi}\,\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi)\right] \\ &\quad + \chi_s\left[\,\frac{qg\mu}{4hc}\,\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) - \frac{q}{2\pi}\,\varepsilon^{ij}\frac{\partial}{\partial\tau}\lambda_j(\xi)\right] + \cdots\ , \end{aligned} \tag{6.51}$$

where $\tau := \xi^0/c$ is the rescaled time variable.

From Eq. (6.38) we find, for example, that for $\mu = 0$ and $A = 3$ (i.e., for the spin density along the 3-direction in the spin - or (co)tangent space)

$$\begin{aligned} \frac{g\mu}{\hbar} \langle S_3^{\,0}\,(\xi)\rangle_{a,\,w} &= \mathcal{M}_c(\xi) + \sigma_H^{\mathrm{spin}} \left[\frac{g\mu}{2\hbar c} \vec{\nabla} \cdot \vec{\tilde{E}}(\xi) - 2\,\mathcal{R}(\xi) \right] + k\,\frac{g^2\mu^2}{4hc} \vec{\nabla} \cdot \vec{\tilde{E}}_c(\xi) \\ &\quad + \chi_\perp \left[\tilde{B}_3(\xi) - \frac{hc}{g\mu\,\pi} \tilde{\Omega}(\xi) \right] + \cdots\,, \end{aligned} \tag{6.52}$$

where $\mathcal{M}_c$ is the *magnetization* of the system in the background field (a_c, w_c) given by (6.23)–(6.25), $\chi_\perp$ is the magnetic susceptibility at (a_c, w_c) defined in (6.50), and

$$\sigma_H^{\mathrm{spin}} := \frac{g\mu}{4\pi}\,k - \frac{g\mu}{8\pi}\,\sigma_s \tag{6.53}$$

is the *Hall conductivity for the spin current*. As Eqs. (6.52) and (6.26) show, σ_H^{spin} is a *pseudoscalar*. Note that (when $\tilde{f}_j = \frac{\partial}{\partial\tau}\,\tilde{E}_j = \frac{\partial}{\partial\tau}\,\lambda_j = 0$) equations (6.51), (6.52) and (6.50) imply the *Hall law* (4.13)! Next, for $\mu = i = 1,\,2$ and $A = 3$ (i.e., for the spin current density in the i-direction in the surface M and polarized along the 3-direction in the spin - or (co)tangent space)

$$\begin{aligned} \langle S_3^{\,i}\,(\xi)\rangle_{a,\,w} &= \sigma_H^{\mathrm{spin}} \left[\varepsilon^{ij}\partial_j \tilde{B}_3(\xi) - \frac{hc}{g\mu\,\pi}\,\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi) - \frac{1}{2c}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) + \frac{h}{g\mu\,\pi}\frac{\partial}{\partial\tau}\lambda^i(\xi) \right] \\ &\quad + k\,\frac{g\mu}{4\pi}\,\varepsilon^{ij}\partial_j B_c(\xi) + \chi_\perp \left[\frac{\hbar c}{g\mu}\,\varepsilon^{ij}\tilde{E}_j(\xi) - \frac{m\hbar c}{qg\mu}\,\frac{\partial}{\partial\tau}\tilde{f}^i(\xi) \right] + \cdots\,, \end{aligned} \tag{6.54}$$

where the dots stand for terms proportional to $\omega_0(\xi)$ and further irrelevant and higher-order terms. Similar equations hold for the remaining $su(2)$ components of $\langle S_A^{\,\mu}\rangle_{a,\,w}$, but we refrain from displaying them explicitly and refer the reader to the discussion in Fröhlich and Studer (1992b).

We encourage the reader to notice how neatly our formulas summarize the laws of the *Hall effect*, including effects due to *tidal forces* coming from *(superfluid) flow* and effects due to *curvature*. [We believe that the tidal terms might be relevant in the study of transitions between certain plateaux of σ_H in very pure samples.]

Let us comment on the relation of our definition of the *Hall conductivity* $\sigma_H = \frac{e^2}{h}\sigma$ as the coefficint of a Chern-Simons term, $\frac{\sigma}{4\pi}\int_{\Lambda_o} \tilde{a} \wedge d\tilde{a}$ in the effective action $S^*_{\Lambda_o}(\tilde{a},\,\tilde{w})$, see (6.26), of an incompressible quantum Hall fluid to the more conventional definition via the *Kubo formula* (see, e.g., Fradkin, 1991). It follows easily from Eqs. (6.4), (6.5), and (6.26) that σ appears in the following *current sum rules*: For every choice of a permutation $(\mu\,\nu\,\rho)$ of $(0\,1\,2)$,

$$\sigma \;=\; 2\pi i \,\mathrm{sign}(\mu\,\nu\,\rho) \int (y-x)^{\mu}\, \langle\, T[\, j^{\nu}(x)\, j^{\rho}(y)]\, \rangle^{\mathrm{con}}_{a,\,w}\; d^3y \;. \tag{6.55}$$

These are *three* equations for one and the same quantity σ. The equation for $(\mu\,\nu\,\rho) = (0\,1\,2)$ is

$$\sigma \;=\; 2\pi i \int (s-t)\, \langle\, T[\, j^{1}(t,\vec{x}\,)\, j^{2}(s,\vec{y}\,)]\, \rangle^{\mathrm{con}}_{a,\,w}\; ds\, d^2\vec{y} \;, \tag{6.56}$$

which is just the Kubo formula (in "mathematical" units); compare, e.g., to Fradkin (1991). The other two equations are an automatic consequence of $U(1)$ gauge invariance.

Thouless and coworkers (Thouless *et al.*, 1982; Niu and Thouless, 1984, 1987; Kohmoto, 1985), and followers (Avron, Seiler, and Simon, 1983; Avron and Seiler, 1985; Dana, Avron, and Zak, 1985; Avron, Seiler, and Yaffe, 1987; Kunz, 1987), have derived from the Kubo formula that

$$\sigma \;=\; \frac{1}{n_o}\, c_1 \;, \tag{6.57}$$

where n_o is the ground-state degeneracy and c_1 is the *first Chern number* of a vector bundle over a two-torus of magnetic fluxes (Φ_1, Φ_2). Thus, c_1 is an integer. Does our formulation "know" that n_o is the degeneracy of the ground state? Yes it does! This follows, e.g., from the material in Sects. 7 and 8, and has been noted in Wen (1989, 1990a), and Wen and Niu (1990).

Bellissard (1988a, 1988b) and Avron, Seiler, and Simon (1990, 1994) have also given a definition of σ as an *index*. Their definition is equivalent to ours, too, and the proof follows from the material in Sects. 7 and 8; see also Sect. 6 in Fröhlich and Kerler (1991).

Finally, we note that from Eqs. (6.4), (6.5), and (6.26) it also follows that $\sigma_H^{\mathrm{spin}} = \frac{g\mu_B}{8\pi}\sigma_s$ (for $k=0$, i.e., fully spin-polarized quantum Hall fluids) is given by a Kubo formula involving spin currents,

$$\sigma_s \;=\; 2\pi i \,\mathrm{sign}(\mu\,\nu\,\rho) \int (y-x)^{\mu}\, \langle\, T[s_3^{\nu}(x)\, s_3^{\rho}(y)]\, \rangle^{\mathrm{con}}_{a,\,w}\; d^3y \;, \tag{6.58}$$

and it can then be shown to be proportional to a first Chern number of a vector bundle over a two-dimensional torus of electric charges per unit length (Q_1, Q_2). We refer the reader to Fröhlich and Studer (1992b) for a more systematic study of current sum rules and "proofs". Here we merely give a last example by expressing the *magnetic susceptibility* $\chi_\perp = -\frac{qg\mu}{2hc}\chi_s$ in the form of a (mixed) sum rule,

$$\chi_s \;=\; 2\pi i \,\mathrm{sign}(\mu\,\nu\,\rho) \int (y-x)^{\mu}\, \langle\, T[\, j^{\nu}(x)\, s_3^{\rho}(y)]\, \rangle^{\mathrm{con}}_{a,\,w}\; d^3y \;. \tag{6.59}$$

6.3 Quasi-Particle Excitations and a Spin-Singlet Electron Pairing Mechanism

In this subsection, we present a first analysis of quasi-particle excitations above the ground state in a two-dimensional, incompressible quantum fluid, whose scaling limit of the effective action is given by the action $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ presented in Eq. (6.26). A systematic, general analysis of quasi-particle excitations in quantum Hall fluids is given in Sect. 8.

At the end of this subsection, we describe a natural mechanism for spin-singlet pairing of electrons that are moving in some two- or three-dimensional magnetic background.

Laughlin Vortices and Fractional Statistics

For simplicity, we begin our analysis by considering a flat, two-dimensional system of charged fermions with vanishing magnetic moment $(\mu = 0)$, so that the $SU(2)$ connection w vanishes identically in an appropriate $SU(2)$ gauge; (the local frames $(e^1(x), e^2(x), e^3(x))$ are chosen to be time-independent, so that there is no tidal Zeeman term; see Sect. 3). We suppose that, in a small neighbourhood of a suitably chosen background potential a_c (typically $a_{c,0} = 0$, $b_c = \partial_1 a_{c,2} - \partial_2 a_{c,1} = \text{const.}$ and of suitable magnitude), the system is incompressible. Then the action in the scaling limit is given by

$$-\frac{1}{\hbar}\, S^*_{\Lambda_o}(\tilde{a}) \;=\; \int_{\Lambda_o} j^\mu_c\, \tilde{a}_\mu \, dv + \frac{\sigma}{4\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{a} \;, \tag{6.60}$$

up to boundary terms. The first term on the r.h.s. is unimportant in the following discussion, and we set $j^\mu_c = 0$.

Let us produce a "*Laughlin vortex*" (Laughlin, 1983a, 1983b, 1990) in this system by turning on a (perturbing) magnetic field $\tilde{b} = \partial_1 \tilde{a}_2 - \partial_2 \tilde{a}_1$ in a small disc. [Actually, $\tilde{b}$ could be a vorticity field of a superfluid flow if, instead of an electronic quantum Hall fluid, we consider a superfluid film. We shall, however, use "magnetic language" in the following discussion.] From our discussion of the Aharonov-Bohm effect in Subsect. 2.2 we know that this excitation only disturbs the system locally, and thus may have a finite energy difference from the ground-state energy, if

$$\frac{1}{2\pi} \int \tilde{b}(t, \vec{\xi}\,)\, d^2\vec{\xi} \;=\; n \;, \qquad \text{with} \qquad n \in \mathbb{Z} \;. \tag{6.61}$$

By Eq. (6.48) [see also Eq. (4.17)], we have that

$$\langle j^0(\xi) \rangle_{a,\, w} \;=\; \frac{\sigma}{2\pi}\, \tilde{b}(\xi) \;, \tag{6.62}$$

and hence the charge of this excitation (in units where $e = 1$ and with the background charge normalized to 0) is given by

$$Q = \int \langle j^0(t, \vec{\xi}) \rangle_{a,w} \, d^2\vec{\xi} = \sigma n \; . \tag{6.63}$$

If σ is not an integer then Q will be *fractional*, in general. Now consider two such excitations localized in two disjoint small disks and interchange them (adiabatically) along some path oriented anticlockwise. According to Subsect. 2.2, the Aharonov-Bohm phase picked up in this process is given by

$$e^{i\pi\theta} := e^{i\pi Q n} = e^{i\pi\sigma n^2} \; , \tag{6.64}$$

where we have normalized the *statistical phase* θ such that $\theta \equiv 1 \pmod 2$ corresponds to *Fermi statistics*, $\theta \equiv 0 \pmod 2$ corresponds to *bosons*, and $\theta \not\equiv 0, 1 \pmod 2$ to *anyons* (Leinaas and Myrheim, 1977; Goldin, Menikoff, and Sharp, 1980, 1981, 1983; Wilczek, 1982a, 1982b; for a review, see Fröhlich, 1990). Thus Laughlin vortices are anyons, unless σn^2 is an integer.

Among the excitations that one can produce in this fashion there should be the particles constituting the system, namely electrons (or holes). Let us suppose that the state of the system is fully spin-polarized, (as is the case for filling factors $\nu \overset{\text{e.g.}}{=} \frac{1}{3}, \frac{1}{5}$ in electronic quantum Hall fluids). Supposing that a magnetic flux n_o (in units where the elementary flux quantum $\frac{hc}{e} = 1$) produces a state of N electrons, we infer, from (6.63), that

$$\sigma = \frac{N}{n_o} \; . \tag{6.65}$$

If N is *odd* this state is composed of N fermions and hence describes an excitation with Fermi statistics, so that by (6.64)

$$e^{i\pi N n_o} = -1 \; . \tag{6.66}$$

Thus n_o must be *odd*, too. In fact, one may show that if N and n_o have no common divisor then n_o is odd. In particular, for $N = 1$, we conclude that

$$\sigma = \frac{1}{n_o} \; , \quad \text{with } n_o \text{ odd} \; . \tag{6.67}$$

This is the famous *odd-denominator rule* (see, e.g., Tao and Wu, 1985). An excitation associated to a magnetic flux (or vorticity) 1 then has fractional charge $Q = \frac{1}{n_o}$ and is an anyon, for $n_o > 1$.

Note that the vector potential $\tilde{a}$ created by a point-like excitation of charge Q located at $\vec{\xi} = \vec{0}$ is given by

$$\tilde{a}_i(t,\vec{\xi}\,) \;=\; -\frac{Q}{\sigma}\sum_{j=1}^{2}\varepsilon_{ij}\,\frac{\xi^j}{\big|\vec{\xi}\,\big|^2}\;, \qquad i=1,\,2\;, \tag{6.68}$$

as follows from (6.61) and (6.63), for $\langle j^0(t,\vec{\xi}\,)\rangle_{a,\,w} = Q\,\delta(\vec{\xi}\,)$. This is the "$U(1)$ Knizhnik-Zamolodchikov connection".

Spinon Quantum Mechanics

Next, we consider an "in vitro" system, namely a "*chiral spin liquid*". [It is not entirely clear that such systems exist in nature, but they might appear as subsystems of superfluid ^3He layers.] A chiral spin liquid is a system of neutral particles of spin $s_o > 0$ and with non-zero magnetic moment (i.e., $\mu \neq 0$) having a spin-singlet ground state for some constant magnetic field, $\mathbf{B}_c$. It is assumed, here, to be incompressible and to exhibit breaking of parity (reflections in lines) and time reversal, but no spontaneous magnetization. In our formalism, the scaling limit ot the effective action of such a system is given by

$$-\frac{1}{\hbar}\,S^*_{\Lambda_o}(w) \;=\; \frac{k}{4\pi}\int_{\Lambda_o}\mathrm{tr}\left(w\wedge \mathrm{d}w + \frac{2}{3}\,w\wedge w\wedge w\right)\;, \tag{6.69}$$

up to boundary terms. Under reflections in lines, w_i transforms as a vector, w_0 as a pseudoscalar, and k as a pseudoscalar. Let us consider an excitation created by turning on an $SU(2)$ gauge field w with curvature (or field strength) h given by

$$\begin{aligned} h(\xi) \;&:=\; \mathrm{d}w(\xi) + w(\xi)\wedge w(\xi) \\ &=\; \frac{i}{2}\sum_{\mu,\nu=0}^{2}\sum_{A=1}^{3} h_{\mu\nu A}(\xi)\,L_A^{(s_o)}\,d\xi^\mu\wedge d\xi^\nu\;, \end{aligned} \tag{6.70}$$

where $h_{\mu\nu A} = \partial_\mu w_{\nu A} - \partial_\nu w_{\mu A} - 2\,\varepsilon_{ABC}\,w_{\mu B}\,w_{\nu C}$; (see also the discussion following (6.36)). For example, we may choose h to be given by

$$\mathbf{h}_{0i}(\xi) \;:=\; (h_{0i\,1}(\xi), h_{0i\,2}(\xi), h_{0i\,3}(\xi)) \;=\; \mathbf{0}\;, \qquad \text{and} \qquad \mathbf{h}_{12}(\xi) \;=\; -\mathbf{n}\,h_o(\vec{\xi}\,)\;, \tag{6.71}$$

where $\mathbf{n}$ is some unit vector in $\mathbb{R}^3$ and h_o is a time-independent function. By Eq. (6.38), the spin density of this excitation is given by

$$\langle \mathbf{s}^0(\xi)\rangle_w \;:=\; \langle (s_1^0(\xi), s_2^0(\xi), s_3^0(\xi))\rangle_w \;=\; -\frac{k}{\pi}\,\mathbf{h}_{12}(\vec{\xi}\,) \;=\; \mathbf{n}\,\frac{k}{\pi}\,h_o(\vec{\xi}\,)\;, \tag{6.72}$$

so that the expectation value of its total spin operator, $\mathbf{S} := \frac{\hbar}{2}\mathbf{L}^{(s)}$ (in the spin-s representation; see (2.5)), is given by

$$\frac{\hbar}{2}\langle\mathbf{L}^{(s)}\rangle_w \;=\; \langle\mathbf{S}\rangle_w \;=\; \mathbf{n}\,\frac{k\hbar}{2\pi}\int h_o(\vec{\xi}\,)\,d^2\vec{\xi}\;. \tag{6.73}$$

Such an excitation is commonly called a "*spinon*". Quantum-mechanically, spin is quantized: $\mathbf{S}\cdot\mathbf{S} = s(s+1)\hbar^2$, with $s\in\frac{1}{2}\mathbb{Z}$. Consider a spinon of spin s *localized* at the point $\vec{\xi} = \vec{\xi}_1$. Then Eq. (6.72) says that $\mathbf{h}_{12}$ has to solve the equation

$$\langle\mathbf{L}^{(s)}\rangle_w\,\delta(\vec{\xi}-\vec{\xi}_1) \;=\; -\frac{k}{\pi}\,\mathbf{h}_{12}(\vec{\xi}\,)\;. \tag{6.74}$$

An $SU(2)$ connection $w = i\sum_\mu \mathbf{w}_\mu\cdot\mathbf{L}^{(s)}\,d\xi^\mu$ for the field strength h satisfying (6.74), with $\mathbf{h}_{0i}=0$, is given by

$$\mathbf{w}_0(\xi) \;=\; \mathbf{0}\;, \qquad \text{and} \qquad \mathbf{w}_j(\xi) \;=\; \frac{1}{2k}\,\langle\mathbf{L}^{(s)}\rangle_w\sum_{l=1}^{2}\varepsilon_{jl}\,\frac{\xi^l-\xi_1^l}{\left|\vec{\xi}-\vec{\xi}_1\right|^2}\;, \qquad j=1,2\;. \tag{6.75}$$

Suppose we now create a second spinon of spin s' moving in the background gauge field w excited by the first spinon. Its dynamics is coupled to w through the covariant derivatives (see Eqs. (3.26) and (3.27))

$$\mathcal{D}_\mu \;=\; \partial_\mu + i\,\mathbf{w}_\mu(\xi)\cdot\mathbf{L}^{(s')}\;, \tag{6.76}$$

with $\mathbf{w}$ as in (6.75). Let us imagine that it makes sense to do "two-spinon quantum mechanics" on a Hilbert space $\mathcal{H}^{(s)}\otimes\mathcal{H}^{(s')}$, with

$$\mathcal{H}^{(s)} \;=\; \mathcal{D}^{(s)}\otimes L^2(M,\,dv)\;,$$

where $\mathcal{D}^{(s)}$ carries the spin-s representation of $SU(2)$. By (6.75) and (6.76), the covariant derivatives on $\mathcal{H}^{(s)}\otimes\mathcal{H}^{(s')}$ are then given by

$$\mathcal{D}_0^1 \;=\; \frac{\partial}{\partial\xi_1^0}\;, \qquad \mathcal{D}_j^1 \;=\; \frac{\partial}{\partial\xi_1^j} + \frac{i}{2k}\sum_{l=1}^{2}\varepsilon_{jl}\,\frac{\xi_1^l-\xi_2^l}{\left|\vec{\xi}_1-\vec{\xi}_2\right|^2}\sum_{A=1}^{3}L_A^{(s)}\otimes L_A^{(s')}\;, \tag{6.77}$$

and

$$\mathcal{D}_0^2 \;=\; \frac{\partial}{\partial\xi_2^0}\;, \qquad \mathcal{D}_j^2 \;=\; \frac{\partial}{\partial\xi_2^j} + \frac{i}{2k}\sum_{l=1}^{2}\varepsilon_{jl}\,\frac{\xi_2^l-\xi_1^l}{\left|\vec{\xi}_1-\vec{\xi}_2\right|^2}\sum_{A=1}^{3}L_A^{(s)}\otimes L_A^{(s')}\;. \tag{6.78}$$

These are the covariant derivatives associated with the celebrated *Knizhnik-Zamolodchikov connection* (Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987). For the "two-spinon quantum mechanics" with parallel transport given by (6.78) to be consistent with unitarity, it is necessary that

$$k \;=\; \pm(\kappa+2)\;, \qquad \kappa = 1,\, 2,\, \dots\;. \tag{6.79}$$

This follows from results in Belavin, Polyakov, and Zamolodchikov (1984), Knizhnik and Zamolodchikov (1984), and Tsuchiya and Kanie (1987). Recalling what we have said in Subsect. 2.3 about the Aharonov-Casher effect, we observe that the "phase factor" arising in the parallel transport of a quantum-mechanical spinon in the field excited by a classical spinon with spin orthogonal to the plane of the system is an Aharonov-Casher phase factor.

Consider an exchange of the positions of two quantum-mechanical, point-like spinons along some anticlockwise oriented path. Then the "Aharonov-Casher phase factor" multiplying the wave function is given by a matrix

$$R^{(\kappa)}_{s\,s'} \;:\; \mathcal{D}^{(s)} \otimes \mathcal{D}^{(s')} \;\rightarrow\; \mathcal{D}^{(s')} \otimes \mathcal{D}^{(s)}\;,$$

which is the *braid matrix* for exchanging a chiral vertex of spin s with a chiral vertex of spin s' in the *chiral Wess-Zumino-Novikov-Witten model* (Belavin, Polyakov, and Zamolodchikov, 1984; Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987; see also Gawedzki, 1990) at level κ. It is given by

$$R^{(\kappa)}_{s\,s'} \;=\; T\,\pi_s \otimes \pi_{s'}\,(\mathcal{R}^{(\kappa)})\;, \tag{6.80}$$

where $\mathcal{R}^{(\kappa)}$ is the universal R-matrix of the quantum group $U_q(sl_2)$, with $q = \exp\left(\frac{i\pi}{\kappa+2}\right)$, and T is the flip (transposition of factors). All this can be extended to "n-spinon quantum mechanics". The matrices $(R^{(\kappa)}_{s\,s'})$ determine an exotic quantum statistics described by non-abelian (for $\kappa > 1$, and $s, s' < \frac{\kappa}{2}$) representations of the braid groups (more precisely, the groupoids of coloured braids) which is commonly called *non-abelian braid statistics* (Fröhlich (1988), Fredenhagen, Rehren, and Schroer, 1989; Fröhlich and Gabbiani, 1990). We note that s and s' are forced to be $\leq \frac{\kappa}{2}$, i.e., there are no spinons of spin $> \frac{\kappa}{2}$. One might call this phenomenon "*spin screening*". If the particles of spin s_o constituting the chiral spin liquid appear as spinon excitations above the ground state and s_o is half-integer then

$$\kappa \;\geq\; 2\,s_o\;.$$

One can argue that the statistics of these particles must be *abelian* braid statistics, i.e., they are anyons. In fact, it then follows that they are semions $(\theta = \frac{1}{2})$. Now, for a given level κ, the matrices $(R_{ss}^{(\kappa)})$ define an abelian representation of the braid groups if, and only if, $2s = \kappa$. Thus it follows that for a chiral spin liquid made of particles of spin s_o

$$\kappa = 2\, s_o \,. \tag{6.81}$$

Any spinon-excitation of spin $0 < s < s_o$ then exhibits *non-abelian* braid statistics!

The reader may feel that our "derivation" of "spinon quantum mechanics" from the effective action $S^*_{\Lambda_o}(w)$ given in (6.69) is based on idealizations – see (6.74) – and jumps in the logics – reasoning between (6.76) and (6.78) – that might be doubtful. Actually, it turns out that our conclusions concerning spinon statistics, in particular Eqs. (6.80) and (6.81), are perfectly correct. This follows from an analysis that we have presented in Sect. VI in Fröhlich and Studer (1993b).

In order to understand quantum Hall fluids with *spin-singlet* ground state, one must glue the Laughlin vortices described in (6.61)–(6.66) to the spinons discussed above. One checks that for $\sigma = \frac{2}{n_o}$, n_o odd, and $\kappa = 2s_o = 1$, a Laughlin vortex of vorticity $n = -\frac{n_o}{2}$ (!) glued to a spinon of spin $s = \frac{1}{2}$ is an excitation of charge $Q = -1$, spin $\frac{1}{2}$ and *Fermi statistics*; see Fröhlich and Kerler (1991), and Sect. VI in Fröhlich and Studer (1993b). These are the properties of an electron. In a quantum Hall fluid (*without* any *very exotic* internal symmetries) one does *not* find any finite-energy excitations with non-abelian braid statistics. However, if one could manufacture a quantum Hall fluid made of charge carriers of spin $s_o = \frac{3}{2}, \frac{5}{2}, \ldots$, with a spin-singlet ground state it would display excitations with non-abelian braid statistics (Fröhlich, Kerler, and Marchetti, 1992). It may appear difficult to build such a system, in practice.

It may be worthwhile emphasizing that in quantum Hall fluids with non-vanishing magnetic susceptibility (spin-polarized Hall fluids) the fractional statistics of Laughlin vortices always appears as a consequence of a *combination* of the Aharonov-Bohm - *and* the Aharonov-Chasher effect; (but notice that, for spin-polarized quantum Hall fluids, the Aharonov-Casher phase factors are automatically abelian). This is a consequence of the fact that electrons have a non-vanishing magnetic moment and follows form Eq. (6.26).

A Spin-Singlet Electron Pairing Mechanism

In this paragraph, we describe a mechanism for spin-singlet pairing of dopant electrons (or holes) moving in an antiferromagnetic - or a resonating valence-bond (RVB) background. Our mechanism may be related to the "spin-bag mechanism" (Schrieffer, Wen, and Zhang, 1988). For definiteness, we consider two-dimensional systems, but our arguments can easily be extended to systems in three dimensions.

The magnetic properties of the system are described by an order parameter, ϕ, transforming under the adjoint representation of $SU(2)_{spin}$. Integrating out the order parameter ϕ, at a fixed temperature $T > 0$, we obtain the free energy, $F_T(w) = -kT \ln Z_T(w)$, as a functional of the gauge field w.

For an antiferromagnet or a system with an RVB ground state, described, e.g., by a Landau-Ginzburg type Lagrangian in which ϕ is coupled minimally to w, the free energy $F_T(w)$ is expected to be smooth in w in a neighbourhood of $w = 0$. This is in contrast to the behaviour of $F_T(w)$ in a system with *ferromagnetic* ordering. In a ferromagnetic system, the order parameter is the spin density which is the time-component of the spin current density. The spin current density is the variable conjugate to the $SU(2)$ gauge field w; see Eq. (6.4). Thus if the expectation value of the total spin operator in an equilibrium state of the system at some temperature T is non-zero then the free energy $F_T(w)$ must have a cusp-like singularity at $w = 0$. But in systems with an antiferromagnetic - or RVB magnetic structure, the order parameter ϕ is *not* given by a component of some current density. Viewing $SU(2)_{spin}$ as an internal symmetry group of the system, ϕ turns out to be a *scalar* field transforming under the adjoint representation of the internal symmetry group. In this situation, it is consistent to assume that $F_T(w)$ is quadratic in w at $w = 0$.

We shall assume that, in the *scaling limit*, the system does not break parity - and time-reversal invariance and is rotation - and translation invariant. It then follows from the assumed smoothness of $F_T(w)$ near $w = 0$, from $SU(2)$ diamagnetism , i.e., $F_T(w)$ is increasing in w, and from the invariance of $F_T(w)$ under time-independent $SU(2)$ gauge transformations that $F_T(w)$ is given by

$$\begin{aligned} F_T(w) \;&=\; -\frac{1}{4}\Big[\,\frac{1}{l(T)}\int \mathrm{tr}\,[\,w_0^2(\vec{\xi}\,)\,]\,d^2\vec{\xi} \;+\; \frac{1}{l'(T)}\sum_{i=1}^{2}\,\int \mathrm{tr}\,[\,(w_i^T)^2(\vec{\xi}\,)\,]\,d^2\vec{\xi} \\ &-\frac{1}{4}\Big[\,l_{\|}(T)\sum_{i=1}^{2}\,\int \mathrm{tr}\,[\,h_{0i}^2(\vec{\xi}\,)\,]\,d^2\vec{\xi} \;+\; l_{\perp}(T)\!\int \mathrm{tr}\,[\,h_{12}^2(\vec{\xi}\,)\,]\,d^2\vec{\xi}\,\Big] \;+\; \cdots\,. \end{aligned} \tag{6.82}$$

Here w is a time-independent, external $SU(2)$ gauge field, and our conventions have been chosen such that the traces in (6.82) are negative; see Eq. (3.27). The third and the fourth term on the r.h.s. of (6.82) are the "Maxwell terms" discussed in (6.36); (note that, because of rotation - and translation invariance, we have $l_{\|}^{(1)} = l_{\|}^{(2)} =: l_{\|}$ and $g_{ij}(\vec{\xi}\,) = \delta_{ij}$, in (3.27)). In the first and second term on the r.h.s. of (6.82), l and l' are constants of dimension cm, and the "transversal" gauge field w_i^T, $i = 1, 2$, is defined by $w_i^T := \Big[\,\delta_i^j - \mathcal{D}_i\,\Delta_{cov}^{-1}\,\mathcal{D}^j\Big]w_j$, with $\Delta_{cov} := \mathcal{D}_j\,\mathcal{D}^j$, where $\mathcal{D}_j$ has been defined in (6.11) and (6.12). Note that the first term is invariant under time-independent $SU(2)$ gauge transformations (6.7), and the second term respects the $SU(2)$ Ward identity (6.11), (i.e., it is invariant under infinitesimal (time-independent) $SU(2)$ gauge transformations), but it is *non-local*. In fact, this term is the non-abelian analogue of the term we have encountered in the free energy (4.7) of a superconductor. Its presence mirrors the fact that we do *not* require the system to be incompressible. $SU(2)$ diamagnetism implies that $l_{\|}, l_{\perp}, l$, and l' are non-negative. [For a system with an RVB - or VBS ground state, the non-local term is expected to be absent.] Finally, the dots in (6.82) stand for terms of higher order in w or involving higher derivatives acting on w.

Given the free energy (6.82), we can study how the system responds to turning on an (external) $SU(2)$ gauge field w. Choosing some w with the property that $\mathbf{w}_j = 0,\ j = 1, 2$, we find, similarly to (6.38) and (6.72), that

$$\langle \mathbf{s}^0(\xi) \rangle_w \;=\; \frac{\delta F_T(w)}{\delta\, \mathbf{w}_0(\xi)} \;=\; -l_\parallel\, \Delta \mathbf{w}_0(\xi) + \frac{1}{l}\, \mathbf{w}_0(\xi) + \cdots \ , \tag{6.83}$$

where Δ is the two-dimensional Laplacian.

In order to create a spin-$\frac{1}{2}$ excitation (a dopant electron) localized at $\vec{\xi} = \vec{\xi}_1$, the three $SU(2)$ gauge field components of $\mathbf{w}_0$ have to solve Eq. (6.83) for a l.h.s. of the form

$$\langle \mathbf{s}^0(\xi) \rangle_w \;=\; \frac{2}{\hbar}\, \langle \mathbf{S} \rangle_w\, \delta(\vec{\xi} - \vec{\xi}_1) \;=:\; \mathbf{\Sigma}_1\, \delta(\vec{\xi} - \vec{\xi}_1)\ . \tag{6.84}$$

If $l_\parallel$ and l are *positive* constants, the solution of (6.83) and (6.84) is given by

$$\mathbf{w}_0(\xi) \;=\; \frac{\mathbf{\Sigma}_1}{l_\parallel}\, K\Big(\frac{1}{\ell}\big|\vec{\xi} - \vec{\xi}_1\big|\Big)\ , \qquad \text{and} \qquad \mathbf{w}_j(\xi) \;=\; \mathbf{0}\ , \qquad j = 1, 2\ , \tag{6.85}$$

where $\ell = \sqrt{l_\parallel\, l}$, and K is a function with the following asymptotic behaviour:

$$K(x) \;=\; \sqrt{\frac{1}{x}}\, e^{-x}\, \Big(\alpha + \mathcal{O}(\tfrac{1}{x})\Big)\ , \qquad \text{if } x \gg 1\ , \tag{6.86}$$

with α a positive constant.

We now consider a second dopant electron moving in the background gauge field w excited by the first one; see (6.85). Recalling the form of the coupling in (6.76), we expect the motion of the second electron to be subject to a force resulting from a “Zeeman term” given by $2c\, \mathbf{w}_0 \cdot \mathbf{S}_2$, where $\mathbf{S}_2$ denotes the spin operator of the second electron. Classically, we find a contribution to the energy of the two-electron system of the form

$$E_{12} \;=\; \hbar c\, \frac{\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2}{l_\parallel}\, K\Big(\frac{1}{\ell}\big|\vec{\xi}_2 - \vec{\xi}_1\big|\Big)\ , \tag{6.87}$$

where $\mathbf{\Sigma}_2$ is the expectation value of the spin operator, $\mathbf{S}_2$, of the second electron which we assume to be localized at $\vec{\xi} = \vec{\xi}_2$. A similar expression to (6.87) is obtained in a “more symmetric” treatment: One solves Eq. (6.83) for several dopant electrons localized at points $\vec{\xi}_1, \ldots, \vec{\xi}_n$ and considers the interaction term in the free energy $F_T(w)$ corresponding to the resulting $SU(2)$ gauge field w.

The form of the energy E_{12} in (6.87) suggests that a term of the form

$$J(\vec{\xi}_1 - \vec{\xi}_2)\, \mathbf{S}_{\vec{\xi}_1} \cdot \mathbf{S}_{\vec{\xi}_2}\ ,$$

must be included in the Hamiltonian of the dopant system, where $J(\vec{\xi}\,)$ is some positive function with $J(\vec{\xi}\,) \sim e^{-|\vec{\xi}\,|/\ell}$, for $|\vec{\xi}\,| \to \infty$.

To summarize, when we consider two dopant electrons (or holes) moving in a magnetic background characterized by the free energy (6.82) then Eq. (6.87) implies that, as a result of the collective response of the background, the two electrons experience a mutual *attraction* if their spins are "*antiparallel*" and a mutual *repulsion* if their spins are "*parallel*". This interaction could result in a superconducting state for spin-singlet pairs of dopant electrons (or holes), as studied in Part II.

7 Anomaly Cancellation and Algebras of Chiral Edge Currents in Two-Dimensional, Incompressible Quantum Fluids

The purpose of this section is to make a first step in discussing the origin of the quantization of the constants σ, χ_s, σ_s, and k which appear as the coefficients of the Chern-Simons terms in the scaling limit of the effective action of two-dimensional, incompressible quantum fluids; see (6.26). From the linear response equations displayed in (6.48) through (6.54) we recall that these constants completely determine the response (on large-distance scales and at low frequencies) of such quantum fluids when perturbed by small external electromagnetic, "tidal", and geometric fields. In particular, they specify the Hall effect for the electric - and for the spin current. The rationality of the constants σ, χ_s, σ_s, and k follows from a *consistency analysis* of the theory presented hitherto. This analysis can be based on a study of the so far mysterious boundary terms, "B.T.", on the r.h.s. of Eq. (6.26), and it has been carried out in full detail in Fröhlich and Studer (1993b). [A complementary approach in which the bulk terms are further exploited is given in Sect. 8 below.] By the requirement of *anomaly cancellation* we find, among the boundary terms, gauge-anomalous contributions which turn out to be the generating functionals of the connected time-ordered Green functions of chiral current operators which generate $\hat{u}(1)$ - and $\hat{su}(2)$ current (Kac-Moody) algebras. Basic, physical requirements on the spectrum of (finite-energy) excitations in incompressible quantum fluids, together with elements of the representation theory of current algebras, enter this consistency analysis. The analysis naturally leads to a list of possible values of quantum numbers (such as (fractional) charges and (fractional) statistical phases) of physical excitations that one expects to find in such quantum fluids.

For the sake of concreteness, we restrict our attention to two-dimensional, incompressible quantum fluids composed of *electrons* (or *holes*). For a different example of a physical system where we can apply similar ideas, see the discussion of superfluid ^3He-A/B-interfaces with broken parity - and time-reversal invariance that we have presented in Subsect. VII.B in Fröhlich and Studer (1993b).

In the first part of this section, we review the physics at the boundary of incompressible quantum Hall fluids (for short, QH fluids) by following some basic ideas of Halperin (1982). Extending these ideas by making use of some facts concerning chiral $\hat{u}(1)$ current algebra and introducing the idea of anomaly cancellation, we present a very natural explanation of the *integer quantum Hall effect*. How to generalize this explanation to cover the *fractional quantum Hall effect* is sketched in the second part of this section. More details on the classification of QH fluids based on a general analysis of $\hat{u}(1)$ - and $\hat{su}(2)$ current algebras describing the boundary excitations of a Hall sample can be found in Subsects. VI.B and VI.C in Fröhlich and Studer (1993b).

Independent work about current algebras in incompressible quantum Hall fluids that bears resemblance with ours (Fröhlich and Kerler, 1991; Fröhlich and Zee, 1991; Fröhlich and Studer, 1992b, 1992c, 1993a, 1993b) has been carried out by Wen and collaborators (Wen, 1990a–1990c, 1991a, 1991b; Block and Wen, 1990a, 1990b; Wen and Niu, 1990). Additional work vaguely or closely related to Wen's and ours can be found in Büttiker (1988), Beenakker (1990), MacDonald (1990), Stone (1991a, 1991b), Balatsky and Fradkin (1991), and Balatsky and Stone (1991).

7.1 Integer Quantum Hall Effect and Edge Currents

We consider a system of electrons confined to a two-dimensional domain Ω in the (x, y)-plane. We choose Ω to be an annulus and denote by $\partial\Omega$ the boundary of Ω. In our example, $\partial\Omega$ consists of two connected components, C_1 and C_2, which are circles of radius R_1 and R_2, respectively. We imagine that there is a (uniform) external magnetic field, $\mathbf{B}_c = (0, 0, B_c)$ (with vector potential $\mathbf{A}_c$, i.e., $B_c = curl\,\mathbf{A}_c$), perpendicular to the plane of the sample. Note that the magnetic field $\mathbf{B}_c$ breaks time-reversal - and parity (reflections-in-lines) invariance.

In Subsect. 4.5, we have mentioned that if the Hall conductivity of the system is on a plateau the longitudinal resistance R_L vanishes and the system is *dissipationless*, or *incompressible*. In this subsection, we intend to study the physics at the boundary of the sample. [If R_L is non-vanishing the physics of the system is complicated, and simple concepts of universality fail to capture the basic properties of the system.]

Classically, in the absence of an external electric field, there are no currents in the system. Quantum-mechanically, however, the picture is different as has been emphasized by Halperin (1982). In the absence of an external electric field, currents supported by the system are localized within approximately one cyclotron radius of the boundary $\partial\Omega$, and they are expected to persist even in the presence of a moderate amount of disorder in the sample. Because of the presence of the external magnetic field $\mathbf{B}_c$ the *edge currents* are *chiral*, i.e., the electrons drifting in the field $\mathbf{B}_c$ can only move in one direction along the boundary components C_1 and C_2 of $\partial\Omega$. We can choose the orientation of the annulus Ω, and therefore of C_1 and C_2, such that the chirality of the edge current localized near C_i is given by the orientation of C_i, $i = 1, 2$.

In order to be more explicit, we temporarily assume that there is no disorder in the system, and the electrons are moving in a confining one-body potential, V, which is constant in the bulk and rises steeply at the boundaries of the sample, i.e., at $|\vec{x}| \approx R_1, R_2$. Furthermore, we assume that electron-electron interactions are turned off (*independent electron approximation*), so that the many-electron states of the system can be constructed by filling up one-electron states, $\psi_{\uparrow/\downarrow}$, in accordance with the Pauli principle. The one-particle wave function $\psi_{\uparrow/\downarrow}$ describes a two-dimensional, charged *scalar* fermion (i.e., a fermion with a fixed spin polarization "up" ($\uparrow$) or "down" ($\downarrow$)). It satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_{\uparrow/\downarrow}(t,\vec{x}) \;=\; \left[-\frac{\hbar^2}{2m^*}\left(\boldsymbol{\nabla} + i\frac{e\hbar}{c}\boldsymbol{A}_c(\vec{x})\right)^2 + V_{\uparrow/\downarrow}(\vec{x})\right]\psi_{\uparrow/\downarrow}(t,\vec{x}) \;, \tag{7.1}$$

where

$$V_{\uparrow/\downarrow}(\vec{x}) \;:=\; V(\vec{x}) \pm \frac{g\mu_B}{2}B_c \;.$$

The second term in $V_{\uparrow/\downarrow}$ is the Zeeman energy ($\mu_B > 0$) whose sign depends on whether the spin of the electron is parallel ($\uparrow$) or antiparallel ($\downarrow$) to the magnetic field $\mathbf{B}_c$. We assume that the effective electron mass, m^*, is less then the vacuum mass, m_o, so that all Landau bands in the combined spectrum of the two Hamiltonians in Eq. (7.1) have different energies. Hence, each Landau band of one-electron states is *fully spin-polarized*.

Because of the rotational symmetry of the annular domain Ω, the eigenvalues, $m \in \mathbb{Z}$, of the z-component, L_z, of the orbital angular momentum operator are good quantum numbers for the one-electron states. For a given Landau band, a one-electron state with magnetic quantum number m is localized, in the radial direction, within about one cyclotron radius, $r_c = \left(\frac{\hbar c}{eB_c}\right)^{1/2}$, from some mean radius r_m, with $r_m = r_c\left(\frac{m}{\pi}\right)^{1/2}$, provided that $R_1 < r_m < R_2$, and $|R_i - r_m| \gg r_c$, for $i = 1, 2$. In the presence of a confining one-body potential, $V(|\vec{x}|)$, the energy, $\mathcal{E}$ (in units of $\hbar\omega_c$, where $\omega_c = \frac{eB_c}{m^* c}$ is the cyclotron frequency), of one-electron states as a function of the angular momentum m (i.e., of the square of their mean radius r_m) is approximately constant in the bulk and rises at the edges; see Halperin (1982).

We define quantities m_i by setting $r_{m_i} \approx R_i$, $i = 1, 2$. Furthermore, the magnetic quantum numbers $m^F_{i,\nu}$, $i = 1, 2$, are determined by requiring that all one-particle levels up to the Fermi energy, E^F, be filled in the Landau band indexed by $\nu = (n, s)$, where $n \in \mathbb{N}_o$, and $s = \uparrow$ or $\downarrow$. [Note that the incompressibility assumption on the system is reflected by the requirement that, in the bulk region (i.e., for $m_1 \ll m \ll m_2$) the Fermi energy E^F lies between two Landau bands.] States corresponding to values m well between m_1 and m_2 carry no net electric current. However, states with m below, but close, to m_1, or above m_2 carry *gapless*, chiral currents localized within approximately one cyclotron radius of the boundaries C_1 and C_2, respectively (Halperin, 1982).

To summarize, we find that the gapless excitations near the Fermi surface of a given filled Landau band are *charged, chiral, scalar fermions* propagating along the boundary of the sample. In order to describe the dynamics of these chiral fermions (chiral Luttinger model) in more detail, we introduce the one-dimensional momenta

$$p_{i,\nu} \;:=\; \frac{m - m^F_{i,\nu}}{R_i}\,\hbar \;, \quad i = 1, 2 \;, \quad \text{and} \quad \nu = (n, s) \in \mathbb{N}_o \times \{\uparrow, \downarrow\} \;, \tag{7.2}$$

and we redefine the energy of the one-electron states relative to the Fermi surface, i.e., we write $E := \mathcal{E} - E^F$. Let us set $R_i = \theta r_i$, with r_i fixed, $i = 1, 2,\ 0 < \theta < \infty$. We scale space- *and* time-coordinates as in (6.14), i.e., $(t, \vec{x}) = \theta(\tau, \vec{\xi})$, where $(\tau, \vec{\xi})$ belongs to a fixed space-time domain $\Lambda_o = \mathbb{R} \times \Omega_o$. Then, as θ grows, we are interested in that part of the spectrum of the edge excitations, $E = E(p_{i,\nu})$, which belongs to an ever smaller interval around $p_{i,\nu} = 0$. Note that, by Eq. (7.2), $p_{i,\nu}$ scales with θ^{-1}. Thus, for the rescaled systems in Ω_o, the spectra of the edge excitations associated with a given Landau band converge towards the *linear* energy-momentum dispersion law of a *massless, chiral,* "*relativistic*" *Fermi field* propagating along a circle of radius $r_i,\ i = 1, 2$.

Before we turn to the description of relativistic Fermi fields we note that, in order to observe the quantum Hall effect experimentally, it is necessary to perturb the system. This can be achieved, for example, by applying a small voltage between the inner and outer edge of the annular sample, thereby changing the chemical potentials of the electrons at the two edges. More generally, we shall couple the system to an additional, external electromagnetic vector potential, A, where $A = A_{\text{tot}} - A_c$ is a small perturbation, and A_c is the vector potential corresponding to $\mathbf{B}_c$. Our next step is thus to recall the description of a massless, chiral, relativistic Fermi field circulating along one component, C_o, of the boundary of the (rescaled) system and coupled to $A|_{\Gamma_o}$, (the restriction of A to $\Gamma_o = \mathbb{R} \times C_o \subset \partial\Lambda_o$).

It is convenient to use "light-cone" coordinates on the $(1+1)$-dimensional boundary space-time Γ_o. We set

$$u_\pm := \frac{1}{\sqrt{2}}(\zeta^0 \pm \zeta^1) = \frac{1}{\sqrt{2}}(v\tau \pm \frac{L}{2\pi}\eta) , \tag{7.3}$$

where $\eta \in [0, 2\pi)$ is an angular variable along C_o (whose length is given by L), τ is (rescaled) time, and the constant v physically corresponds to the propagation speed of *charged waves at the edge of the sample*. [The value of the velocity v does not matter in the following.] We write

$$A|_{\Gamma_o} = A_+(u)\, du_+ + A_-(u)\, du_- , \tag{7.4}$$

where

$$A_\pm(u) = \frac{1}{\sqrt{2}} \Big(A_0|_{\Gamma_o}(\zeta) \pm A_1|_{\Gamma_o}(\zeta) \Big)|_{\zeta=\zeta(u)} ,$$

with $u := (u_+, u_-)$ and $\zeta := (\zeta^0, \zeta^1)$. The (1+1)-dimensional d'Alembertian, $\Box := \partial_0^2 - \partial_1^2 = \frac{1}{v^2}\frac{\partial^2}{\partial\tau^2} - \left(\frac{2\pi}{L}\right)^2 \frac{\partial^2}{\partial\eta^2}$, is given in "light-cone" coordinates by

$$\Box = 2\, \partial_+ \partial_- , \tag{7.5}$$

where $\partial_\pm := \frac{\partial}{\partial u_\pm}$.

In $1+1$ dimensions, a relativistic Fermi field (fermion, for short) is described by a two-component Dirac (i.e., complex) spinor, ψ. We choose the chiral representation of Dirac matrices:

$$\gamma^0 = \sigma_1 , \quad \gamma^1 = -i\sigma_2 , \quad \text{and} \quad \gamma_5 := \gamma^0\gamma^1 = \sigma_3 , \tag{7.6}$$

where σ_1, σ_2, σ_3 are the standard Pauli matrices. Moreover, we define by $\overline{\psi} := \psi^*\gamma^0$ the conjugate of the Dirac spinor ψ and identify its left-/right-handed component with

$$\psi_{L/R}(\zeta) := \frac{1}{2}(1 \mp \gamma_5)\psi(\zeta) . \tag{7.7}$$

Our aim is to write down an action for a massless, chiral fermion, $\psi_{L/R}$, coupled to the vector potential $A|_{\Gamma_o}$, and to calculate an effective gauge field action, $\Gamma_{L/R}(A|_{\Gamma_o})$, by integrating out the chiral fermion degrees of freedom located at the edge of the sample. Naively, one might try to start with an action of the form

$$\text{“ } S_{L/R}(\overline{\psi}_{L/R}, \psi_{L/R}; A|_{\Gamma_o}) = i\int_{\Gamma_o} \overline{\psi}_{L/R}(\zeta)\, D(A|_{\Gamma_o})\, \psi_{L/R}(\zeta)\, d^2\zeta \text{ ” } , \tag{7.8}$$

where the *Dirac operator*, $D(A|_{\Gamma_o})$, is defined by

$$D(A|_{\Gamma_o}) := \gamma^\mu \left(\partial_\mu + i\frac{e}{\hbar c}A_\mu|_{\Gamma_o}(\zeta)\right) = \not{\partial} + i\frac{e}{\hbar c}\not{A}|_{\Gamma_o}(\zeta) . \tag{7.9}$$

However, it is *not* possible to compute the effective action of a massless, chiral fermion coupled to $A|_{\Gamma_o}$ by a fermionic (Berezin) path integral based on the action (7.8). Put differently, it is *not* possible to define a determinant of the Dirac operator $D(A|_{\Gamma_o})$ *restricted* to the subspace of either only left- or only right-handed field modes. This is because of the simple fact that the Dirac operator $D(A|_{\Gamma_o})$ maps left- to right-handed modes and vice versa, i.e., the chirality subspaces are not invariant under the action of $D(A|_{\Gamma_o})$; see, e.g., Alvarez-Gaumé and Ginsparg (1984). Using (7.4) and (7.7), we can rewrite the standard action of a massless, two-component Dirac field ψ on Γ_o in terms of its components ψ_L and ψ_R. We find that

$$\begin{aligned} &i\int_{\Gamma_o} \overline{\psi}(\zeta)\, D(A|_{\Gamma_o})\, \psi(\zeta)\, d^2\zeta \\ &= i\sqrt{2}\int_{\Gamma_o} \left[\psi_L^* \left(\partial_- + i\frac{e}{\hbar c}A_-\right)\psi_L + \psi_R^*\left(\partial_+ + i\frac{e}{\hbar c}A_+\right)\psi_R\right](\zeta)\, d^2\zeta . \end{aligned} \tag{7.10}$$

[The equation of motion for $\psi_{L/R}$ following from (7.10) reads $\left(\partial_\mp + i\frac{e}{\hbar c}A_\mp\right)\psi_{L/R} = 0$. Hence, in $1+1$ dimensions the left-/right-handed modes are actually left-/right-moving excitations, provided $A_\mp = 0$.] By inspecting the coupling structure of $A_\mp$ to $\psi_{L/R}$ in Eq. (7.10), we are led to the following expression for the *effective gauge field action of a massless, chiral (left-/right-moving) relativistic Fermi field coupled to the external vector potential* $A|_{\Gamma_o}$:

$$\begin{aligned}\frac{i}{\hbar}\Gamma_{L/R}(A|_{\Gamma_o}) &:= \Big[\ln\det D(A|_{\Gamma_o})\Big]_{A_\pm=0} + \frac{a}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\Gamma_o} A_+(u)A_-(u)\,d^2u \\ &= \ln\det\left[\not{\partial} + i\frac{e}{\hbar c}\not{A}|_{\Gamma_o}\,\frac{1}{2}\,(1\mp\gamma_5)\right]\,, \end{aligned} \tag{7.11}$$

see Jackiw (1985), and Jackiw and Rajaraman (1985). In Eq. (7.11), the determinant of the Dirac operator $D(A|_{\Gamma_o})$ is calculated on the full mode-space of a $(1+1)$-dimensional, two-component Dirac field, and its evaluation goes back to Schwinger (1962). In the second term, a is an *arbitrary* real constant. This term stands for a finite renormalization ambiguity and is related to the fact that one cannot invoke $U(1)$ gauge invariance as a guiding principle in the calculation of *chiral* effective actions (Jackiw and Rajaraman, 1985; Leutwyler, 1986 and references therein). Chiral effective actions are *anomalous*, a fact that we will exploit shortly! We set $a=1$, which is a particularly convenient choice for the subsequent discussion (see also Jackiw, 1985), and find that

$$\frac{1}{\hbar}\Gamma_{L/R}(A|_{\Gamma_o}) = \frac{1}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\Gamma_o}\left[A_+(u)\,A_-(u) - 2A_\mp(u)\,\frac{\partial_\pm^2}{\Box}\,A_\mp(u)\right]d^2u\,. \tag{7.12}$$

Let us define the left-/right-handed current (operator), $j^\mu_{L/R}$, by

$$j^\mu_{L/R}(\zeta) := \,:\overline{\psi}(\zeta)\,\gamma^\mu\frac{1}{2}\,(1\mp\gamma_5)\,\psi(\zeta):\,, \tag{7.13}$$

where $:\ :$ indicates normal ordering. Then, using Eqs. (7.10) and (7.11), we observe that the effective action $\Gamma_{L/R}(A|_{\Gamma_o})|_{A_\pm=0}$ is the *generating functional* for time-ordered, connected Green functions of $j^\mu_{L/R}$. Both currents, j^μ_L and j^μ_R, independently generate a *chiral* $\hat{u}(1)$ *current algebra*. This will be discussed in more detail in the next subsection. We note that the choice of the *left*- or *right*-handed current, j^μ_L or j^μ_R, depends on the physics of the given system, namely, on the sign of the external magnetic field and on whether the physical edge currents are carried by *electrons* or *holes*.

Next, we exploit the fact that effective actions for *chiral* fermions are breaking $U(1)$ gauge invariance, i.e., that they are *anomalous*! Performing a $U(1)$ gauge transformation, $A \mapsto A + \mathrm{d}\chi$, the explicit expression for the chiral effective action $\Gamma_{L/R}(A|_{\Gamma_o})$ given by (7.12) implies that

$$\frac{1}{\hbar}\Gamma_{L/R}([A+\mathrm{d}\chi]|_{\Gamma_o}) \;=\; \frac{1}{\hbar}\Gamma_{L/R}(A|_{\Gamma_o}) \pm\frac{1}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\Gamma_o}\Big[A_+(u)\,\partial_-\chi(u) - A_-(u)\,\partial_+\chi(u)\Big]\,d^2u\;. \tag{7.14}$$

Thus, the *chiral anomaly* of the effective action produced by the quantum-mechanical degrees of freedom located at the edge of the sample takes the form

$$\pm\frac{1}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\Gamma_o}\Big[A_+(u)\,\partial_-\chi(u) - A_-(u)\,\partial_+\chi(u)\Big]\,d^2u \;=\; \pm\frac{1}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\Gamma_o}\mathrm{d}\chi\wedge A\;. \tag{7.15}$$

Remember that a Landau band gives rise to an algebra of chiral edge currents at *each* connected component, C_o, of the boundary, $\partial\Omega_o$, of the sample. Hence the total chiral effective boundary action resulting from a given Landau band is obtained by simply adding up the contributions of the form (7.12) for the different connected components of $\partial\Omega_o$. We recall our assumption that, in the bulk, the Fermi energy, E^F, lies well *between* two Landau bands (incompressibility) and that there are $N = 0, 1, 2, \ldots$ *filled* Landau bands *below* the Fermi energy, each of fixed spin-polarization (i.e., all the electrons can be treated as scalar fermions). Then, in the scaling limit, all the quantum-mechanical degrees of freedom localized near the boundary $\partial\Omega_o$ of the sample together give rise to the following boundary contribution, $\zeta^{L/R}_{\partial\Lambda_o}(A|_{\partial\Lambda_o})$, to the *total partition function*, $Z_{\Lambda_o}(A)$, of the system:

$$\zeta^{L/R}_{\partial\Lambda_o}(A|_{\partial\Lambda_o}) \;=\; \prod_{j=0}^{N}\;\prod_{\Gamma_o\subset\partial\Lambda_o}\exp\Big[\frac{i}{\hbar}\Gamma_{L/R}(A|_{\Gamma_o})\Big]\;. \tag{7.16}$$

From Eqs. (7.14) and (7.15) it follows that the *total chiral anomaly* of the boundary contribution (7.16) is given by

$$\pm\frac{N}{4\pi}\left(\frac{e}{\hbar c}\right)^2\int_{\partial\Lambda_o}\mathrm{d}\chi\wedge A\;. \tag{7.17}$$

We are now in a position to describe the basic idea of *anomaly cancellation*: In Sects. 2 and 3, we have seen that non-relativistic quantum mechanics is $U(1)$ gauge invariant which means that the total partition function, $Z_{\Lambda_o}(A)$, of the system *is* $U(1)$ gauge invariant! In other words, the total chiral anomaly (7.17) due to the *degrees of freedom localized at the boundary* has to be *cancelled* by the one of an anomalous term in the total effective action resulting from the *degrees of freedom in the bulk* of the system. The term with the required anomaly under $U(1)$ gauge transformations is the (by now familiar) *abelian Chern-Simons* term,

$$\mp \frac{N}{4\pi}\left(\frac{e}{\hbar c}\right)^2 \int_{\Lambda_o} A \wedge \mathrm{d}A \ . \tag{7.18}$$

Thus, to leading order in the scale parameter θ, the total partition function, $Z_{\Lambda_o}(A)$, of a (non-interacting) quantum Hall fluid with $N = 0, 1, 2, \ldots$ filled Landau bands coupled to a small perturbing vector potential A takes the form

$$Z_{\Lambda_o}(A) \ = \ \zeta^{L/R}_{\partial\Lambda_o}(A|_{\partial\Lambda_o}) \ \exp\left[\mp i\frac{N}{4\pi}\left(\frac{e}{\hbar c}\right)^2 \int_{\Lambda_o} A \wedge \mathrm{d}A + \mathrm{G.I.}\right] \ , \tag{7.19}$$

where "G.I." stands for possible $U(1)$ gauge-invariant terms. As we have seen in Eqs. (4.27) and (4.31), it is the Chern-Simons term (7.18) in the total (bulk plus boundary) effective action which reproduces the basic response equations, (4.13) and (4.17), of the quantum Hall effect. In particular, comparing Eqs. (4.27) and (7.19), we identify the coefficient of the Chern-Simons term (7.18) with the Hall conductivity σ_H of the system, i.e.,

$$\sigma_H \ = \ \pm N\,\frac{e^2}{h} \ , \qquad N = 0, 1, 2, \ldots \ . \tag{7.20}$$

These considerations yield a natural description of the physics underlying the *integer quantum Hall effect*, provided the system consists of non-interacting electrons.

One can argue that the picture of chiral edge excitations given above, and hence the form of the anomaly, still hold when the system is perturbed by a small amount of disorder. A *chiral* Luttinger liquid perturbed by a weak random potential will *not* exhibit Anderson localization, because there is no interference between left- and right-moving waves! This is in contrast to what happens in the bulk where a moderate amount of disorder leads to plenty of localized states; (we then require that the Fermi energy, E^F, lie in a region of localized states, in order to conclude incompressibility of the fluid). In fact, we recall that Anderson localization in the bulk is crucial in order for the (integer) quantum Hall effect to be observable experimentally (Halperin, 1983; Morandi, 1988; Prange, 1990; and references therein). More precisely, the width of a Hall plateau depends on the amount of disorder in the system – which determines the density of localized states – and, in the thermodynamic limit, would tend to 0 as the strength of the disorder tends to 0.

Taking electron-electron interactions into account, the *form* of the anomaly will not change, because it is universal. However, the value of the Hall conductivity σ_H can and will change.

In the following subsection, we discuss spin-polarized, two-dimensional, incompressible systems of electrons which do not necessarily have integer filling factors ν, due to *electron-electron interactions*; see (4.12).

However, in contrast to the logic of the present subsection, we shall start from the *universal* form, $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, given in Eq. (6.26), that the effective action for the *bulk* degrees of freedom takes in the scaling limit. [We recall that expression (6.26) of $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ takes the spin degrees of freedom into account and that it does *not* exclude any effects of electron-electron interactions that respect $U(1) \times SU(2)$ gauge invariance and are compatible with incompressibility of the quantum fluid!] We will identify those terms in $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ which exhibit an anomalous behaviour under $U(1)$ and $SU(2)$ gauge transformations not vanishing at the boundary $\partial\Lambda_o$. The idea of *anomaly cancellation* then will lead us to the study of the dynamics of degrees of freedom at the *boundary* of the system compensating the gauge non-invariance of the bulk terms in $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$. Similarly as above, we shall find charge - *and* spin carrying chiral edge currents forming $\hat{u}(1)$ - and $\hat{su}(2)$ current (Kac-Moody) algebras, respectively. The representation theory of these current algebras then captures the *universal features* of systems exhibiting a *fractional quantum Hall effect for the electric and for the spin current.*

In the following subsection, we shall illustrate this logic by treating the simplest situation of fully spin-polarized (or scalar) quantum Hall fluids. For a complete implementation of the above logic, see Subsects. VI.B and VI.C in Fröhlich and Studer (1993b). There properties of "chiral boundary systems" ($\hat{u}(1)$ current algebras) associated with fully spin-polarized quantum Hall fluids are discussed in detail, and the additional features arising for systems exhibiting spin - ($\hat{su}(2)$ current algebra) and possibly internal symmetries ($\hat{g}$ current algebra associated to some compact Lie group G) are explained.

7.2 Edge Excitations in Spin-Polarized Quantum Hall Fluids

In this subsection, we consider *interacting*, spin-polarized, two-dimensional, incompressible quantum Hall fluids, where "spin-polarized" means that the spin degrees of freedom are "frozen". Moreover, we shall neglect the magnetic moments of the electrons. [For a treatment of spin-polarized Hall fluids including the effects of the magnetic moments of the electrons, see Subsect. VI.C in Fröhlich and Studer (1933b).] For such systems, the *universal* form of the scaling limit $S^*_{\Lambda_o}(\tilde{a})$ of the effective action is thus obtained by discarding all terms depending on the $SU(2)$ gauge fields w or $\tilde{w}$ in expression (6.26). We find that

$$-\frac{1}{\hbar} S^*_{\Lambda_o}(\tilde{a}) \;=\; \int_{\Lambda_o} j^\mu_c(\xi)\, \tilde{a}_\mu(\xi)\, d^3\xi + \frac{\sigma}{4\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{a} \;+\; \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o}) \;, \tag{7.21}$$

where $\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o})$ stands for boundary terms only depending on the restriction of the perturbing, external vector potential $\tilde{a}$ to the boundary of the system. [Note that we have returned to working in "mathematical units"; see, e.g., (3.36). Moreover, we consider a euclidean background metric, $\gamma_{ij} = \delta_{ij}$; see (6.13).] So far, the coefficient σ of the Chern-Simons term on the r.h.s. of Eq. (7.21) can be an arbitrary real constant.

The particular situation where σ takes *integer* values has been identified in the previous subsection as describing the *integer quantum Hall effect*. The main purpose of this subsection is to understand the basic aspects of the more general situation of the *fractional* quantization of the values of the constant σ arising in systems where electron-electron interactions cannot be neglected. We recall that, by Eq. (6.49), σ determines the value of the Hall conductivity, $R_H^{-1} = \sigma \frac{e^2}{h}$, which encodes the linear response properties of incompressible quantum Hall fluids, provided we neglect the spin degrees of freedom; see Eqs. (6.48) and (6.51). In this subsection, we develop a picture of universal aspects of the *fractional quantum Hall effect* in spin-polarized, two-dimensional electronic systems by focusing on the physics at their edges.

We follow the ideas described in Subsect. 7.1: In a first step, we make use of *anomaly cancellation* in the direction opposite to the one explained in Subsect. 7.1, i.e., we start from the bulk terms. As explained in Fröhlich and Studer (1992b, 1993a), $S^*_{\Lambda_o}(\tilde{a})$ must be invariant under $U(1)$ gauge transformations

$$\tilde{a} \;\mapsto\; {}^{\chi}\tilde{a} \;:=\; \tilde{a} + \mathrm{d}\chi \;, \tag{7.22}$$

in spite of the fact that $\tilde{a}$ is the vector potential of a *perturbation* of the electromagnetic field (i.e., $\tilde{a} = a - a_c$ is the *difference* of two connection 1-forms)! For a trivial $U(1)$ bundle, always realized for our choice of space-time domains $\mathbb{R} \times \Omega_o$, the space of connections (vector potentials) is a real *vector space*. In particular, any gauge transformation of a sum of connections can be rewritten as the result of gauge transforming each summand in the sum *separately*. This is in contrast to the situation encountered for non-abelian gauge fields. Performing a gauge transformation (7.22) on $\tilde{a}$ with χ non-vanishing at $\partial\Lambda_o$, we find from (7.21) that

$$\begin{aligned}\frac{1}{\hbar}\, S^*_{\Lambda_o}(\tilde{a} + \mathrm{d}\chi) \;&=\; \frac{1}{\hbar}\, S^*_{\Lambda_o}(\tilde{a}) + \int_{\partial\Lambda_o} (*j_c)\, \chi \\ &\quad + \frac{\sigma}{4\pi} \int_{\partial\Lambda_o} \mathrm{d}\chi \wedge \tilde{a} - \;\mathrm{B.T.}([\tilde{a} + \mathrm{d}\chi]|_{\partial\Lambda_o}) \;+\; \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o}) \;,\end{aligned} \tag{7.23}$$

where $*j_c = \frac{1}{2} \sum_{\mu,\,\nu} \varepsilon_{\mu\nu\sigma}\, j_c^{\sigma}\,(\xi)\, \mathrm{d}\xi^{\mu} \wedge \mathrm{d}\xi^{\nu}$ is the (Hodge) dual of the current j_c.

Note that $S^*_{\Lambda_o}(\tilde{a}+\mathrm{d}\chi)$ would be equal to $S^*_{\Lambda_o}(\tilde{a})$, and hence we could set $\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o}) = 0$ (up to gauge-invariant terms at $\partial\Lambda_o$), if

$$*j_c|_{\partial\Lambda_o} \;=\; \frac{\sigma}{4\pi}\, \mathrm{d}\tilde{a}|_{\partial\Lambda_o} \;, \tag{7.24}$$

since $\int_{\partial\Lambda_o} \mathrm{d}\chi \wedge \tilde{a} = -\int_{\partial\Lambda_o} \chi\, \mathrm{d}\tilde{a}$. However, j_c is an arbitrary current in the Hall fluid when $a = a_c$ (i.e., $\tilde{a} = 0$), and $\tilde{a}$ is the potential of a small but *arbitrary*, external perturbation. Therefore the relation (7.24) cannot hold in general.

Experimentally, for a Hall fluid in a heterostructure or MOSFET, the boundary $\partial\Lambda_o$ of the sample is such that there is no leakage of electric charge through $\partial\Lambda_o$, which means that the normal component of j_c^μ at $\partial\Lambda_o$ has to vanish, or equivalently,

$$*j_c|_{\partial\Lambda_o} \;=\; 0\ . \tag{7.25}$$

In this case, the second term on the r.h.s. of (7.23) *vanishes*, and requiring the total effective action $S^*_{\Lambda_o}(\tilde{a})$ to be $U(1)$ gauge-invariant implies the following functional equation for the boundary terms:

$$\mathrm{B.T.}([\tilde{a} + \mathrm{d}\chi]|_{\partial\Lambda_o}) - \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o}) \;=\; \frac{\sigma}{4\pi}\int_{\partial\Lambda_o} \mathrm{d}\chi \wedge \tilde{a}\ , \tag{7.26}$$

which must hold for arbitrary $\tilde{a}$ and χ.

From the discussion in the previous subsection we can immediately infer the solution to Eq. (7.26). Introducing "light-cone" coordinates on each connected component, Γ_o, of the $(1+1)$-dimensional boundary space-time $\partial\Lambda_o$ (see Eq. (7.3)), we have, as in Eq. (7.4), that

$$\begin{aligned} \tilde{a}|_{\Gamma_o} \;&=\; \frac{e}{\hbar c}A_+(u)\,du_+ + \frac{e}{\hbar c}A_-(u)\,du_- \\ &=:\; \alpha_+(u)\,du_+ + \alpha_-(u)\,du_-\ . \end{aligned} \tag{7.27}$$

From Eqs. (7.14) and (7.15) it follows that the solution to Eq. (7.26) is given by

$$\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_o}) \;=\; \sigma_L \sum_{\Gamma_o\subset\partial\Lambda_o} \frac{1}{\hbar}\Gamma_L(\alpha) \;+\; \sigma_R \sum_{\Gamma_o\subset\partial\Lambda_o} \frac{1}{\hbar}\Gamma_R(\alpha) \;+\; \mathrm{G.I.}(\alpha)\ , \tag{7.28}$$

with

$$\sigma \;=\; \sigma_L - \sigma_R\ , \tag{7.29}$$

where $\Gamma_{L/R}(\alpha)$ is as in Eq. (7.12), σ_L and σ_R are non-negative constants, and "G.I." stands for manifestly gauge-invariant terms only depending on $\tilde{a}|_{\partial\Lambda_o}$. We recall from Eq. (7.14) that the contributions to the total anomaly of the two terms $\Gamma_L(\alpha)$ and $\Gamma_R(\alpha)$ are of opposite sign which explains the sign in (7.29). We do not need to discuss the terms in "G.I." any further and hence omit them in the following discussion.

In Subsect. 7.1, we have seen that, for $\sigma_{L/R} = N$ a (positive) *integer*, there is a straightforward interpretation of the boundary terms in Eq. (7.28) in terms of N bands of *non-interacting*, chiral (left-/right-moving) fermions propagating along the different connected components of the sample boundary $\partial\Omega_o$. In order to understand the more general, *fractional* quantization of the values of the constant σ, we have to generalize this physical picture. For this purpose, we use bosonization techniques always available in two space-time dimensions; see Sect. 5. In this subsection, we derive an expression for $\Gamma_{L/R}(\alpha)$ (see (7.11)) in terms of one chiral Bose field. In a second step, this bosonic expression would have to be generalized to several bands of excitations at the boundary. For details on this second step, see Sect. VI.B in Fröhlich and Studer (1993b).

In the following we assume, for simplicity, that the topology of the sample Ω_o is that of a disk, i.e., the boundary $\partial\Omega_o$ consists of but *one* connected component C_o.

Let us first suppose that the external gauge field $\alpha := \tilde{a}|_{\partial\Lambda_o}$ is set to 0. In Eq. (7.13) we have introduced the currents

$$j^\mu_{L/R}(\zeta) \;=\; \frac{1}{2}\,[\,j^\mu(\zeta) \mp j^\mu_5(\zeta)]\;, \tag{7.30}$$

of a massless Dirac spinor ψ. Recalling the equation of motion satisfied by the left-/right-handed component of ψ (see (7.7), (7.6) and the remark after (7.10)) we see that j^μ and j^μ_5 are conserved currents, i.e.,

$$\partial_\mu j^\mu(\zeta) \;=\; 0 \;=\; \partial_\mu j^\mu_5(\zeta)\;. \tag{7.31}$$

The general solution to (7.31) is

$$j^\mu(\zeta) \;=\; \sqrt{2}\,\varepsilon^{\mu\nu}\partial_\nu\phi(\zeta)\;, \qquad \text{and} \qquad j^\mu_5(\zeta) \;=\; \sqrt{2}\,\varepsilon^{\mu\nu}\partial_\nu\phi_5(\zeta)\;, \tag{7.32}$$

where ϕ and ϕ_5 are scalar fields, and $\varepsilon^{01} = -\varepsilon^{10} = 1$. However, in $1+1$ dimensions, it follows form (7.13) and (7.6) that $j^0_5 = j^1$ and $j^1_5 = -j^0$. Thus, $j^\mu_5 = -\sqrt{2}\,\partial^\mu\phi$, and (7.31) implies that

$$\partial_\mu\partial^\mu\phi(\zeta) = \Box\phi(\zeta) \;=\; 0\;, \tag{7.33}$$

i.e., ϕ is a *free, massless, relativistic Bose field.* Any solution to Eq. (7.33) has the form (see (7.3))

$$\phi(\zeta(u)) \;=\; \phi_L(u_+) + \phi_R(u_-)\;. \tag{7.34}$$

Moreover, since j^μ and j_5^μ are currents propagating along the boundary $\partial\Omega_o$, ϕ has to satisfy the periodicity conditions

$$\partial_\pm \phi(\zeta^0, \zeta^1 + L) \;=\; \partial_\pm \phi(\zeta^0, \zeta^1) \; . \tag{7.35}$$

In terms of the chiral components ϕ_L and ϕ_R of the Bose field ϕ, we can define the *chiral currents* J_L and J_R,

$$\begin{aligned} \frac{2\pi}{\ell} J_L(u_+) \;&:=\; \partial_+ \phi_L(u_+) \;=\; j_L^0(\zeta(u)) \; , \qquad \text{and} \\ \frac{2\pi}{\ell} J_R(u_-) \;&:=\; -\partial_- \phi_R(u_-) \;=\; j_R^0(\zeta(u)) \; , \end{aligned} \tag{7.36}$$

where $\ell = \frac{L}{\sqrt{2}}$. Clearly $\partial_\mp J_{L/R} = 0$. We emphasize that Eqs. (7.30)–(7.36) hold at the level of *quantized* fields. They are at the origin of *abelian bosonization* in two space-time dimensions.

The currents J_L and J_R both generate a chiral $\hat{u}(1)$ current algebra as follows: We can decompose the current J_L (and similarly J_R) into its Fourier modes:

$$J_L(u_+) \;=\; \frac{\ell}{2\pi} \partial_+ \phi_L(u_+) \;=\; \frac{1}{\kappa} p + \frac{1}{\sqrt{\kappa}} \sum_{n \neq 0} \alpha_n \, e^{-2\pi i n u_+/\ell} \; , \tag{7.37}$$

where κ is a positive normalization constant whose meaning will become apparent shortly. Integrating relation (7.36) we find that

$$\phi_L(u_+) \;=\; q + \frac{2\pi}{\ell\kappa} p \, u_+ + \frac{i}{\sqrt{\kappa}} \sum_{n \neq 0} \frac{1}{n} \alpha_n \, e^{-2\pi i n u_+/\ell} \; , \tag{7.38}$$

where q is a real integration constant. Since ϕ_L is a real quantum field, the operator α_{-n} is the adjoint of α_n, i.e., $\alpha_{-n} = (\alpha_n)^\dagger$, $n \in \mathbb{Z} \backslash \{0\}$. Moreover, the Fourier coefficients in (7.38) are subject to the following commutation relations

$$[q, p] \;=\; i \; , \qquad \text{and} \qquad [\alpha_m, \alpha_n] \;=\; m \, \delta_{m,-n} \; , \quad m, n \in \mathbb{Z} \backslash \{0\} \; . \tag{7.39}$$

Eq. (7.38) and the relations (7.39) define $\hat{u}(1)$ *current (Kac-Moody) algebra* (at level κ); see, e.g., Goddard and Olive (1986), Buchholz, Mack, and Todorov (1990), Ginsparg (1990), and also Frenkel (1981) and Kac (1983). The vacuum (charge-0) sector of this quantum field theory is the Fock space built from the vacuum state, $|0\rangle$, which is defined by

$$\alpha_n |0\rangle \;=\; 0 \; , \quad n > 0 \; , \qquad \text{and} \qquad p|0\rangle \;=\; 0 \; , \tag{7.40}$$

by applying polynomials in the creation operators α_{-n}, $n = 1, 2, \ldots$, to $|0\rangle$.

These constructions are well-known in string - and conformal field theory. We describe them below. First, however, we derive an expression, in terms of Bose fields, for the boundary term "B.T.", given in Eq. (7.28).

In the path integral formulation, a free, massless Bose field ϕ, propagating along the boundary $\partial\Omega_o$ of the sample, is described by a gaussian action

$$\frac{1}{\hbar} S(\phi) \;:=\; \frac{\kappa}{4\pi} \int_{\partial\Lambda_o} \partial_-\phi(u)\, \partial_+\phi(u)\, d^2u \;, \tag{7.41}$$

where $d^2u := \mathrm{d}u_- \wedge \mathrm{d}u_+$ is the (oriented) space-time measure on $\partial\Lambda_o$, and κ is the same normalization constant that appeared already in Eqs. (7.37) and (7.38). If we consider but one chiral component of the field ϕ we have to supplement the action (7.41) by a chirality constraint

$$\partial_-\phi(u) \;=\; 0 \;, \qquad \text{or} \qquad \partial_+\phi(u) \;=\; 0 \;. \tag{7.42}$$

Next, we couple ϕ to an external gauge potential $\alpha = \tilde{a}|_{\partial\Lambda_o}$. We present a formal argument allowing us to identify the correct way of coupling ϕ to α and providing a path integral derivation of the fermion-boson equivalence in $1+1$ dimensions. The state sum (a divergent constant) for a free, chiral (left-moving), massless Bose field reads

$$Z_L \;:=\; \int \mathcal{D}\phi \, \exp\Big[\frac{i}{\hbar} S(\phi)\Big] \; \delta(\partial_-\phi) \;, \tag{7.43}$$

where $S(\phi)$ is given by (7.41). Since the field ϕ is an angle variable, we may shift it according to $\phi \mapsto {}^{\chi}\phi := \phi + \frac{1}{\kappa}\chi|_{\partial\Lambda_o}$. Using the fact that the integration measure is gauge invariant, i.e., $\mathcal{D}\,{}^{\chi}\phi = \mathcal{D}\phi$, we find, after partial integration, that

$$\begin{aligned} Z_L \;&=\; \exp\Big[\frac{i}{\hbar\kappa^2}\, S(\chi)\Big] \\ &\times \int \mathcal{D}\phi \, \exp\Big[\frac{i}{\hbar} S(\phi) + \frac{i}{2\pi\hbar} \int_{\partial\Lambda_o} \partial_+\phi(u)\, \partial_-\chi(u)\, d^2u\Big] \; \delta\Big(\partial_-\phi + \frac{1}{\kappa}\, \partial_-\chi|_{\partial\Lambda_o}\Big) \;. \end{aligned} \tag{7.44}$$

Let us choose χ such that $\partial_-\chi|_{\partial\Lambda_o} = -Q\,\alpha_-$, where Q is some constant. Then Eq. (7.44) and expression (7.12) for the action functional $\Gamma_L(\alpha)$ imply that the following identity holds

$$\exp\Big[\frac{iQ^2}{\hbar\kappa}\, \Gamma_L(\alpha)\Big] \;=\; \frac{1}{Z_L} \int \mathcal{D}\phi \, \exp\Big[\frac{i}{\hbar} S_{WZNW}(\phi;\alpha)\Big] \; \delta\Big(\partial_-\phi - \frac{Q}{\kappa}\, \alpha_-\Big) \;, \tag{7.45}$$

where

$$
\begin{aligned}
\frac{1}{\hbar} S_{WZNW}(\phi;\alpha) \;&:=\; \frac{\kappa}{4\pi}\int_{\partial\Lambda_o} \partial_-\phi(u)\,\partial_+\phi(u)\,d^2u \\
&-\; \frac{Q}{2\pi}\int_{\partial\Lambda_o}\Big[\partial_+\phi(u)\,\alpha_-(u) - \Big(\partial_-\phi - \frac{Q}{\kappa}\,\alpha_-\Big)(u)\,\alpha_+(u)\Big]\,d^2u \\
&+\; \frac{Q^2}{4\pi\,\kappa}\int_{\partial\Lambda_o} \alpha_-(u)\,\alpha_+(u)\,d^2u\ , \qquad (7.46)
\end{aligned}
$$

with α_+ as in (7.27). We note that the α_+-term in the second line of (7.46) vanishes when the constraint in (7.45) is imposed. It has been added for symmetry reasons discussed below. Moreover, the third term on the r.h.s. of (7.46) is independent of ϕ. It has been added in order for the l.h.s. of (7.45) to coincide with the expression given in (7.12).

The field theory with action (7.46) is known as the *gauged, abelian Wess-Zumino-Novikov-Witten (WZNW) model*; see, e.g., Gawedzki (1990) (and Floreanini and Jackiw (1987), Sonnenschein (1988), and Harada (1990)). The *chirality constraint* in Eq. (7.45),

$$
\partial_-\phi(u) - \frac{Q}{\kappa}\,\alpha_-(u) \;=\; 0\ , \qquad (7.47)
$$

is invariant under gauge transformations

$$
\begin{aligned}
\phi(u) \;&\mapsto\; {}^{\chi}\phi(u) \;:=\; \phi(u) + \frac{Q}{\kappa}\,\chi|_{\partial\Lambda_o}(u)\ , \\
\alpha \;&\mapsto\; {}^{\chi}\alpha \;:=\; \alpha + \mathrm{d}\chi|_{\partial\Lambda_o}\ . \qquad (7.48)
\end{aligned}
$$

By Eqs. (7.14) and (7.45), the theory specified by the r.h.s. of (7.45), with an action as given in Eq. (7.46), gives rise, under a gauge transformation (7.48), to the anomaly

$$
\frac{\sigma}{4\pi}\int_{\partial\Lambda_o} \mathrm{d}\chi\wedge\tilde{a}\ , \qquad \text{with} \qquad \sigma \;=\; \frac{Q^2}{\kappa}\ . \qquad (7.49)
$$

It is clear from (7.46) that, physically, Q specifies the charge of the left-moving component of the Bose field ϕ in units where the elementary charge $-e = 1$; see (7.27) and (7.36). If $\frac{Q^2}{\kappa}$ equals 1, then Eq. (7.45) can be used to prove the equivalence of the theory of one chiral fermion, given in (7.11), to the theory of one chiral boson, given in (7.46).

Replacing L by R on the l.h.s. of (7.45) corresponds to interchanging $+$ and $-$, and replacing Q by $-Q$, and κ by $-\kappa$ on the r.h.s. of (7.45). [Interchanging $+$ and $-$, the measure d^2u goes over into $-d^2u$. In order for this symmetry property to become evident, we have included, on the r.h.s. of (7.46), the α_+-term that vanishes upon imposing the constraint (7.47).] Clearly the resulting anomaly has the opposite sign of the one given in (7.49). Physically, this symmetry property of (7.45) corresponds to replacing *electrons* (L) by *holes* (R) as the elementary charge carriers for the edge currents at the boundary of the sample. It reflects the fact that, for a given external magnetic field $\mathbf{B}_c$, electrons and holes circulate in opposite directions. From this it follows that we can continue our discussion by only considering the left-moving excitations along the boundary $\partial\Omega_o$, the corresponding equations for the right-moving ones following by applying this symmetry.

There is yet another way of deriving expression (7.46) that should be mentioned: We start from the Chern-Simons term $\exp\Big[-\frac{i\sigma}{4\pi}\int_{\Lambda_o}\tilde{a}\wedge \mathrm{d}\tilde{a}\Big]$ and perform a gauge transformation, $\tilde{a}\mapsto {}^{\varphi}\tilde{a} := \tilde{a} + \mathrm{d}\varphi$. Then, integrating the gauge-transformed Chern-Simons functional, $\exp\Big[-\frac{i\sigma}{4\pi}\int_{\Lambda_o}{}^{\varphi}\tilde{a}\wedge \mathrm{d}^{\varphi}\tilde{a}\Big]$, over all gauge transformations φ satisfying the boundary condition, $\partial_-\varphi - \alpha_- = 0$, where $\alpha = \tilde{a}|_{\partial\Lambda_o}$, we reproduce the same expressions as in Eqs. (7.45) and (7.46) by setting $\varphi = \frac{\kappa}{Q}\phi$. This procedure can also be applied in the non-abelian situation; see Fröhlich and Studer (1993b).

So far, the constants κ and Q are arbitrary real numbers. Next, we sketch how to find natural constraints on these two constants by making use of the representation theory of $\hat{u}(1)$ current algebra and some basic requirements on the spectrum of physical excitations in a Hall fluid. As a consequence of these constraints we find that σ has to take *rational* values!

The basic objects in the representation theory of chiral $\hat{u}(1)$ current algebra are the chiral vertex (Weyl) operators which play the role of Clebsch-Gordan operators. We recall some basic properties of these operators. It is convenient (see (7.37) and (7.38)) to introduce the coordinates

$$z := e^{2\pi i u_+/\ell}\,, \qquad \bar{z} := e^{2\pi i u_-/\ell}\,. \tag{7.50}$$

In terms of the left-moving Bose field ϕ_L given in (7.38), we define the *chiral vertex operators*

$$V_n(z) := \,:e^{in\phi_L(z)}:\,, \quad \text{with} \quad n\in\mathbb{R}\,, \tag{7.51}$$

where $:\;:$ denotes normal ordering (moving all α_n with $n>0$ to the right of α_m with $m<0$, and p to the right of q). Applying the commutation relations (7.39), one obtains the basic exchange (or Weyl) relations of chiral vertex operators,

$$V_n(z)\, V_m(w) \;=\; e^{\pm i\pi \frac{nm}{\kappa}}\, V_m(w)\, V_n(z)\;, \qquad z \neq w\;, \tag{7.52}$$

where the sign, $+$ or $-$, depends on the sign of $(\arg z - \arg w)$ relative to a fixed choice of a reference direction.

In the presence of an external gauge field α, the *charge operator* $\mathcal{Q}$ is defined by

$$\mathcal{Q} \;:=\; \oint_{|z|=\mathrm{const.}} \left[J_L(z) - \frac{Q}{\kappa}\, \alpha_z(z) \right] \frac{dz}{2\pi i\, z}\;, \tag{7.53}$$

where $\alpha_z = \alpha_+ \frac{\partial u_+}{\partial z}$. This operator is manifestly gauge invariant. Recalling Eqs. (7.37) and (7.39), we find that

$$[\mathcal{Q},\, V_n(z)] \;=\; \frac{n}{\kappa}\, V_n(z)\;. \tag{7.54}$$

Thus, the chiral vertex operator $V_n(z)$ creates a left-moving excitation of charge $q = \frac{n}{\kappa}$. It follows from (7.53) that, in order to change the charge of the system by an amount $\frac{n}{\kappa}$, the magnetic flux penetrating the system has to be changed by an amount n (in units where $-\frac{hc}{e} = 1$).

Next, we attempt to identify those chiral vertex operators (7.51) that when applied to the ground state create states with properties that are *consistent* with two basic assumptions about the physics of two-dimensional, incompressible quantum Hall fluids whose elementary charge carriers are (spin-polarized) electrons.

(a1) The system describes finite-energy excitations with the quantum numbers of a *(scalar) electron*, i.e., with *charge* 1 and obeying *Fermi statistics*.

(a2) The quantum-mechanical state vector describing an *arbitrary* physical state of the system is *single-valued* in the position coordinates of all those (finite-energy) excitations that are composed of *electrons* and/or *holes*.

Exploiting formulae (7.52) and (7.54) in the light of the two assumptions **(a1)** and **(a2)**, we find that $Q = \pm 1$ and $\kappa = 2p+1$, with $p = 0, 1, 2, \ldots$; see Fröhlich and Studer (1993b). Hence, by (7.49), we find that $\sigma = \frac{1}{2p+1}$, and the *Hall conductivity* $R_H^{-1} = \sigma \frac{e^2}{h}$ is a *rational* multiple of $\frac{e^2}{h}$. Our analysis thus provides a description of the celebrated Laughlin fluids (Laughlin, 1983a, 1983b) in an "edge-current picture". In particular, in the simplest example of $p = 0$, we are describing an integer quantum Hall fluid with $\sigma = \pm 1$, and the discussion above coincides with the one given in Subsect. 7.1.

In order to describe quantum Hall fluids with more general, rational values of the Hall conductivity (see Fig. 4.1), the ideas presented above must be generalized to "multi-channel chiral boundary systems"; see Fröhlich and Studer (1993b). This generalization leads naturally to the notion of a "*quantum Hall lattice*". In Subsects. 8.2 and 8.3, we review the defining properties of a quantum Hall lattice in a "bulk picture" of incompressible quantum Hall fluids. This complementary picture has been shown to be equivalent to the above "edge-current picture" (Fröhlich and Thiran, 1994). Equipped with the notion of a quantum Hall lattice, we shall describe, in Subsects. 8.3–8.6, the main classification results of incompressible quantum Hall fluids that have been derived in Fröhlich, Studer, and Thiran (1995), and Fröhlich, Kerler *et al.* (1995). For additional ideas worth thinking about, see Cappelli *et al.* (1995).

Our results can be interpreted as a "*gap labelling theorem*": Assuming *incompressibility* of a two-dimensional quantum Hall fluid, we prove that its Hall conductivity R_H^{-1} has to belong to a certain set of *rational* multiples of $\frac{e^2}{h}$. Conversely, if R_H^{-1} is *not* a rational multiple of $\frac{e^2}{h}$, the corresponding two-dimensional electron system *cannot* be incompressible, i.e., there *cannot* be a positive energy gap δ above the ground-state energy in the spectrum of the many-body Hamiltonian of this system; (see also Subsect. 4.5).

8 Classification of Incompressible Quantum Hall Fluids

In this section, we review a classification of *incompressible (dissipation-free) quantum Hall fluids* (or, for short, *QH fluids*) in terms of integral lattices and arithmetical invariants thereof. Our classification enables us to characterize the plateau values of the Hall conductivity σ_H in the interval $(0, 1]$ (in units where $e^2/h = 1$) corresponding to "stable" incompressible quantum Hall fluids. Theoretical results are carefully compared to experimental data, and various predictions and experimental implications of our theory are discussed.

8.1 QH Fluids and QH Lattices: Basic Concepts

Our analysis of the fractional quantum Hall effect is based on a precise notion of incompressible (dissipation-free) quantum Hall fluids – see assumptions **(A1)** through **(A4)** below – from which their main features can be derived; see Fröhlich and Studer (1992b, 1992c, 1993b), Fröhlich and Thiran (1994), Fröhlich, Studer, and Thiran (1995), and Fröhlich, Kerler *et al.* (1995). Our mathematical characterization, in terms of "QH lattices", of (universality classes of) QH fluids enables us to enumerate and classify QH fluids. In this subsection, we review the defining properties of QH lattices.

We recall from Subsect. 4.5 that the fractional QH effect is observed in two-dimensional gases of electrons at temperatures $T \approx 0\,K$ subject to a very nearly constant magnetic field, $\mathbf{B}_c$, transversal to the plane of the system. In Subsect. 6.1, we have characterized a QH fluid by the property that connected Green functions of the electric charge - and current densities have cluster decomposition properties stronger than those encountered in a metal or in a system where the electric charge - and current densities couple the ground state of the system to a Goldstone boson (as in a London superconductor). One can then show (see Subsect. 6.2) that the *longitudinal resistance*, R_L, of a QH fluid *vanishes*, which can also be used as a *definition* of an (incompressible) QH fluid.

The longitudinal resistance R_L is a complicated function of the filling factor ν (see (4.12)), and it is a difficult problem of many-body theory to predict where R_L vanishes as a function of ν. We do not solve this problem in this section. Instead, we show that *if* R_L *vanishes* then the *Hall conductivity* R_H^{-1} necessarily belongs to a certain set of *rational multiples* of e^2/h. Furthermore, given such a value of R_H^{-1}, we can determine the possible types of quasi-particles, i.e., the different "Laughlin vortices" of the system, their electric charges and their statistical phases.

Next, we describe the basic assumptions and physical principles underlying our analysis of QH fluids.

(A1) The temperature T of the system is close to $0\,K$. For an (incompressible) QH fluid at $T = 0\,K$, the *total electric charge* is a good quantum number to label physical states of the system describing finite-energy excitations above the ground state. The charge of the ground state of the system is normalized to be zero; see Sect. 5, Eqs. (5.11) - (5.12), and Fröhlich, Götschmann and Marchetti (1995a).

(A2) In the regime of very short wave vectors and low frequencies, the *scaling limit*, the total electric current density is the sum of $N = 1, 2, 3, \ldots$ *separately conserved* $u(1)$ current densities, describing electron and/or hole transport in N separate "channels" distinguished by conserved quantum numbers. [For a finite, but macroscopic sample, this assumption implies that there are N approximately separately conserved chiral $u(1)$ edge currents circulating around the boundary of the system.] In our analysis, we regard N as a free parameter. – Physically, N turns out to depend on the filling factor ν and other parameters characterizing the system.

(A3) In the units chosen in this section where $h = -e = 1$, the electric charge of an *electron/hole* is $+1/-1$. Any local finite-energy excitation (quasi-particle) above the ground state of the system with *integer* total electric charge q_{el} satisfies *Fermi-Dirac statistics* if q_{el} is *odd*, and *Bose-Einstein statistics* if q_{el} is *even*.

(A4) The quantum-mechanical state vector describing an *arbitrary* physical state of an (incompressible) QH fluid is *single-valued* in the position coordinates of all those local excitations that are composed of *electrons* and/or *holes*.

The basic contention advanced in Fröhlich and Studer (1992b, 1992c, 1993b) and Fröhlich and Thiran (1994) (see also Sect. 6) is that if a QH fluid is interpreted as a two-dimensional system of electrons with vanishing longitudinal resistance R_L satisfying assumptions **(A1)** through **(A4)** above then, in the scaling limit, its quantum-mechanical description is completely coded into a *quantum Hall lattice (QH lattice)*. A QH lattice $(\Gamma, \mathbf{Q})$ consists of an odd, integral lattice Γ and an integer-valued, linear functional $\mathbf{Q}$ on Γ. In general, the metric on Γ need not be positive definite. The number of positive eigenvalues of the metric corresponds, physically, to the number of edge currents of one chirality, the number of negative eigenvalues correponds to the number of egde currents of opposite chirality. In this section, we present results mainly on QH fluids with edge currents of just *one* chirality. These QH fluids correspond to QH lattices $(\Gamma, \mathbf{Q})$ where Γ is a *euclidian* lattice, i.e., a lattice with a *positive-definite* metric. These special QH lattices are called *chiral QH lattices (CQHLs)*.

Our study of *chiral* QH lattices is motivated by a physical hypothesis expressing a "*chiral factorization*" property of QH fluids.

(A5) The fundamental charge carriers of a QH fluid are electrons and/or holes. We assume that, in the scaling limit, the dynamics of electron-rich subfluids of a QH fluid is *independent* of the dynamics of hole-rich subfluids, and, up to charge conjugation, the theoretical analysis of an electron-rich subfluid is *identical* to that of a hole-rich subfluid.

A discussion of the "working hypothesis" **(A5)**, including some proposals for its experimental testing, is given in Fröhlich, Studer, and Thiran (1995). There, it is also shown that all "hierarchy fluids" of the Haldane-Halperin (Haldane, 1983; Halperin, 1984) and Jain-Goldman (1992) scheme, respectively, satisfy our assumptions **(A1–4)** and (a slightly weaker form of) assumption **(A5)**.

Chiral Quantum Hall Lattices (CQHLs)

Let V be an N-dimensional, real vector space equipped with a *positive-definite* inner product, $<.,.>$. In V we choose a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ such that

$$K_{ij} \;=\; K_{ji} \;:=\; <\mathbf{e}_i, \mathbf{e}_j> \,\in \mathbb{Z}\;, \qquad \text{for all } \; i,j = 1,\ldots,N\;, \tag{8.1}$$

i.e., *all* matrix elements of (K_{ij}) are *integers*. The basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ generates an *integral, euclidean* (i.e., positive-definite) *lattice*, Γ, defined by

$$\Gamma \;:=\; \{\, \mathbf{q} = \sum_{i=1}^{N} q^i \mathbf{e}_i \mid q^i \in \mathbb{Z},\; \text{for all } i = 1,\ldots,N \,\}\;. \tag{8.2}$$

The matrix K in (8.1) is called the *Gram matrix* of the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ generating the lattice Γ. Let $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^N\}$ be the basis of V *dual* to $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$, i.e., the basis satisfying

$$<\boldsymbol{\varepsilon}^i, \mathbf{e}_j> \;=\; \delta^i_j\;, \qquad \text{for all } \; i,j = 1,\ldots,N\;. \tag{8.3}$$

Then,

$$\boldsymbol{\varepsilon}^i \;=\; \sum_{j=1}^{N} (K^{-1})^{ij} \mathbf{e}_j\;, \tag{8.4}$$

where K^{-1} is the inverse of the matrix $K = (K_{ij})$, with K_{ij} given in (8.1), and $(K^{-1})^{ij} = <\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j>$. The basis $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^N\}$ generates the *dual lattice*, Γ^*, defined by

$$\begin{aligned} \Gamma^* \;&:=\; \{\, \mathbf{n} \in V \mid \; <\mathbf{n}, \mathbf{q}> \in \mathbb{Z},\; \text{for all } \mathbf{q} \in \Gamma \,\} \\ &=\; \{\, \mathbf{n} = \sum_{i=1}^{N} n_i\, \boldsymbol{\varepsilon}^i \mid n_i \in \mathbb{Z},\; \text{for all } i = 1,\ldots,N \,\}\;, \end{aligned} \tag{8.5}$$

and, by (8.4), Γ^* contains Γ. Denoting by $\Delta := \det K \in \mathbb{Z}$ the determinant of the matrix K ($\det K > 0$ for euclidean lattices), the matrix elements $(K^{-1})^{ij}$ corresponding to the scalar products between basis vectors in Γ^* are, in general (for $\Delta \neq 1$), *rational* numbers; since, by *Kramer's rule*, we have

$$(K^{-1})^{ij} = (\widetilde{K})^{ij}/\Delta \ , \tag{8.6}$$

with $\widetilde{K}$ the integer-valued matrix of cofactors of the matrix K.

The (equivalence) classes of elements in Γ^* modulo Γ form an abelian group, $\Gamma^*/\,\Gamma$. There are $\Delta = |\Gamma^*/\,\Gamma|$ distinct classes; Δ is often referred to as the lattice *discriminant*. An integral lattice Γ is said to be *odd* if it contains a vector $\mathbf{q}$ such that $<\mathbf{q}\,,\mathbf{q}>$ is an *odd* integer. Thus Γ is odd if, and only if, $K_{ii} = <\mathbf{e}_i\,,\mathbf{e}_i>$ is odd for at least one i. (Otherwise, Γ is said to be *even*.)

Given a set of integers, $n_1, \ldots, n_N$, we denote by $\gcd(n_1, \ldots, n_N)$ their *greatest common divisor*. A vector $\mathbf{n} = \sum_{i=1}^{N} n_i\, \boldsymbol{\varepsilon}^i \in \Gamma^*$ is called *primitive* (or "visible") if, and only if,

$$\gcd(n_1, \ldots, n_N) \ = \ \gcd(<\mathbf{n}\,,\mathbf{e}_1>, \ldots, <\mathbf{n}\,,\mathbf{e}_N>) \ = \ 1 \ . \tag{8.7}$$

In geometrical terms, a vector $\mathbf{n} \in \Gamma^*$ is primitive if, and only if, the line segment joining the origin to the point $\mathbf{n}$ is free of any lattice point. In particular, one can always take a primitive vector as the first vector of a lattice basis.

A lattice Γ is said to be *decomposable* if it can be written as an orthogonal direct sum of sublattices,

$$\Gamma \ = \ \bigoplus_{j=1}^{r} \Gamma_j \ , \qquad \text{for some } \ r \geq 2 \ , \tag{8.8}$$

with the property that $<\mathbf{q}^{(i)}, \mathbf{q}^{(j)}> = 0$, for arbitrary vectors $\mathbf{q}^{(i)}$ in Γ_i and $\mathbf{q}^{(j)}$ in Γ_j, and for arbitrary $i \neq j$. Otherwise, Γ is said to be *indecomposable*.

If m and n are two integers we shall write $m \equiv n \bmod p$ if, and only if, $m - n$ is an integer multiple of p.

We are now prepared to define what we mean by a *chiral QH lattice (CQHL)*:

Definition 8.1.1. *A CQHL is a pair* $(\Gamma, \mathbf{Q})$ *where* Γ *is an odd, integral euclidean lattice, and* $\mathbf{Q}$ *is a primitive vector in the dual lattice* Γ^* *with the property that*

$$<\mathbf{Q}\,,\mathbf{q}> \equiv <\mathbf{q}\,,\mathbf{q}> \mod 2 \ , \qquad \textit{for all } \ \mathbf{q} \in \Gamma \ , \tag{8.9}$$

i.e., $\mathbf{Q}$ *is an* odd *vector in the dual lattice* Γ^*.

Let $(\Gamma, \mathbf{Q})$ be a CQHL for which $\Gamma = \bigoplus_{j=1}^{r} \Gamma_j$ is decomposable. Then $\Gamma^* = \bigoplus_{j=1}^{r} \Gamma_j^*$ is the associated decomposition of the dual lattice. We say that $(\Gamma, \mathbf{Q})$ is *proper* if, in the decomposition of $\mathbf{Q}$

$$\mathbf{Q} \;=\; \sum_{j=1}^{r} \mathbf{Q}^{(j)} \;, \qquad \text{with} \qquad \mathbf{Q}^{(j)} \;:=\; \mathbf{Q}|_{\Gamma_j^*} \in \Gamma_j^* \;, \tag{8.10}$$

corresponding to the decomposition of Γ^*, every $\mathbf{Q}^{(j)}$ is *non-zero*. In this section we discard improper CQHLs; see the discussion at the end of Subsect. 8.2.

Definition 8.1.2. *A CQHL is called* primitive *if, in the decomposition (8.10) of* $\mathbf{Q}$*, the component* $\mathbf{Q}^{(j)}$ *is a* primitive *vector in* Γ_j^**, for all* $j = 1, \ldots, r$*.*

We note that any *indecomposable* CQHL (i.e., one with $r = 1$ in (8.8) and (8.10)) is proper and primitive.

8.2 A Dictionary Between the Physics of QH Fluids and the Mathematics of QH Lattices

In this subsection, we provide a precise dictionary between mathematical properties of QH lattices and physical properties of QH fluids. Such a dictionary has been presented in Fröhlich and Studer (1993a, 1993b), and Fröhlich and Thiran (1994). Here we recall its main contents and significance. The starting point is the idea to describe the physics of a QH fluid in the scaling limit in terms of an effective field theory of its conserved current densities; see Sect. 5. Since a QH fluid has a strictly positive mobility gap δ, the scaling limit of the effective theory of its conserved current densities must be a *topological field theory* (Fröhlich and Zee, 1991; Fröhlich and Studer, 1992b). The presence of a non-zero external magnetic field transversal to the plane to which the electrons are confined implies that the quantum dynamics of the system *violates* the symmetries of parity (reflections-in-lines) and time reversal. Thus the topological field theory will *not* be parity - and time-reversal invariant.

In $2+1$ space-time dimensions, the continuity equation

$$\frac{\partial}{\partial t} j^0 + \vec{\nabla} \cdot \vec{j} \;=\; 0 \;, \tag{8.11}$$

obeyed by a conserved current density $j^\mu = (j^0, \vec{j}\,)$, implies that j^μ is the curl of a vector potential, i.e., $j^\mu = \varepsilon^{\mu\nu\lambda}\, \partial_\nu b_\lambda$, for a vector potential $b = (b_0, \vec{b}\,)$ that is unique up to the gradient of a scalar field (gauge invariance!); see Sect. 5.

Let us consider a QH fluid which, in the scaling limit, has N independent, conserved current densities, $j^{\mu 1}, \ldots, j^{\mu N}$. The electric current density, J^{μ}_{el}, is then a linear combination, $J^{\mu}_{el} = \sum_{i=1}^{N} Q_i \, j^{\mu i}$, of these current densities. When formulated in terms of the vector potentials $b^1, \cdots, b^N$ of these conserved current densities, the topological field theory describing the physics of the QH fluid in the scaling limit can only be a *pure, abelian Chern-Simons theory* in the fields $b^1, \cdots, b^N$, because it must violate parity - and time-reversal invariance. The bulk action is given by

$$S(\mathbf{b}) \;=\; \frac{1}{4\pi} \int C_{ij} \, b^i_\mu \, \partial_\nu b^j_\lambda \, \varepsilon^{\mu\nu\lambda} \, dt d^2x \;=:\; \frac{1}{4\pi} \int <\mathbf{b}_\mu \, , \partial_\nu \mathbf{b}_\lambda> \, \varepsilon^{\mu\nu\lambda} \, dt d^2x \;, \tag{8.12}$$

where $C_{ij} = C_{ji}$ is some non-degenerate quadratic form (metric) on $\mathbb{R}^N$.

Physical states of a pure, abelian Chern-Simons theory describe static N-tuples of "charges" localized in bounded disks of space. Each N-tuple, $(q^1, \ldots, q^N)$, of "charges" localized in some disk D of space is an N-tuple of eigenvalues of the operators

$$\int_D j^{0i}(\vec{x}, t) \, d^2x \;=\; \oint_{\partial D} \vec{b}^i(\vec{x}, t) \cdot d\vec{x} \;, \qquad i = 1, \ldots, N \;. \tag{8.13}$$

The equations of motion of pure, abelian Chern-Simons theory, with the currents $j^{\mu i}$ minimally coupled to N external gauge fields $a_{\mu i}$, $i = 1, \ldots, N$, read

$$\varepsilon_{\mu\nu\rho} j^{\rho i} \;=\; \partial_\mu b^i_\nu - \partial_\nu b^i_\mu \;=\; (C^{-1})^{ij} \, (\partial_\mu a_{\nu j} - \partial_\nu a_{\mu j}) \;=:\; (C^{-1})^{ij} f_{\mu\nu j} \;, \tag{8.14}$$

These equations imply that, to an N-tuple of charges, $(q^1, \ldots, q^N)$, there corresponds an N-tuple of "fluxes", $(\varphi_1, \ldots, \varphi_N)$, with $\varphi_j = \int_D f_{12j} \, d^2x$, related to q^i through

$$q^i \;=\; (C^{-1})^{ij} \varphi_j \;, \qquad i = 1, \ldots, N \;. \tag{8.15}$$

Consider a state of Chern-Simons theory describing two excitations with *identical* charges $(q^1, \ldots, q^N)$ localized in disjoint, congruent disks of space, D_1 and D_2. From the theory of the *Aharonov-Bohm effect* we know that if the positions of the two disks are exchanged adiabatically along counter-clockwise oriented paths, the state vector only changes by a phase factor, $\exp i\pi\theta$, given by

$$\exp i\pi\theta \;=\; \exp i\pi <\mathbf{q} \, , \mathbf{q}> \;, \tag{8.16}$$

where

$$\theta \;=\; \sum_{i=1}^{N} \varphi_i \, q^i \;=\; \sum_{i=1}^{N} (C^{-1})^{ij} \varphi_i \, \varphi_j \;=\; \sum_{i=1}^{N} C_{ij} \, q^i q^j \;=:\, <\mathbf{q} \, , \mathbf{q}> \;. \tag{8.17}$$

It is well known that $\exp i\pi <\mathbf{q}\,,\mathbf{q}>$ has the meaning of a *statistical phase* of the excitation corresponding to the charges $(q^1, \ldots, q^N)$. The N-dimensional vector $\mathbf{q}$ introduced here is equivalently defined through its components $(q^1, \ldots, q^N)$, or, through (8.15), in terms of its dual (w.r.t. the metric C_{ij}) components $(\varphi_1, \ldots, \varphi_N)$, the "fluxes". An excitation labelled by $\mathbf{q}$ with $<\mathbf{q}\,,\mathbf{q}> \equiv 1 \bmod 2$ is a fermion, while if $<\mathbf{q}\,,\mathbf{q}> \equiv 0 \bmod 2$ it represents a boson.

Next, we consider a state describing two excitations with two *different* charge vectors $\mathbf{q}_{(1)}$ and $\mathbf{q}_{(2)}$ localized in disjoint disks D_1 and D_2. Imagine that D_2 is transported around D_1 adiabatically, along a counter-clockwise oriented loop. Then the theory of the Aharonov-Bohm effect teaches us that the state vector only changes by a phase factor given by

$$\exp 2i\pi <\mathbf{q}_{(1)}\,,\mathbf{q}_{(2)}> \ , \tag{8.18}$$

where

$$<\mathbf{q}_{(1)}\,,\mathbf{q}_{(2)}> \ := \ \sum_{i=1}^{N} C_{ij}\, q^i_{(1)} q^j_{(2)} \ = \ \sum_{i=1}^{N} \varphi_{(1)i}\, q^i_{(2)} \ = \ \sum_{i=1}^{N} (C^{-1})^{ij} \varphi_{(1)i}\, \varphi_{(2)j} \tag{8.19}$$

Since the charges, q^i, of physical states – i.e., the eigenvalues of the operators $\int_D j^{0i}(\vec{x},t)\, d^2x$, $i = 1, \ldots, N$ – are *additive* quantum numbers, the set of vectors $\mathbf{q}$ of physical excitations of a QH fluid form a *lattice*, Γ_{phys}, whose dimension can be taken to be N. Expressing the electric current density J^μ_{el} as a linear combination, $J^\mu_{el} = \sum_{i=1}^N Q_i\, j^{\mu i}$, of the current densities $j^{\mu 1}, \ldots, j^{\mu N}$, the *total electric charge*, $q_{el}(\mathbf{q})$, of an excitation of the system with charge vector $\mathbf{q}$ localized in some disk D is found to be given by

$$q_{el}(\mathbf{q}) \ = \ \sum_{i=1}^{N} Q_i\, q^i \ =: <\mathbf{Q}\,,\mathbf{q}> \ , \tag{8.20}$$

where the electric charge assignments Q_i of each current have been collected in an N-dimensional vector $\mathbf{Q}$ henceforth referred to as the "charge vector". Note that, according to the above definition, the Q_i are the *dual* components of the charge vector $\mathbf{Q}$. The electric charge $q_{el}(\mathbf{q})$ is an eigenvalue of the electric charge operator

$$\int J^0_{el}(\vec{x},t)\, d^2x \ = \ \int \left(\sum_{i=1}^{N} Q_i\, j^{0i}(\vec{x},t) \right) d^2x \ . \tag{8.21}$$

We define Γ as the set of all physical excitations $\mathbf{q}$ with *integral* electric charge, i.e.,

$$\Gamma \ := \ \{\, \mathbf{q} \in \Gamma_{phys} \,|\ <\mathbf{Q}\,,\mathbf{q}> = q_{el}(\mathbf{q}) \in \mathbb{Z} \,\} \ . \tag{8.22}$$

Clearly Γ is a sublattice of Γ_{phys}. If Γ has at least one excitation $\mathbf{q}$ of electric charge $q_{el}(\mathbf{q}) = 1$ then the dimension of Γ is equal to N. Assumption **(A3)** entails that arbitrary vectors $\mathbf{q}_{(i)}$ in Γ must have integer statistical phases, $<\mathbf{q}_{(i)}\,,\mathbf{q}_{(i)}>$, related to their electric charges by the congruence $q_{el}(\mathbf{q}_{(i)}) \equiv <\mathbf{q}_{(i)}, \mathbf{q}_{(i)}> \mod 2$. Writing the scalar product $<\mathbf{q}_{(1)}\,, \mathbf{q}_{(2)}>$ between two arbitrary charge vectors in Γ as

$$<\mathbf{q}_{(1)}\,,\mathbf{q}_{(2)}> \;=\; \frac{1}{2}\Big(<\mathbf{q}_{(1)}+\mathbf{q}_{(2)}\,,\mathbf{q}_{(1)}+\mathbf{q}_{(2)}> \;-\; <\mathbf{q}_{(1)}\,,\mathbf{q}_{(1)}> \;-\; <\mathbf{q}_{(2)}\,,\mathbf{q}_{(2)}>\Big)\;,$$

we see that this is always an integer and conclude that Γ is an *integral* lattice. Since there is at least one electron-like excitation $\mathbf{q}$ in Γ with $q_{el}(\mathbf{q}) = 1$, **(A3)** also constrains the lattice Γ to be *odd*, because $<\mathbf{q}\,,\mathbf{q}> \equiv q_{el}(\mathbf{q}) \equiv 1 \bmod 2$. For an arbitrary lattice basis $\{\mathbf{q}_{(1)}, \ldots, \mathbf{q}_{(N)}\}$ of Γ, all the scalar products $Q_i = <\mathbf{Q}\,,\mathbf{q}_{(i)}>$, $i = 1, \ldots, N$, are integers, and hence the vector $\mathbf{Q}$ belongs to the *dual*, Γ^*, of Γ; (see (8.5)). The charge vector $\mathbf{Q}$ must necessarily be *primitive*–i.e., $\gcd(Q_1, \ldots, Q_N) = 1$–in order for an excitation of electric charge 1 to exist.

Every vector $\mathbf{q}$ of the sublattice Γ of Γ_{phys} can now be consistently interpreted as labelling a physical excitation describing a (multi-)electron and/or (multi-)hole configuration (depending on the integral value of the electric charge). Note that, starting from an electron-like excitation $\mathbf{q}_{(1)}$ in Γ, one can construct a basis of Γ consisting of electron vectors $\mathbf{q}_{(i)}$, with $q_{el}(\mathbf{q}_{(i)}) = 1$, for $i = 1, \ldots, N$. We call such a basis a "symmetric" lattice basis.

The next step consists in finding restrictions on the lattice Γ_{phys} by making use of our last assumption **(A4)**. Consider, for each $i = 1, \ldots, N$, a physical state of the system describing an excitation with "charges" $\mathbf{q}'$ localized in a disk D_1 and an excitation corresponding to an electron with vector $\mathbf{q}_{(i)}$ localized in a disk D_2 disjoint from D_1. Then we derive from assumption **(A4)** and (8.18) that

$$\exp 2\pi i <\mathbf{q}', \mathbf{q}_{(i)}> \;=\; 1\;, \tag{8.23}$$

i.e., $<\mathbf{q}', \mathbf{q}_{(i)}>$ must be an *integer*, for all $i = 1, \ldots, N$. From this it follows that $\mathbf{q}'$ belongs to the dual lattice Γ^*. Thus, vectors $\mathbf{q}'$ of *arbitrary* physical excitations of a QH fluid described by the lattice Γ and the charge vector $\mathbf{Q}$ must all belong to the lattice Γ^* dual to Γ. We then have the following inclusions between the various lattices introduced so far:

$$\Gamma^* \;\supseteq\; \Gamma_{phys} \;\supseteq\; \Gamma\;. \tag{8.24}$$

To discover how the data $(\Gamma, \mathbf{Q})$ predict the value of the Hall conductivity σ_H, we consider the effect of turning on a perturbing external magnetic field $\mathbf{B} = \mathbf{B}_{total} - \mathbf{B}_c$ localized in a small disk D inside the system and creating a total magnetic flux Φ, with

$$\Phi \;=\; \int_D B(\vec{x})\, d^2x \;, \tag{8.25}$$

where B is the component of $\mathbf{B}$ perpendicular to the plane of the system. Since the total electric current density J^μ_{el} is given by $J^\mu_{el} = \sum_{i=1}^N Q_i\, j^{\mu i}$, the "flux" φ_i created by $\mathbf{B}$ is given by

$$\varphi_i \;=\; Q_i\, \Phi \;, \tag{8.26}$$

for $i = 1, \ldots, N$. This follows from the way in which, in Chern-Simons theory, the currents $j^{\mu i}$ are coupled to an external gauge field. Let $A = (A_0, \vec{A}\,)$ denote the electromagnetic vector potential of the external magnetic field $\mathbf{B}$. Then from the fact that the total electric current J^μ_{el} couples to the vector potential A_μ through the term $J^\mu_{el}\, A_\mu$, it follows that the currents $j^{\mu i}$ are minimally coupled to the field $Q_i\, A_\mu$. Thus the gauge fields $a_{\mu i}$ appearing on the r.h.s. of (8.14) are given by $a_{\mu i} = Q_i\, A_\mu$ which implies (8.26); see Fröhlich and Studer (1992b, 1992c), Fröhlich and Thiran (1994), and Subsect. 7.2. Since the charges $q^i = \int_D j^{0\,i}(\vec{x}, t)\, d^2x = \oint_{\partial D} \vec{b}^i(\vec{x}, t)\cdot d\vec{x}$, corresponding to the fluxes φ_k, are equal to $(C^{-1})^{ik}\, \varphi_k$, by the equations of motion (8.14) of Chern-Simons theory, we conclude that the *total excess electric charge* created by the excess magnetic flux Φ is given by

$$\begin{aligned} q_{el}(\Phi) \;&=\; \sum_{i=1}^N Q_i\, q^i \;=\; \sum_{i,k=1}^N Q_i\, (C^{-1})^{ik}\, \varphi_k \\ &=\; \left(\sum_{i,k=1}^N Q_i\, (C^{-1})^{ik}\, Q_k \right) \Phi \;=:\; <\mathbf{Q}\, , \mathbf{Q}>\, \Phi \;. \end{aligned} \tag{8.27}$$

Comparing this equation to the equations describing the electrodynamics of a QH fluid which have been described in Subsect. 4.5, in particular, to Eq. (4.17),

$$J^0_{el} \;=\; \sigma_H\, B \;,$$

we find that the coefficient of Φ on the r.h.s. of (8.27) is the *Hall conductivity*, σ_H, of the QH fluid. Thus we arrive at the fundamental equation

$$\sigma_H \;=\; <\mathbf{Q}\, , \mathbf{Q}> \quad . \tag{8.28}$$

Since, as shown above, $\mathbf{Q}$ belongs to the dual lattice Γ^*, we have that

$$\mathbf{Q} \;=\; \sum_{i=1}^{N} Q_i \, \boldsymbol{\varepsilon}^i \;,$$

with $Q_i \in \mathbb{Z}$, for $i = 1, \ldots, N$, where $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^N\}$ is the basis of Γ^* dual to a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ of the *integral* lattice Γ with Gram matrix $K_{ij} = <\mathbf{e}_i, \mathbf{e}_j> \in \mathbb{Z}$. We could choose, for example, a symmetric basis of electron-like excitations, i.e., $\mathbf{e}_i = \mathbf{q}_{(i)}$, for all $i = 1, \ldots, N$. In this situation the electric charge requirements fix the coefficients $Q_i = 1$, for all $i = 1, \ldots, N$, since we must have that $q_{el}(\mathbf{e}_i) = <\mathbf{Q}\,, \mathbf{e}_i> = Q_i = 1$. But any other choice of basis is admissible as well. The all important consequence of $\mathbf{Q}$ being a vector in the dual of Γ is that its squared length $<\mathbf{Q}\,,\mathbf{Q}>$ is a *rational number*. To see this, we return to the discussion after (8.5). We have that $<\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j> = (K^{-1})^{ij} = (\widetilde{K})^{ij}/\Delta$, where $\Delta = \det K$ (the lattice discriminant) is an integer, and $(\widetilde{K})^{ij}$ are integers. It follows that

$$<\mathbf{Q}\,,\mathbf{Q}> \;=\; \sum_{i,j=1}^{N} Q_i\, Q_j <\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j> \;=\; \left(\sum_{i,j=1}^{N} Q_i\, (\widetilde{K})^{ij}\, Q_j \right) \Delta^{-1} \tag{8.29}$$

is a rational number whose denominator is a *divisor* of Δ; (in general there will be non-trivial common divisors of Δ and of the numerator of the expression on the r.h.s. of (8.29)). We conclude that the Hall conductivity σ_H of a QH fluid satisfying assumptions **(A1)** through **(A4)** is necessarily a *rational number*.

The analysis just completed shows that, in the scaling limit, the physics of a QH fluid is described by a pair $(\Gamma, \mathbf{Q})$ of an integral, odd lattice Γ and a primitive vector $\mathbf{Q}$ in the dual lattice, with $<\mathbf{Q}\,,\mathbf{q}> \equiv <\mathbf{q}\,,\mathbf{q}>$ mod 2, for all vectors $\mathbf{q}$ in Γ, i.e., by data that we have termed *QH lattice* in Subsect. 8.1.

In the following, we mainly specialize to *chiral* QH lattices (CQHLs) which are QH lattices where the lattice Γ is *euclidean*. They describe (universality classes of) QH fluids with edge currents of *definite chirality*, the basic charge carriers being, say, electrons. The theory of an electron-rich QH fluid differs from that of a hole-rich QH fluid only in relative minus signs in all equations relating charges to fluxes and involving the electric charge of the basic charge carriers. One simply reverses the sign of the charge vector $\mathbf{Q}$ and the metric C : $\mathbf{Q} \to -\mathbf{Q}$ and $C_{ij} \to -C_{ij}$, i.e., $<.\,,.> \to -<.\,,.>$. A more general type of QH fluids consisting of electron- *and* hole-rich subsystems is therefore described by *two* CQHLs, $(\Gamma_e, \mathbf{Q}_e)$ and $(\Gamma_h, \mathbf{Q}_h)$, with

$$\sigma_H \;=\; <\mathbf{Q}_e, \mathbf{Q}_e> \;-\; <\mathbf{Q}_h, \mathbf{Q}_h> \quad . \tag{8.30}$$

In such QH fluids, the subsystems described by $(\Gamma_e, \mathbf{Q}_e)$ and by $(\Gamma_h, \mathbf{Q}_h)$ are independent of each other; (in particular, left- and right-moving edge excitations are independent of each other). The mathematical properties of $(\Gamma_e, \mathbf{Q}_e)$ and of $(\Gamma_h, \mathbf{Q}_h)$ are analogous. It is therefore sufficient to study the properties of $(\Gamma_e, \mathbf{Q}_e)$, say, and we shall omit the subscript "e" henceforth. Note that this "factorized" situation implements assumption **(A5)** of Subsect. 8.1.

As in (8.8) and (8.10), let $\Gamma = \bigoplus_{j=1}^r \Gamma_j$ be the decomposition of the lattice Γ into an orthogonal direct sum of indecomposable sublattices Γ_j, and let $\mathbf{Q} = \sum_{j=1}^r \mathbf{Q}^{(j)}$, with $\mathbf{Q}^{(j)} \in \Gamma_j^*$, be the associated decomposition of the charge vector $\mathbf{Q}$. Then, by (8.28),

$$\sigma_H = <\mathbf{Q}\,, \mathbf{Q}> = \sum_{j=1}^{r} <\mathbf{Q}^{(j)}, \mathbf{Q}^{(j)}> = \sum_{j=1}^{r} \sigma_H^{(j)} \tag{8.31}$$

is the corresponding decomposition of the Hall conductivity (or *Hall fraction*) as a sum of Hall fractions of subfluids described by the pairs $(\Gamma_j, \mathbf{Q}^{(j)})$, $j = 1, \ldots, r$. Let us imagine that, for some j, $\mathbf{Q}^{(j)} = 0$. Then $\sigma_H^{(j)} = 0$, and the subfluid corresponding to $(\Gamma_j, \mathbf{Q}^{(j)})$ does not have any interesting electric properties. For the purpose of describing electric properties of QH fluids and classifying the possible values of the Hall fraction σ_H of QH fluids, subfluids described by $(\Gamma_j, \mathbf{Q}^{(j)})$, with $\mathbf{Q}^{(j)} = 0$, can therefore be discarded. Thus we may henceforth assume that $\mathbf{Q}^{(j)} \neq 0$, for all $j = 1, \ldots, r$, i.e., we may limit our analysis to *proper* CQHLs; see (8.10). [However, if there are QH fluids with spin-charge separation, subfluids $(\Gamma_j, \mathbf{Q}^{(j)})$ with $\mathbf{Q}^{(j)} = 0$ will appear, and spin currents could receive contributions from such neutral subfluids; see Subsect. VI.C in Fröhlich and Studer (1993b).]

Next, suppose that, for some j with $1 \leq j \leq r$, $\mathbf{Q}^{(j)}$ is an integer multiple of a *primitive* vector $\mathbf{Q}_*^{(j)} \in \Gamma_j^*$, i.e., $\mathbf{Q}^{(j)} = \nu_j\, \mathbf{Q}_*^{(j)}$, with $\nu_j \geq 2$; see (8.7). Then, for *any* $\mathbf{q} \in \Gamma_j$, $<\mathbf{Q}^{(j)}, \mathbf{q}>$ is an *integer multiple* of ν_j. This would mean that the electric charge of an *arbitrary* quasi-particle of the subfluid described by $(\Gamma_j, \mathbf{Q}^{(j)})$ would be an integer multiple of ν_j (in units where $-e = 1$), i.e., only bound states of electrons and holes of electric charge $n\nu_j$, $n \in \mathbb{Z}$, would appear as quasi-particles of such a subfluid. There appear to exist QH fluids where this situation arises (e.g. "hierarchy QH fluids", see Appendix E in Fröhlich, Studer, and Thiran (1995), or films of superfluid He^3, see Subsect. VII.B in Fröhlich and Studer (1993b)). But for simplicity, we shall assume henceforth that $\nu_j = 1$, i.e., that $\mathbf{Q}^{(j)}$ is a *primitive* vector in Γ_j^*, for each $j = 1, \ldots, r$. In other words, we limit our analysis of CQHLs to what we have called *primitive* CQHLs; see Def. 8.1.2 at the end of the previous subsection.

8.3 Basic Invariants of Chiral QH Lattices (CQHLs) and their Physical Interpretations

Let $(\Gamma, \mathbf{Q})$ be a *chiral* QH lattice (CQHL), i.e., a QH lattice describing (a universality class of) QH fluids with edge currents of a *definite chirality.* In this subsection, we describe elementary mathematical properties of $(\Gamma, \mathbf{Q})$. They are formulated in terms of *invariants* of $(\Gamma, \mathbf{Q})$. Numerical invariants of a CQHL $(\Gamma, \mathbf{Q})$ are numbers which only depend on its *intrinsic* properties. They are *independent* of the choice of a basis in Γ and of a "reshuffling" of electric charge assignments corresponding to a transformation of $\mathbf{Q}$ by an orthogonal symmetry of Γ.

Choosing a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$, in Γ, the CQHL is specified by the (integral) Gram matrix $K_{ij} = <\mathbf{e}_i, \mathbf{e}_j>$, $i, j = 1, \ldots, N$, and by the row vector $\underset{\rightarrow}{Q} = (Q_1, \ldots, Q_N)$ which specifies the components of the charge vector $\mathbf{Q}$ in the dual basis $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^N\}$ of Γ^*, i.e., $\mathbf{Q} = \sum_{j=1}^N Q_j \, \boldsymbol{\varepsilon}^j$; see Eqs. (8.1) through (8.5). Choosing a different basis in Γ, the resulting pair $(K', \underset{\rightarrow}{Q}')$ is related to the pair $(K, \underset{\rightarrow}{Q})$ by

$$K' \; = \; S^T \, K \, S \; , \qquad \text{and} \qquad \underset{\rightarrow}{Q}' \; = \underset{\rightarrow}{Q} \, S \; , \tag{8.32}$$

where S is an element of $GL(N, \mathbb{Z})$, the group of integral, non-degenerate $N \times N$-matrices. Note that, for S^{-1} to be an element of the group, the determinant of any element S of $GL(N, \mathbb{Z})$ has to be ± 1.

A concise presentation of the numerical invariants of a CQHL, $(\Gamma, \mathbf{Q})$, is given by the *symbol*

$${}_N\!\left(\sigma_H = \frac{n_H}{d_H}\right)_{\lambda}^{g} \, [\ell_{min}, \ell_{max}] \; , \tag{8.33}$$

where the invariants summarized in the symbol have the following mathematical definitions and physical interpretations:

(1) $N := \dim \Gamma = \operatorname{rank} \Gamma$, the *lattice dimension* N, is equal to the number of separately conserved $u(1)$ current densities of the corresponding QH fluid, (in the scaling limit). Although no rigorous results are known, a mean field theory of the FQHE suggests that N depends on the filling factor ν and on the density and strength of impurities (disorder) in the system. We expect that the absolute upper bound, N_*, on the dimension N of physically realizable CQHLs tends to ∞, as the density and strength of impurities tend to 0; see Fröhlich, Kerler *et al.* (1995).

(2) By (8.28) and (8.32), the *Hall conductivity* (or *Hall fraction*) σ_H is clearly a CQHL invariant: $\sigma_H = <\mathbf{Q}\,, \mathbf{Q}> = \underset{\rightarrow}{Q} \cdot K^{-1} \underset{\rightarrow}{Q}^T$. By (8.6) and the definition of $\underset{\rightarrow}{Q}$, it is a positive *rational* number.

(3) Writing $\sigma_H = n_H/d_H$, with $\gcd(n_H, d_H) = 1$, the important invariant of the lattice given by its *discriminant* Δ can be written as

$$\Delta \;:=\; \det K \;=\; l\, d_H \;, \tag{8.34}$$

where the invariant l is called the *level* of the CQHL $(\Gamma, \mathbf{Q})$; see (8.6).

(4) The level l satisfies an interesting factorization property, namely $l = g\lambda$, where g is defined by $g := \gcd(Q^1, \ldots, Q^N)$, with $Q^j := \Delta < \mathbf{Q}\,, \boldsymbol{\varepsilon}^j >$, $j = 1, \ldots, N$, and $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^N\}$ *any* dual basis of Γ^*. Thus, by (8.34), the discriminant is given by $\Delta = g\lambda d_H$. The invariant λ is called the *charge parameter*, and its physical significance derives from the following fact: one can prove (Fröhlich and Thiran, 1994) that, in our units where $e = -1$, the smallest possible (fractional) electric charge of a quasi-particle of the corresponding QH fluid is given by

$$e^* \;:=\; \min_{\mathbf{n}\in\Gamma^*,\, <\mathbf{Q}\,,\mathbf{n}>\neq 0} |<\mathbf{Q}\,, \mathbf{n}>| \;=\; \frac{1}{\lambda d_H} \;. \tag{8.35}$$

(5) Next, we define the *relative-angular-momentum invariants* ℓ_{min} and ℓ_{max}. Since $\mathbf{Q}$ is a primitive vector in Γ^* (see (8.7)) there is a basis of Γ, $\{\mathbf{q}_1, \ldots, \mathbf{q}_N\}$, such that $<\mathbf{Q}\,, \mathbf{q}_i> = 1$, $i = 1, \ldots, N$. The set of all such "symmetric" bases is denoted by $\mathcal{B}_{\mathbf{Q}}$. Then, for any CQHL $(\Gamma, \mathbf{Q})$, we define the invariants

$$L_{min} \;:=\; \min_{\mathbf{q}\in\Gamma,\, <\mathbf{Q}\,,\mathbf{q}>=1} <\mathbf{q}, \mathbf{q}> \;, \tag{8.36}$$

and

$$L_{max} \;:=\; \min_{\{\mathbf{q}_1,\ldots,\mathbf{q}_N\}\in\mathcal{B}_{\mathbf{Q}}} \left(\max_{1\leq i\leq N} <\mathbf{q}_i, \mathbf{q}_i> \right) \;. \tag{8.37}$$

In the situation where $(\Gamma, \mathbf{Q})$ is a *primitive, decomposable* CQHL with decomposition $(\Gamma, \mathbf{Q}) = \oplus_{j=1}^k (\Gamma_j, \mathbf{Q}^{(j)})$ (see Definition 8.1.2) it is natural to refine the definitions (8.36) and (8.37) as follows:

$$\ell_{min}(\Gamma, \mathbf{Q}) \;:=\; \min_{1\leq j\leq k} L_{min}(\Gamma_j, \mathbf{Q}^{(j)}) \;\geq\; L_{min}(\Gamma, \mathbf{Q}) \;, \tag{8.38}$$

and

$$\ell_{max}(\Gamma, \mathbf{Q}) \;:=\; \max_{1\leq j\leq k} L_{max}(\Gamma_j, \mathbf{Q}^{(j)}) \;\geq\; L_{max}(\Gamma, \mathbf{Q}) \;. \tag{8.39}$$

We note that, because $\mathbf{Q}$ is an odd vector in Γ^* (see (8.9)), the relative-angular-momentum invariants (8.36) through (8.39) are positive, *odd* integers satisfying

$$L_{min} \le L_{max} \,, \quad \text{and} \quad \ell_{min} \le \ell_{max} \,. \tag{8.40}$$

Exploiting the Chern-Simons description of QH fluids, it has been argued in Fröhlich and Thiran (1994), and Fröhlich, Kerler *et al.* (1995) that, for an *elementary* chiral QH fluid corresponding to the *indecomposable* CQHL $(\Gamma, \mathbf{Q})$, $\ell_{min} = L_{min}$ is the smallest possible relative angular momentum of a state of two electrons excited above the ground state of the fluid. The physical significance of the quantity ℓ_{max} as well as its role in the classification of CQHLs will be expounded in Subsect. 8.4; (see also Fröhlich, Studer, and Thiran (1995), and Fröhlich, Kerler *et al.* (1995)).

The elementary invariants defined in **(1–4)**, above, are well-defined for general QH lattices; see Fröhlich and Thiran (1994).

Examples. We conclude this subsection by considering some examples of QH fluids in the language of QH lattices, thereby illustrating the usefulness of the above invariants.

(a) *QH fluids with* $\sigma_H = N,\ N = 1, 2, \ldots,$ *in the non-interacting electron approximation.* These integer QH fluids correspond to the (self-dual) simple euclidean lattices in N dimensions: $\Gamma = \Gamma_{phys} = \Gamma^* = \mathbb{Z}^N = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Here, N is the number of separately conserved edge currents (Halperin, 1982) or filled Landau levels. Denoting by $\mathbf{e}_i$ the generator of the ith summand, $i = 1, \ldots, N$, we have $K_{ij} = <\mathbf{e}_i, \mathbf{e}_j> = \delta_{ij}$. By the primitivity condition on the charge vector $\mathbf{Q}$, we have $\mathbf{Q} = \mathbf{e}_1 + \cdots + \mathbf{e}_N$, and $\sigma_H = <\mathbf{Q}\,,\mathbf{Q}> = 1 + \cdots + 1 = N$. We note that, by the self-duality of $\mathbb{Z}^N$, there are no fractionally charged excitations with fractional statistics ("anyons") in these fluids.

The symbols characterizing these integer QH fluids read

$$ {}_{N}\!\left(\frac{n_H}{d_H}\right)^{g}_{\lambda} [\ell_{min}, \ell_{max}] \;=\; {}_{N}(N)^1_1\,[1,1]\,, \qquad N = 1, 2, \ldots \,. \tag{8.41}$$

Note that, by the decomposability of these CQHLs, we can write ${}_{N}(N)^1_1 = {}_{1}(1)^1_1 \oplus \cdots \oplus {}_{1}(1)^1_1$, in accordance with the physical picture of N independent, filled Landau levels.

(b) *The Laughlin fluids* (Laughlin, 1983a, 1983b) *at* $\sigma_H = 1/m$, *where* $m = 2p+1,\ p = (0), 1, 2, \ldots.$ Here $\Gamma = \sqrt{m}\,\mathbb{Z}$, which is the one-dimensional lattice generated by $\mathbf{e}$ with squared length $<\mathbf{e}\,,\mathbf{e}> = m$. The dual lattice $\Gamma^* = (1/\sqrt{m})\,\mathbb{Z}$ which is generated by $\boldsymbol{\varepsilon} = \mathbf{e}/m$. The charge vector, being primitive in Γ^*, takes the form $\mathbf{Q} = \boldsymbol{\varepsilon}$, and thus $\sigma_H = <\mathbf{Q}\,,\mathbf{Q}> = 1/m$. The quasi-particles are labelled by $n\boldsymbol{\varepsilon} \in \Gamma_{phys} = \Gamma^*,\ n \in \mathbb{Z}$. By (8.20), they have fractional electric charges $q_{el}(n) = <\mathbf{Q}\,, n\boldsymbol{\varepsilon}> = n/m$, and, by Eqs. (8.16) and (8.17), they have fractional statistical phases $\theta(n) \equiv <n\boldsymbol{\varepsilon}, n\boldsymbol{\varepsilon}> \equiv n^2/m \pmod 2$.

In this example, the knowledge of the electric charges q_{el} of the excitations uniquely determines their statistical phases θ. Such a charge-statistics relation is a property of many interesting higher-dimensional QH lattices; see Thm. 8.4.5 below. However, a relation of this kind does *not* hold for arbitrary QH lattices.

The symbols of the CQHLs describing the Laughlin fluids are

$$ {}_N\!\left(\frac{n_H}{d_H}\right)_\lambda^g [\ell_{min}, \ell_{max}] \;=\; {}_1\!\left(\frac{1}{2p+1}\right)_1^1 [2p+1, 2p+1] \,, \qquad p = 1, 2, \dots \,. \tag{8.42} $$

(c) For each $p = 1, 2, \dots$, there is the series of Hall fractions $\sigma_H = N/(2pN+1)$ with $N = 1, 2, \dots$. From the data presented in Fig. 4.1, it is clear that many of the experimentally most prominent Hall fractions belong to these series (or to the charge-conjugated partner series of the one with $p = 1$; see Subsect. 8.6). These fractions also figure prominently in Jain's work (Jain, 1989, 1990, 1992) – the Jain-Goldman (1992) hierarchy scheme –, and we refer to Thm. 8.4.8 below where, from a classification point of view, the uniqueness of the associated CQHLs is proven. The above series of Hall fractions can be obtained from the following series of *indecomposable* CQHLs: the data pairs $(K, \underset{\to}{Q})$ which determine these CQHLs are given, in some bases that we call "normal", by

$$ K \;=\; \left.\left(\begin{array}{c|ccccc} 2p+1 & -1 & 0 & \cdot & \cdot & 0 \\ \hline -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & 0 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 0 & 0 & \cdot & 0 & -1 & 2 \end{array}\right)\right\}_N \,, \qquad \text{and} \qquad \underset{\to}{Q} \;=\; (\underbrace{1, 0, \dots, 0}_{N}) \,, \tag{8.43} $$

and the associated symbols read

$$ {}_N\!\left(\frac{n_H}{d_H}\right)_\lambda^g [\ell_{min}, \ell_{max}] \;=\; {}_N\!\left(\frac{N}{2pN+1}\right)_1^1 [2p+1, 2p+1] \,, \qquad p, N = 1, 2, \dots \,. \tag{8.44} $$

Note that the $(N-1)$-dimensional submatrix in the lower right of K is the Cartan matrix of the simple Lie algebra $A_{N-1} = su(N)$, $N = 2, 3, \dots$. For $N = 1$, we recognize in (8.43) and (8.44) the expressions corresponding to the Laughlin fluids; see example **(b)**. In connection with the QH effect, the matrices in (8.43) first appeared in Read (1990). Combining results of Read (1990) and Fröhlich and Zee (1991) (see Appendix E in Fröhlich, Studer, and Thiran (1995)) the CQHLs specified by (8.43) can be seen to correspond to the "basic Jain states" (Jain, 1989, 1990, 1992) at $\sigma_H = N/(2pN+1)$. Moreover, it has been shown in Fröhlich and Zee (1991) that the QH fluids corresponding to (8.43) exhibit large symmetries, namely $\widehat{su}(N)$ current algebras at level 1. In Subsect. 8.5, we show that the above CQHLs belong to an interesting class of CQHLs with "large" symmetries, so-called "maximally symmetric" CQHLs.

By extending definitions (8.43) and (8.44) to $p = 0$, the *composite* integer QH fluids of example **(a)** can also be included as special cases of **(c)**. (The Chern-Simons mean field theory of the FQHE offers an explanation of this fact.)

8.4 General Theorems and Classification Results for CQHLs

The purpose of this subsection is to review general facts and classification results for CQHLs. We summarize, in the form of eight theorems, results that have been presented in Fröhlich and Thiran (1994), Fröhlich, Studer, and Thiran (1994, 1995), and Fröhlich, Kerler *et al.* (1995) where more details can be found. We sketch some of the proofs and discuss phenomenological implications of the theorems.

The first two theorems are based on *CQHL inequalities* that establish relations between some of the numerical invariants introduced in Subsect. 8.3.

Theorem 8.4.1. *The set of (proper) CQHLs* $(\Gamma, \mathbf{Q})$ *with rank* $N \leq N_*$ *and relative-angular-momentum invariant* $\ell_{max} \leq \ell_*$, *where* N_* *and* ℓ_* *are two arbitrary, finite integers, is* finite.

This theorem implies that the set of Hall fractions σ_H that can be realized by CQHLs satisfying the above bounds on N and ℓ_{max} is finite. However, the number of possible fractions is growing superexponentially in N_* and ℓ_*; (e.g., for $N_* = 2$ and $\ell_* = 3$, there are 10 CQHLs, while for $N_* = 3$ and $\ell_* = 5$, one finds already more than 250 CQHLs). Fortunately, in the physically relevant situation where one also has a natural upper bound, σ_*, on the Hall fractions to be considered, the number of CQHLs satisfying this bound and the ones in Thm. 8.4.1 is drastically reduced!

The basic tools in proving Thm. 8.4.1 are *Hadamard's inequality* for positive-definite quadratic forms (see, e.g., Siegel, 1989), which implies that

$$\lambda g\, d_H \;=\; \Delta \;=\; \det K \;\leq\; \ell_{max}^{N} \;, \tag{8.45}$$

and the fact that (Fröhlich, Kerler and Thiran, 1995)

$$\lambda g\, n_H \;\leq\; C(N)\, \ell_{max}^{N-1} \;, \tag{8.46}$$

where $C(N)$ is a constant depending on the lattice dimension N (e.g., for two-dimensional CQHLs, one finds that $C(2) = 4$). ■

Recall that N is the number of separately conserved $u(1)$ current densities in a QH fluid (in the scaling limit). Increasing the amount of disorder (the density or strength of impurities) in the system is expected to reduce N_* because of "channel-mixing" effects.

Theoretical arguments and numerical simulations (Meissner, 1992,1993) suggest that the relative-angular-momentum invariant ℓ_{max} obeys a universal upper bound $\ell_{max} \leq \ell_*$, with $\ell_* = 7$ or 9. This can be understood as follows: If ℓ_{max} were larger than 7 or 9, say, the density of electrons in the ground state of a (pure) QH fluid would be so small that it would be energetically more favourable for the electrons to form a Wigner crystal, thereby destroying the incompressibility of the system (Lam and Girvin, 1984; Levesque, Weiss, and MacDonald, 1984). This remark shows that the following basic CQHL inequality is of interest.

Theorem 8.4.2. *For a CQHL* $(\Gamma, \mathbf{Q})$*, the Hall fraction* σ_H *and the relative-angular-momentum invariants* L_{min}*,* ℓ_{min}*, and* ℓ_{max} *satisfy*

$$\frac{1}{\sigma_H} \leq L_{min} \leq \ell_{min} \leq \ell_{max} . \tag{8.47}$$

This theorem is a direct consequence of the *Cauchy-Schwarz inequality* (in the real vector space $V \supset \Gamma$), $<\mathbf{Q}, \mathbf{q}>^2 \leq <\mathbf{q}, \mathbf{q}><\mathbf{Q}, \mathbf{Q}>$, and the fact that, for any vector $\mathbf{q} \in \Gamma$, with $q_{el}(\mathbf{q}) = <\mathbf{Q}, \mathbf{q}> \neq 0$, we have that $<\mathbf{Q}, \mathbf{q}>^2 \geq 1$. ■

If we suppose that chiral QH fluids satisfy a universal bound $\ell_{max} \leq \ell_*$ then (8.47) tells us that CQHLs with $\sigma_H < 1/\ell_*$ cannot be realized. The data in Fig. 4.1 of Subsect. 4.5 are consistent with a choice of $\ell_* = 7$.

Given these observations on the quantities N and ℓ_{max}, one is led to the following heuristic principle:

Stability Principle. *A QH fluid described (in the scaling limit) by a CQHL* $(\Gamma, \mathbf{Q})$ *is the* more stable, *the* smaller *the value of the invariant* ℓ_{max} *and, given the value of* ℓ_{max}*, the* smaller *the dimension* N *of* Γ.

This principle has been discussed in detail in Fröhlich, Studer, and Thiran (1995), and in Fröhlich, Kerler *et al.* (1995). It is also exemplified by the results summarized in Fig. 8.1 below.

Theorem 8.4.3. *Let* $(\Gamma, \mathbf{Q})$ *be a CQHL with* even *Hall denominator* d_H*. Then the charge parameter* λ *has to be* even, *too.*

For a proof of this theorem, we first define the vector $\mathbf{v} := \lambda d_H \mathbf{Q} \in \Gamma^*$. Then, for all dual vectors $\mathbf{n} = \sum_{j=1}^{N} n_j \boldsymbol{\varepsilon}^j \in \Gamma^*$, we find that $<\mathbf{v}, \mathbf{n}> = \lambda d_H <\mathbf{Q}, \mathbf{n}> = \sum_{j=1}^{N} (\Delta <\mathbf{Q}, \boldsymbol{\varepsilon}^j> / g)\, n_j = \sum_{j=1}^{N} (Q^j$ by using $\Delta = g \lambda d_H$ and the definition of g; see points **(3)** and **(4)** in Subsect. 8.3. Thus $\mathbf{v}$ is actually an element of $(\Gamma^*)^* \simeq \Gamma$ and hence, by the oddness of $\mathbf{Q}$ (see (8.9)), the congruence $<\mathbf{Q}, \mathbf{v}> \equiv <\mathbf{v}, \mathbf{v}>$ (mod 2) holds. From the definitions of σ_H and $\mathbf{v}$ it follows that the l.h.s. of the congruence equals λn_H and the r.h.s. equals $\lambda^2 d_H n_H$, i.e., $\lambda n_H \equiv \lambda^2 d_H n_H$ (mod 2). Finally, since for d_H even n_H is odd (see point **(3)** in Subsect. 8.3), the latter congruence implies that λ is even. ■

The phenomenologically most interesting implication of Thm. 8.4.3 is that, in QH fluids with an even Hall denominator d_H, there exist quasi-particle excitations with "fractional" fractional charges, i.e., since $\lambda = 2, 4, \ldots$,

$$e^* \;=\; \frac{1}{\lambda d_H} \;\leq\; \frac{1}{2 d_H} \;. \tag{8.48}$$

It would be interesting to test this model-independent prediction experimentally for even-denominator QH fluids at $\sigma_H = 1/2$ and 5/2: We predict that $e^* \leq 1/4$ (in units where $e = -1$)!

Theorem 8.4.4. *At every Hall fraction $\sigma_H = 1/m$, m odd, there is a* unique *indecomposable CQHL with the property that its level $l = \lambda g = 1$. This CQHL is one-dimensional and corresponds to the Laughlin fluid at $\sigma_H = 1/m$. Moreover, any CQHL with $\sigma_H = 1/m$, m odd, and $N \geq 2$ has a charge parameter $\lambda \geq 2$.*

A proof of this theorem has been given in Subsect. 7.5 in Fröhlich and Thiran (1994).

The CQHLs corresponding to the Laughlin fluids have been described explicitly in example **(b)** at the end of Subsect. 8.3. By an argument similar to the one in (8.48), the last statement in Thm. 8.4.4 has implications that are, in principle, observable! In Subsect. 8.6, an example illustrating this point is discussed in connection with possible phase transitions at $\sigma_H = 1$.

An interesting subclass of CQHLs is formed by CQHLs with level $l = 1$, i.e., their lattice discriminant Δ equals the Hall denominator d_H. Indecomposable CQHLs with level $l = 1$, and thus $\lambda = g = 1$, have been classified for $d_H \leq 25$ and N below some fairly high $N_c(\sigma_H)$, typically around 10; see Fröhlich and Thiran (1994), and Fröhlich, Studer, and Thiran (1994).

This partial classification has been achieved by combining the recent lattice-classification results of Conway and Sloane (1988a) with a systematic investigation of the possible charge vectors $\mathbf{Q}$ in the dual of all the classified (odd, integral, euclidean) lattices. For the latter search, one makes use of the following fact: from the Cauchy-Schwarz inequality and the defining relation $\sigma_H = <\mathbf{Q}\,,\mathbf{Q}>$, one infers that, for a CQHL $(\Gamma,\mathbf{Q})$, the dual components Q_j of the charge vector $\mathbf{Q} = \sum_{j=1}^{N} Q_j\, \boldsymbol{\varepsilon}^j$ are constrained by

$$Q_j^2 \;\leq\; \sigma_H\, \ell_{max} \;, \qquad \text{for all } \; j = 1, \ldots, N \;. \tag{8.49}$$

Thus, focussing attention on CQHLs with $\ell_{max} \leq \ell_*$ and $\sigma_H \leq \sigma_*$, Eq. (8.49) implies that the search for all possible charge vectors $\mathbf{Q}$ in the dual of a given lattice Γ is a *finite* problem. ■

Next, we summarize a few general properties of CQHLs with level $l = 1$:

Theorem 8.4.5. *Let $(\Gamma, \mathbf{Q})$ be a (proper) CQHL with level $l = \lambda g = 1$. Then*

(i) d_H *is* odd, *and* $\Gamma^*/\Gamma \simeq \mathbb{Z}_{d_H}$;

(ii) *in order to realize a Hall fraction* σ_H *with* n_H *even (odd),* N *has to be even (odd, and* $N \equiv n_H \pmod 4$*));*

(iii) *for quasi-particles labelled by* $\mathbf{n} \in \Gamma^*$*, a charge-statistics relation holds: if* $q_{el}(\mathbf{n}) = \varepsilon/d_H$ *then* $\theta(\mathbf{n}) \equiv (n_H)^{-1}\,\varepsilon^2/d_H \pmod 2$*.*

In the last statement of this theorem, the number $(n_H)^{-1}$ is defined as follows: if n_H is odd, then $n_H(n_H)^{-1} \equiv 1 \pmod{2d_H}$, and if n_H is even, then $(n_H)^{-1} := 2(2n_H)^{-1} + d_H$, with $2n_H(2n_H)^{-1} \equiv 1 \pmod{d_H}$. A proof of this theorem can be found in Sect. 5 in Fröhlich and Thiran (1994).

Shift Maps and their Implications

In the remaining part of this subsection, we study "structurally similar" chiral QH fluids. At the level of CQHLs, "structural" relationships are realized by particular maps, called *shift maps*. From a classification point of view, shift maps allow – under suitable conditions – to immediately carry over classification results for CQHLs with Hall fractions in a certain interval to corresponding results for other intervals. Phenomenologically interesting implications of structural relationships are outlined in Subsect. 8.6.

First, we divide the interval $(0,\infty)$ of possible Hall fractions σ_H into a sequence of suitable subintervals, "*windows*", $\Sigma_p^{\pm}$ defined by

$$\Sigma_p^{+} \;:=\; \{\, \sigma_H \mid \frac{1}{2p+1} \leq \sigma_H < \frac{1}{2p} \,\} \,, \qquad p = 1,2,\ldots\;,$$

and

$$\Sigma_p^{-} \;:=\; \{\, \sigma_H \mid \frac{1}{2p} \leq \sigma_H < \frac{1}{2p-1} \,\} \,, \qquad p = 1,2,\ldots\;. \tag{8.50}$$

The "+" superscripts are chosen because these subintervals contain the "first main series" of Hall fractions, $\sigma_H = N/(2pN+1)$, $N = 1,2,\ldots$. Similarly, the "−" superscripts for the "complementary" windows indicate that these windows contain the "second main series" of Hall fractions, $\sigma_H = N/(2pN-1)$, $N = 2,3,\ldots$. Moreover, we denote by Σ_0^{+} the interval $[1,\infty)$, and by Σ_p the union of the two complementary subintervals Σ_p^{+} and Σ_p^{-}, i.e., $\Sigma_p := \Sigma_p^{+} \cup \Sigma_p^{-}$, $p = 1,2,\ldots$.

Second, we define a class of CQHLs that will figure prominently in the sequel.

Definition 8.4.1. *A primitive CQHL* $(\Gamma,\mathbf{Q})$ *(see Def. 8.1.2) with Hall fraction* $\sigma_H \in \Sigma_p$ *is called* L-minimal *if, and only if,* ℓ_{max} *takes the smallest possible value consistent with (8.47), namely* $\ell_{max} = 2p+1$, $p = 1,2,\ldots$.

By (8.38)–(8.40), L-minimal CQHLs satisfy $L_{min} = \ell_{min} = L_{max} = \ell_{max} = 2p+1$. General, powerful implications that follow from L-minimality are summarized below in Thms. 8.4.6–8.4.8; for proofs, see Fröhlich, Kerler *et al.* (1995).

Theorem 8.4.6. *For* $p = 1, 2, \ldots,$ *let* $(\Gamma, \mathbf{Q})$ *be a (proper) CQHL with* $\sigma_H \in \Sigma_p$ *and* $L_{max} = 2p + 1$. *Then* $(\Gamma, \mathbf{Q})$ *is primitive and* L-minimal, *i.e., we also have that* $L_{min} = \ell_{max} = 2p + 1$. *Moreover,* $(\Gamma, \mathbf{Q})$ *is* indecomposable *if* $\sigma_H < 2/3$.

The bound $\sigma_H < 2/3$ for indecomposability is sharp. As a matter of fact, at $\sigma_H = 2/3$, there is an L-minimal ($\ell_{max} = 3$) *composite* CQHL. It is given by the direct sum of two Laughlin fluids at $\sigma_H = 1/3$; see example **(b)** in Subsect. 8.3.

Next, "shift maps" are defined.

Definition 8.4.2. *Shift maps, denoted by* $\mathcal{S}_p$, $p = 1, 2, \ldots,$ *are maps between (proper) CQHLs of equal dimensions,* $\mathcal{S}_p : (\Gamma, \mathbf{Q}) \mapsto (\Gamma', \mathbf{Q}')$. *Starting from an arbitrary basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ *of* $(\Gamma, \mathbf{Q})$, *the image* $(\Gamma', \mathbf{Q}')$ *is uniquely specified by constructing a basis* $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_N\}$ *and a charge vector* $\mathbf{Q}'$ *that satisfy the conditions*

$$K'_{ij} = <\mathbf{e}'_i, \mathbf{e}'_j> = <\mathbf{e}_i, \mathbf{e}_j> + 2p <\mathbf{Q}, \mathbf{e}_i><\mathbf{Q}, \mathbf{e}_j> = K_{ij} + 2p\, q_{el}(\mathbf{e}_i) q_{el}(\mathbf{e}_j) \ ,$$

and

$$Q'_i = <\mathbf{Q}', \mathbf{e}'_i> = <\mathbf{Q}, \mathbf{e}_i> = Q_i \ , \quad \text{for all } i, j = 1, \ldots, N \ . \tag{8.51}$$

Although the conditions in (8.51) are formulated w.r.t. given bases, they specify the image $(\Gamma', \mathbf{Q}')$ uniquely, since different choices of bases and charge vectors in (8.51) simply lead to data pairs $(K', \vec{Q}')$ for $(\Gamma', \mathbf{Q}')$ which are related by the equivalence transformations (8.32).

Denoting by $\Gamma_0 \subset \Gamma$ the *neutral sublattice* of a CQHL $(\Gamma, \mathbf{Q})$, i.e.,

$$\Gamma_0 := \{ \mathbf{q} \in \Gamma \mid <\mathbf{Q}, \mathbf{q}> = q_{el}(\mathbf{q}) = 0 \} \ , \tag{8.52}$$

it is straightforward to show that shift maps leave neutral sublattices invariant,

$$\Gamma'_0 = \Gamma_0 \ . \tag{8.53}$$

As will be explained in more detail in Subsect. 8.5, Eq. (8.53) implies that (in the scaling limit) the corresponding chiral QH fluids exhibit the *same symmetries*. This fact is the mathematical basis for calling two chiral QH fluids *structurally similar*.

What is the action of the shift map $\mathcal{S}_p : (\Gamma, \mathbf{Q}) \mapsto (\Gamma', \mathbf{Q}')$, for $p = 1, 2, \ldots,$ on the space of invariants introduced in Subsect. 8.3?

(i) The discriminant Δ' of the (odd, integral, euclidean) lattice Γ' is given by

$$\Delta' = \Delta\,(1 + 2p\,\sigma_H) \ . \tag{8.54}$$

(ii) The Hall conductivity changes according to

$$\frac{1}{\sigma'_H} \;=\; \frac{1}{\sigma_H} + 2p \;, \tag{8.55}$$

which corresponds to the "D-operation" in the Jain-Goldman (1992) hierarchy scheme; see also Jain (1989, 1990, 1992) and Fröhlich and Zee (1991). Note that Eq. (8.55) implies that any CQHL which is the image under a shift map $\mathcal{S}_p,\; p = 1, 2, \ldots,$ necessarily has a Hall fraction strictly below $1/(2p)$.

(iii) The level l, g, and the charge parameter λ are all invariant under the action of a shift map $\mathcal{S}_p$.

We summarize **(i)–(iii)** by giving a succinct representation of the action of the shift map $\mathcal{S}_p$ at the level of the CQHL symbol,

$$ {}_N\!\left(\sigma_H = \frac{n_H}{d_H}\right)^{g}_{\lambda} \;\overset{\mathcal{S}_p}{\longmapsto}\; {}_N\!\left(\sigma'_H = \frac{n_H}{d_H + 2p\, n_H}\right)^{g}_{\lambda} \,,\; p = 1, 2, \ldots \;. \tag{8.56}$$

(iv) The name "shift map" for $\mathcal{S}_p$ is motivated by the fact that the relative-angular-momentum invariants L_{min} and L_{max} are simply shifted by $2p$,

$$L'_{min} \;=\; L_{min} + 2p \;, \qquad \text{and} \qquad L'_{max} \;=\; L_{max} + 2p \;. \tag{8.57}$$

Unfortunately, for the invariants ℓ_{min} and ℓ_{max} of *generic* primitive CQHLs, there does *not, in general*, hold a transformation rule similarly simple to (8.57)! – However, for *indecomposable* CQHLs, the identities $\ell_{min} = L_{min}$ and $\ell_{max} = L_{max}$ hold.

From the definitions above one sees that the shift maps $\mathcal{S}_p$ are invertible on the set of (proper) CQHLs with Hall fractions $\sigma_H \leq 1/(2p),\; p = 1, 2, \ldots$. – From (8.51) it follows that $\mathcal{S}_p^{-1} = \mathcal{S}_{-p}$. – The preimages of these CQHLs are readily seen to be (proper) CQHLs. The set of (proper) CQHLs is closed under the action of the maps $\mathcal{S}_p$ and their inverses (when defined).

However, the maps $\mathcal{S}_p$ and their inverses do *not* necessarily preserve the decomposability properties of CQHLs. – E.g., composite CQHLs can be mapped into indecomposable ones, as illustrated in Thm. 8.4.8 below. – Moreover, the maps $\mathcal{S}_p$ and their inverses do *not*, in general, preserve the primitivity property we have imposed on (composite) CQHLs; see Def. 8.1.2. – For an example of a primitive CQHL with a preimage that is non-primitive, see Sect. 4 in Fröhlich, Kerler *et al.* (1995). – From these remarks and definitions (8.38) and (8.39) of the invariants ℓ_{min} and ℓ_{max} it is clear that the transformation properties of these invariants under shift maps are not as straightforward as the ones in (8.57).

The main objective of the present section is the classification of primitive CQHLs. Although this set is not closed under the action of shift maps and their inverses, it is remarkable that a subset of the primitive CQHLs, the class of *L-minimal* CQHLs (see Def. 8.4.1) *is closed* under the action of shift maps and their inverses. This is the key to powerful classification results described presently.

It is convenient to partition the class of L-minimal CQHLs into the following subsets:

$$\mathcal{H}_p^{\pm} := \{ (\Gamma, \mathbf{Q}) \mid \sigma_H \in \Sigma_p^{\pm},\ L\text{-minimal, i.e., primitive and } \ell_{min} = \ell_{max} = 2p+1 \} , \quad (8.58)$$

where $p = (0), 1, 2, \ldots$, in accordance with the definition of the windows $\Sigma_p^{\pm}$ given in (8.50).

The next two theorems show that, on the one hand, the sets $\mathcal{H}_p := \mathcal{H}_p^+ \cup \mathcal{H}_p^-$ are structurally similar for different p's, while, on the other hand, there is an essential *structural asymmetry* between the sets $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$, for a given p.

Theorem 8.4.7 (Bijection Theorem). *The sets $\mathcal{H}_p$ of L-minimal CQHLs with $\sigma_H \in \Sigma_p$, for $p = 2, 3, \ldots$, are in* one-to-one correspondence *with the set $\mathcal{H}_1$. The corresponding bijections are realized by the shift maps $\mathcal{S}_{p-1} : \mathcal{H}_1 \to \mathcal{H}_p$.*

The proof of this theorem rests on Thm. 8.4.6 above, and it should be emphasized that chirality and L-minimality are crucial for the theorem to hold; see Fröhlich, Kerler *et al.* (1995). The bijection theorem (Thm. 8.4.7) implies that, for the classification of L-minimal CQHLs, we can restrict our analysis to the lattices with Hall fractions σ_H in the "fundamental domain" $\Sigma_1 := [1/3, 1)$!

In Fröhlich, Kerler *et al.* (1995), the set $\mathcal{H}_0^+$ of L-minimal CQHLs with $\sigma_H \in [1, \infty)$ has been constructed. Applying the shift map $\mathcal{S}_1$ to it, we obtain the set $\mathcal{H}_1^+$ of L-minimal CQHLs in the window $\Sigma_1^+ := [1/3, 1/2) \subset \Sigma_1$. Hence, by the bijection theorem, all the sets $\mathcal{H}_p^+$, $p \geq 1$, are known. In fact, we have the following result.

Theorem 8.4.8 (Uniqueness Theorem). *For each $p = 0, 1, 2, \ldots$, the set $\mathcal{H}_p^+$ of L-minimal CQHLs with $\sigma_H \in \Sigma_p^+$ coincides with the (infinite) series, $N = 1, 2, \ldots$, of indecomposable, N-dimensional, maximally symmetric CQHLs with SU(N) symmetry of N-ality* 1, *meaning that the elementary charge-1 fermions (electrons) described by these CQHLs transform under the fundamental representation of SU(N). For a given p, the corresponding symbols read*

$${}_N\!\left(\sigma_H = \frac{N}{2pN+1}\right)_1^1 [\ell_{min} = \ell_{max} = 2p+1] , \qquad N = 1, 2, \ldots . \quad (8.59)$$

The maximally symmetric CQHLs of this theorem have been described explicitly in example **(c)** at the end of Subsect. 8.3. Each Hall fraction appearing in (8.59) is realized by a unique L-minimalCQHL with $SU(N)$ symmetry (of N-ality 1). In the notation of the next subsection (see (8.64) below), the sets $\mathcal{H}_p^+$ can be written as

$$\mathcal{H}_p^+ \;=\; \{\, (2p+1 \,|\, {}^1\!A_{N-1}) \;|\; N = 1, 2, \ldots \,\} \;. \tag{8.60}$$

In Thm. 8.4.6, it has been stated that all CQHLs in (8.60) with $p > 0$ are *indecomposable*. Furthermore, since their level l equals unity, Thm. 8.4.5 states that a *charge-statistics relation* holds for the quasi-particle excitations of the corresponding QH fluids.

A first comparison of the Hall fractions appearing in (8.59) with the experimental data presented in Fig. 4.1 of Subsect. 4.5 reveals an impressive agreement between the Hall fractions predicted by L-minimal CQHLs and the experimentally observed values in the windows $\Sigma_p^+,\ p = 1, 2,$ and 3. Note that CQHLs with higher dimensions and/or higher values of ℓ_{max} are associated with less stable QH fluids, which is in accordance with our stability principle advocated at the beginning of this subsection. In the windows, $\Sigma_p^+,\ p = 1, 2,$ and 3, there is *only one* Hall fraction, 4/11, for which there are some experimental indications (however, only very weak ones!) that *cannot* be realized by an L-minimal CQHL.

8.5 Maximally Symmetric CQHLs

Maximally symmetric CQHLs form the most natural generalizations of the "elementary" A- (or $su(N)$-) fluids that appeared in the uniqueness theorem (Thm. 8.4.8) of the last subsection and which have been shown to encompass the Laughlin fluids as well as the "basic" Jain fluids. Before we can give a precise definition of the class of maximally symmetric CQHLs, we need to investigate a general geometrical feature of CQHLs, namely their "Witt sublattices". We use technical language and then translate our definitions into explicit statements at the level of the data pairs $(K, \vec{Q})$ associated with CQHLs; for the definition of these pairs, see the beginning of Subsect. 8.3.

Let $(\Gamma, \mathbf{Q})$ be a CQHL. Then the *Witt sublattice*, $\Gamma_W \subset \Gamma$, is defined to be the sublattice of Γ generated by all vectors of length squared 1 and 2. The general theory of integral, euclidean lattices (Conway and Sloane, 1988a, 1988b) tells us that Γ_W is of the form

$$\Gamma_W \;=\; \Gamma_A \oplus \Gamma_D \oplus \Gamma_E \oplus I_l \;, \tag{8.61}$$

where I_l denotes the (self-dual) simple euclidean lattice in l dimensions and Γ_A, Γ_D, and Γ_E are direct sums of root lattices of the simple Lie algebras $A_{m-1} = su(m)$, $D_{m+2} = so(2m+4),\ m = 2, 3, \ldots$, and $E_m,\ m = 6, 7, 8$, respectively. The subscripts in the symbols A_n, D_n, and E_n indicate the *ranks* of these algebras. We note that all the root lattices of these Lie algebras are generated by vectors of only one length, namely of length squared 2. [In the mathematical literature, the A-, D-, and E-Lie algebras are called simply-laced.] For a general reference on Lie algebras and their representation theory, see Slansky (1981).

Denoting by $\mathcal{O}$ the orthogonal complement of Γ_W in Γ, whose dimension satisfies $\dim \mathcal{O} = N - \dim \Gamma_W \geq 1$, the sublattice $\Gamma_W \oplus \mathcal{O}$ is called the *Kneser shape* of Γ, and one has the following embeddings of lattices:

$$\Gamma_W \oplus \mathcal{O} \ \subseteq \ \Gamma \ \subseteq \ \Gamma^* \ \subseteq \ \Gamma_W^* \oplus \mathcal{O}^* \ , \tag{8.62}$$

where * denotes the dual of a lattice, as defined in (8.5).

It can be shown (Fröhlich and Thiran, 1994) that, for *indecomposable* CQHLs $(\Gamma, \mathbf{Q})$, Γ_W does *not* contain any I_l and Γ_{E_8} sublattices. In the following, we will concentrate on indecomposable CQHLs, or, correspondingly, on "elementary" chiral QH fluids.

Theorem 8.5.1. *Let* $(\Gamma, \mathbf{Q})$ *be an indecomposable CQHL with* $\sigma_H < 2$. *Then* $\mathbf{Q}$ *is* orthogonal *to* Γ_W, *i.e.,* $\mathbf{Q} \in \mathcal{O}^*$, *and* $\Gamma_W \subseteq \Gamma_0$ *where* Γ_0 *is the neutral sublattice of* $(\Gamma, \mathbf{Q})$. *Moreover, if* $\Gamma_W \neq \emptyset$ *all the inclusions in (8.62) are* proper.

For a proof of this theorem and more details on the constructions above – which constitute the basis of the complete classification program of (general) CQHLs –, see Sect. 6 in Fröhlich and Thiran (1994).

Thm. 8.5.1 has an interesting corollary concerning the *symmetry properties* of the chiral QH fluid corresponding to $(\Gamma, \mathbf{Q})$: To every point in Γ there corresponds a vertex operator of the algebra of edge currents (compare with Subsect. 7.2). Let $\mathcal{G}$ denote the Lie algebra – a direct sum of simple algebras A_n, D_n, and $E_{6,7}$ – whose root lattice is given by the Witt sublattice Γ_W of $(\Gamma, \mathbf{Q})$. It is not hard to show (Fröhlich and Zee, 1991; Fröhlich and Thiran, 1994) that the algebra generated by the vertex operators corresponding to the Witt sublattice, Γ_W, of Γ and the neutral $u(1)$ currents is the enveloping algebra of the Kac-Moody current algebra $\hat{\mathcal{G}}$ at level 1 (denoted $\hat{\mathcal{G}}_1$).

The (infinite-dimensional) *symmetry algebra* $\hat{\mathcal{G}}_1$ canonically contains the (finite dimensional) Lie algebra $\mathcal{G}$ that consists of global symmetry generators. Thus the Lie group G corresponding to $\mathcal{G}$ is the *group of global symmetries* of the QH fluid. This implies that, given m electrons – fermionic quasi-particles with charge 1 and labelled by, say, $\mathbf{q}_1, \ldots, \mathbf{q}_m \in \Gamma \subset \Gamma^*$ –, they transform under particular unitary irreducible representations (irreps.) of G. These unitary irreps. are specified as follows: Let

$$\mathbf{q}_i \ = \ \mathbf{q}_{i,W} + \mathbf{q}'_i \,, \qquad \text{with} \qquad \mathbf{q}_{i,W} \in \Gamma_W^* \,, \qquad \text{and} \qquad \mathbf{q}'_i \in \mathcal{O}^* \ , \tag{8.63}$$

be the decomposition, according to (8.62), of the ith electron vector, $i = 1, \ldots, m$. Then we may write $\mathbf{q}_{i,W} = \boldsymbol{\omega}_i + \mathbf{r}_i$, with $\mathbf{r}_i \in \Gamma_W$ a *root vector*, and with $\boldsymbol{\omega}_i \in \Gamma_W^*$ an *elementary weight*, i.e., a smallest length representative of the cosets (or "congruence classes" in Lie group terminology) in the quotient Γ_W^*/Γ_W (see, e.g., Slansky, 1981). Furthermore, by the general representation theory of Lie - and Kac-Moody algebras (see, e.g., Goddard and Olive, 1986), the elementary weight $\boldsymbol{\omega}_i$ determines uniquely a unitary irrep., $\pi_{\boldsymbol{\omega}_i}$, of G according to which the one-electron state labelled by $\mathbf{q}_i$ transforms, $i = 1, \ldots, m$.

From the general results about lattices given in Conway and Sloane (1988a), it follows that all the elementary weights $\boldsymbol{\omega}_i \in \Gamma_W^*$ that can appear in (8.63) are such that the corresponding irreps. of $\mathcal{G}$ *can be extended* to unitary highest-weight representations of $\hat{\mathcal{G}}$ at level 1. For a discussion of the latter point, see, e.g., Subsect. 3.4 in Goddard and Olive (1986). We call these elementary weights "*admissible*" weights, and the ones that can occur for the simple algebras A_n, D_n, and $E_{6,7}$ have been given explicitly in Appendix A in Fröhlich, Studer, and Thiran (1995).

One can show (Fröhlich and Thiran, 1994) that if $\dim \mathcal{O} = 1$ and $\Gamma_0 = \Gamma_W$ then all one-electron states transform under the *same* unitary irrep. $\pi_{\boldsymbol{\omega}}$ of G, i.e., $\mathbf{q}_{1,W} \equiv \cdots \equiv \mathbf{q}_{m,W} \equiv \boldsymbol{\omega} \mod \Gamma_W$.

These results motivate the following definition of *maximally symmetric CQHLs.*

Definition 8.5.1. *A (proper) CQHL* $(\Gamma, \mathbf{Q})$ *is called* maximally symmetric *iff* $\dim \mathcal{O} = 1$ *and* $\Gamma_0 = \Gamma_W$*, i.e., the neutral sublattice of* $(\Gamma, \mathbf{Q})$ *and its Witt sublattice coincide. Furthermore, denoting by* $\mathcal{G}$ *the Lie algebra associated with the root lattice* Γ_W*, the one-electron states described by* $(\Gamma, \mathbf{Q})$ *are required to transform under a unitary irrep. of the Lie algebra* $\mathcal{G}$ *which can be extended to a unitary highest-weight representation of* $\hat{\mathcal{G}}_1$*.*

Maximally symmetric CQHLs $(\Gamma, \mathbf{Q})$ are specified by the data

$$(L \,|\, {}^{\boldsymbol{\omega}}\Gamma_W) \,, \tag{8.64}$$

where L is an odd, positive integer, $\Gamma_W = \Gamma_A \oplus \Gamma_D \oplus \Gamma_{E\neq 8}$ is the Witt sublattice of $(\Gamma, \mathbf{Q})$, and $\boldsymbol{\omega} \in \Gamma_W^*/\Gamma_W$ is an admissible weight labelling an irrep. of the Lie algebra $\mathcal{G}$ associated with Γ_W. The possible weights $\boldsymbol{\omega}$ are further restricted by the value of L, namely $<\boldsymbol{\omega}, \boldsymbol{\omega}> < L$; see (8.66) below.

If the Witt sublattice is a direct sum of simple root lattices, $\Gamma_W = \Gamma_{W_1} \oplus \cdots \oplus \Gamma_{W_k}$, $k \geq 2$, then the associated Lie algebra $\mathcal{G}$ is semi-simple with decomposition $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_k$, and, correspondingly, the admissible weight reads $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_k$ where $\boldsymbol{\omega}_i \in \Gamma_{W_i}^*/\Gamma_{W_i}$, $i = 1, \ldots, k$. In order to get an *indecomposable* lattice, every projection $\boldsymbol{\omega}_i$ must represent a non-trivial coset in $\Gamma_{W_i}^*/\Gamma_{W_i}$. This can also be shown to be sufficient; see Fröhlich, Kerler, and Thiran (1995). In the sequel, we always assume that admissible weights $\boldsymbol{\omega}$ fulfill this requirement. Hence *all* the maximally symmetric CQHLs mentioned in this section are *indecomposable*.

Equivalently to (8.64), we can also specify maximally symmetric CQHLs $(\Gamma, \mathbf{Q})$ by their corresponding data pair $(K, \underset{\rightarrow}{Q})$, once a basis has been chosen in Γ; see the beginning of Subsect. 8.3. Relative to a "normal" basis $\{\mathbf{q}, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ, $(\Gamma, \mathbf{Q})$ is specified by

$$K \;=\; \left.\left(\begin{array}{c|c} L & \underset{\rightarrow}{\omega} \\ \hline \underset{\rightarrow}{\omega}^T & C(\Gamma_W) \end{array} \right)\right\}\text{N} \,, \qquad \text{and} \qquad \underset{\rightarrow}{Q} \;=\; (\underbrace{1, 0, \ldots, 0}_{\text{N}}) \,, \tag{8.65}$$

where $L = <\mathbf{q}\,,\mathbf{q}>$ is the same odd integer as in (8.64), $C(\Gamma_W)$ is the Gram matrix of the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ_W – in a convenient normal basis, it equals the *Cartan matrix* of the Lie algebra $\mathcal{G}$ associated with Γ_W –, and, finally, $\underset{\rightarrow}{\omega} = (\omega_1, \ldots, \omega_{N-1})$ is the vector of the dual components of $\boldsymbol{\omega}$ which are given by $\omega_j = <\boldsymbol{\omega}, \mathbf{e}_j>$, $j = 1, \ldots, N-1$. According to the decomposition (8.62), the basis vector $\mathbf{q}$ can be written as $\mathbf{q} = {\sigma_H}^{-1}\mathbf{Q} + \boldsymbol{\omega}$.

If Γ_W is a direct sum of simple root lattices then $C(\Gamma_W) = C(\Gamma_{W_1}) \oplus \cdots \oplus C(\Gamma_{W_k})$ is a block-diagonal matrix, and $\underset{\rightarrow}{\omega} = \underset{\rightarrow}{\omega}_1 + \cdots + \underset{\rightarrow}{\omega}_k$. An example of data pairs (8.64) and (8.65) has been given by (8.60) and (8.43), respectively. The explicit forms of the Cartan matrices for the simple algebras A_n, D_n, and $E_{6,7}$, and of the dual vectors $\underset{\rightarrow}{\omega}$ for the admissible weights $\boldsymbol{\omega}$ have been given in Appendix A of Fröhlich, Studer, and Thiran (1995).

We denote by $\Delta(\Gamma_W)$ the discriminant of the Witt sublattice, Γ_W, of Γ, i.e., $\Delta(\Gamma_W) := \det C(\Gamma_W) = |\Gamma_W^*/\Gamma_W|$. From (8.65), it immediately follows that for maximally symmetric CQHLs

$$\begin{aligned} \Delta \;=\; \det K \;&=\; \Delta(\Gamma_W)\,[L - \underset{\rightarrow}{\omega} \cdot C(\Gamma_W)^{-1} \underset{\rightarrow}{\omega}^T] \\ &=\; \Delta(\Gamma_W)\,[L - <\boldsymbol{\omega}, \boldsymbol{\omega}>] \quad (>0)\,, \end{aligned} \tag{8.66}$$

and

$$\sigma_H \;=\; <\mathbf{Q}\,,\mathbf{Q}> = \underset{\rightarrow}{Q} \cdot K^{-1} \underset{\rightarrow}{Q}^T \;=\; \frac{\Delta(\Gamma_W)}{\Delta} \;>\; \frac{1}{L}\,. \tag{8.67}$$

These two equations are basic for proving the following theorem.

Theorem 8.5.2. *The symbol of a maximally symmetric CQHL* $(\Gamma, \mathbf{Q})$ *specified by (8.64) or, equivalently, by (8.65), takes the form*

$$_{1+\operatorname{rank}\Gamma_W = N}\left(\sigma_H = \frac{1}{L - <\boldsymbol{\omega}, \boldsymbol{\omega}>}\right)^{g = \Delta(\Gamma_W)/h_{\boldsymbol{\omega}}}_{\lambda = h_{\boldsymbol{\omega}}/n_H}\,. \tag{8.68}$$

where $h_{\boldsymbol{\omega}}$ *is the order of the elementary weight* $\boldsymbol{\omega}$ *in* Γ_W^*/Γ_W*. Furthermore, for the relative-angular-momentum invariants* ℓ_{max} *and* ℓ_{min}*, the equalities* $\ell_{min} = \ell_{max} = L$ *hold.*

If Γ_W is a direct sum of simple root lattices, $\Gamma_W = \Gamma_{W_1} \oplus \cdots \oplus \Gamma_{W_k}$, $k \geq 2$, and $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_k$, as above, then the following identities hold:

$$\begin{aligned} \operatorname{rank}\Gamma_W \;&=\; \sum_{i=1}^{k} \operatorname{rank}\Gamma_{W_i}\,, \\ <\boldsymbol{\omega}, \boldsymbol{\omega}> \;&=\; \sum_{i=1}^{k} <\boldsymbol{\omega}_i, \boldsymbol{\omega}_i>\,, \\ \Delta(\Gamma_W) \;=\; \det C(\Gamma_W) \;&=\; \prod_{i=1}^{k} \det C(\Gamma_{W_i})\,, \end{aligned}$$

and

$$h_{\boldsymbol{\omega}} \;=\; \mathrm{lcm}(h_{\boldsymbol{\omega}_1}, \ldots, h_{\boldsymbol{\omega}_k}) \;, \tag{8.69}$$

where the *least common multiple* (lcm) of two integers a and b is defined by $\mathrm{lcm}(a,b) := ab/\gcd(a,b)$, and similarly for more than two integers.

For the simple Lie algebras $A_{m-1} = su(m)$, $D_{m+2} = so(2m+4)$, $m = 2, 3, \ldots$, and $E_{6,7}$, all the ranks and determinants of their Cartan matrices, as well as all the lengths squared and orders of their admissible weights can be found in Appendix A of Fröhlich, Studer, and Thiran (1995).

Classification

Exploiting the results of Thm. 8.5.2 and the identities in (8.69), it is possible to list all maximally symmetric CQHLs which have a fixed value of L and whose Hall fractions σ_H ($> 1/L$; see (8.67)) belong to a given interval. In Fröhlich, Studer, and Thiran (1995), *all* maximally symmetric CQHLs with $\ell_{min} = \ell_{max} = L = 3$ and $\sigma_H < 1$ have been listed. They are organized in four infinite one-parameter, one infinite two-parameter, and six finite series of CQHLs. For a physically relevant subset of Hall fractions (with $d_H < 21$ and odd), the resulting CQHLs are indicated in Fig. 8.1 below.

The most powerful implication of this classification result is obtained by combining it with the discussion about the shift maps in the second part of Subsect. 8.4: In Fröhlich, Studer, and Thiran (1995), we have shown that this leads to *the complete classification of L-minimal, maximally symmetric CQHLs.* For some alternative ideas on the classification of QH fluids, see also Cappelli et al. (1995).

Some of our main insights will be summarized in the following subsection; see, in particular, Fig. 8.1. For a detailed discussion of this classification result and comparison with experimental data, however, we refer to the second part of Sect. 5 in Fröhlich, Studer, and Thiran (1995).

8.6 Summary and Physical Implications of the Classification Results

The purpose of this subsection is to summarize phenomenological implications of the theoretical results presented in the previous subsections. We confront these results with the experimental data presented in Subsect. 4.5; see, in particular, Fig. 4.1 which summarizes data on single-layer QH systems. So far, we have focused our attention mainly on *chiral* QH lattices with Hall fractions $\sigma_H \leq 1$. We recall that such lattices describe QH fluids that have only electrons (or holes) as fundamental charge carriers and whose edge currents are of *definite chirality*. When restricting considerations to CQHLs, we do *not* make use of the idea of "charge conjugation", i.e., $\sigma_H = 1 - \sigma'_H$, in the discussion of QH fluids with $1/2 < \sigma_H < 1$. In this interval, charge conjugation is usually invoked in other approaches to the fractional QH effect; see Haldane (1983), Halperin (1984), Jain (1989, 1990, 1992), and Jain and Goldman (1992). There is no difficulty, within our framework, to go beyond the chirality assumption and to consider the general classification problem of mixed-chirality QH lattices including, e.g., those corresponding to the QH fluids proposed by the charge-conjugation picture; see Fröhlich, Studer, and Thiran (1995). The *general* problem, however, is exorbitantly involved, and conclusions lack the simplicity and transparency of the results obtained when restricting the analysis to the class of *chiral* QH lattices. Experimentally, it would be interesting to test the chirality assumption by direct edge-current measurements, e.g., of the type reported in Ashoori *et al.* (1992); (which, by the way, are compatible with a *purely chiral* structure for the QH fluid at $\sigma_H = 2/3$ that has been studied there).

General Structuring Results

• *The Inequality* $\ell_{max} \geq \sigma_H^{-1}$. The first important result of our analysis is that, for an *arbitrary* CQHL $(\Gamma, \mathbf{Q})$, the associated invariants σ_H and ℓ_{max} satisfy the fundamental inequality $\ell_{max} \geq \sigma_H^{-1}$; see (8.47). This result implies a first *organizing principle* for CQHLs with $0 < \sigma_H \leq 1$. It offers a natural splitting of the interval $(0, 1]$ into successive windows, Σ_p, defined by $1/(2p+1) \leq \sigma_H < 1/(2p-1)$, $p = 1, 2, \ldots$. The relative-angular-momentum invariant ℓ_{max} characterizing an arbitrary CQHL with $\sigma_H \in \Sigma_p$ is thus greater or equal to $2p+1$, $p = 1, 2, \ldots$. Adopting the (heuristic) stability principle of Subsect. 8.4, CQHLs in successive windows Σ_p, for *increasing* values of p, are expected to describe QH fluids of *decreasing* stability.

Combining the fundamental inequality above with a (universal) upper bound ℓ_* on the relative-angular-momentum invariant ℓ_{max}, we conclude that, physically, *no* chiral QH fluid can form with a Hall conductivity $\sigma_H < 1/\ell_*$. Theoretical and numerical arguments (see the remarks preceding Thm. 8.4.2) as well as the present-day experimental data (see Fig. 4.1 in Subsect. 4.5) suggest that $\ell_* = 7$. In the sequel, we shall impose this bound.

Thus, for the classification of *physically relevant* CQHLs, we have to consider only the first three windows, Σ_1, Σ_2, and Σ_3, because only there CQHLs with $\ell_{max} \leq 7$ can be found. In particular, in the third window Σ_3, any physically relevant CQHL $(\Gamma, \mathbf{Q})$ *must* have $\ell_{max} = 7$, meaning that $(\Gamma, \mathbf{Q})$ has to be *L-minimal*, i.e., all relative-angular-momentum invariants take their smallest possible value; see Def. 8.4.1.

- *Uniqueness Theorem (Thm. 8.4.8).* The uniqueness theorem of the previous subsection classifies all *L-minimal* CQHLs in the left halfs, Σ_p^+, of the windows Σ_p, i.e., in the subintervals $1/(2p+1) \leq \sigma_H < 1/(2p)$, $p = 1, 2, \ldots$. A unique, N-dimensional, L-minimal CQHL is found at every Hall fraction $\sigma_H = N/(2pN+1)$, $p, N = 1, 2, \ldots$. Given this uniqueness theorem and the bound $\ell_* = 7$, we find the following remarkable result: The classification problem of *physically relevant* CQHLs can be *completely* solved in the small subwindow $\Sigma_3^+ := [1/7, 1/6)$. In Σ_3^+, the only fractions at which such lattices can be found are $\sigma_H = N/(6N+1)$, $N = 1, 2, \ldots$, and the electrons of the corresponding unique, chiral QH fluids carry an $SU(N)$-symmetry.

In Thm. 8.4.1 we have mentioned that any set of CQHLs satisfying upper bounds on their invariants ℓ_{max} and N is finite. It is interesting to note that, *independently* of any upper bound, N_*, on the dimension N, every *closed* subinterval in Σ_3^+ contains only a *finite* number of physically relevant ($\ell_{max} \leq 7$) CQHLs. Presently we do not know to which extent this is also true in other (sub)windows. The relevance of this remark derives from the fact that it is rather difficult to find satisfactory bounds N_* on the dimension N of physically relevant QH lattices; see the remarks preceding Thm. 8.4.2. Thus, by-passing the need for a bound on the dimension N represents progress in the theory.

So far, the only indication of a QH fluid in Σ_3^+ has been found at $\sigma_H = 1/7$, corresponding to the first member of this series. It coincides with the Laughlin fluid at $m = 7$; see example **(b)** in Subsect. 8.3. Further probing (although difficult experimentally) of the subwindow Σ_3^+ would clearly be interesting!

- *L-Minimality.* Next, we comment on the *assumption of L-minimality* that we impose when trying to classify CQHLs that correspond to stable physical QH fluids. As discussed above, this assumption is strictly justified only in the window Σ_3. In general, it is an implementation of the *heuristic stability principle* described in Subsect. 8.4 stating that the *lower* the values of ℓ_{max} and N of a CQHL, the *more stable* the corresponding QH fluid. Thus, the justification of the assumption of L-minimality in the other two windows, Σ_2 and Σ_1, is directly connected to the validity of this stability principle. Thanks to the precise predictions it yields, experimental tests can be proposed to settle its validity. Such tests are discussed in detail by Fröhlich, Studer, and Thiran (1995).

In order to formulate such tests, one has to go beyond the assumption of L-minimality in classifying CQHLs, which requires much more work and can be achieved, in full generality, only for small values of the bounds N_* and ℓ_*. In Fröhlich, Studer, and Thiran (1995), we have fully classified all CQHLs with upper bounds on ℓ_{max} and N given by $(\ell_*, N_*) = (7,2)$, $(\ell_*, N_*) = (5,3)$, and $(\ell_*, N_*) = (3,4)$. The general problem is bound to be very complicated, since it involves as an input the knowledge of the complete classification of integral lattices with given bounds on the lattice discriminant, and this is a very intricate mathematical problem (unsolved, in general, for euclidean lattices, as needed here). With more patience and computing skill one might extend the above results slightly (at most, however, up to cases where $N_* = 6$ or 7) because, in low dimensions, complete lists of lattices can be computed (Conway and Sloane, 1988b; Dickson, 1930); see Sect. 6 in Fröhlich, Studer, and Thiran (1995). Partial results (for small discriminants) derived from the lattice classification in Conway and Sloane (1988b) have already been given in Table 2 in Fröhlich and Thiran (1994). In addition to these results, however, we have also classified all L-minimal, maximally symmetric CQHLs with $\sigma_H \leq 1$, (*independent* of any upper bound on the dimension N) in Fröhlich, Studer, and Thiran (1995), as mentioned in Subsect. 8.5.

The upshot of all the results in Fröhlich, Studer, and Thiran (1995) is that *the assumption of L-minimality for physically relevant CQHLs is well supported by the presently available experimental data.* Hence, in the sequel, we adopt it as our basic working hypothesis.

- *Bijection Theorem (Thm. 8.4.7).* An important theorem described in the previous subsection, the bijection theorem (for a proof, see Fröhlich, Kerler *et al.*, 1995), asserts that there are one-to-one correspondences between the sets of L-minimal CQHLs in the different windows Σ_p, $p = 1, 2, \ldots$. Although this theorem does not classify CQHLs, it is a powerful *structuring device* for L-minimal CQHLs with $\sigma_H \leq 1$. In particular, it reduces the classification of L-minimal CQHLs with Hall fractions in the entire interval $0 < \sigma_H < 1$ to the classification of such lattices in the "fundamental domain" $\Sigma_1 := [1/3, 1)$! For a pictorial summary of results for this fundamental domain, see Fig. 8.1.

Theoretical Implications versus Experimental Data

In the remaining part of this subection, we shall discuss explicit consequences of the uniqueness and bijection theorems (Fröhlich, Kerler *et al.*, 1995), complement them with classification results given in Fröhlich, Studer, and Thiran (1995), and confront our conclusions with the experimental data summarized in Fig. 4.1 in Subsect. 4.5.

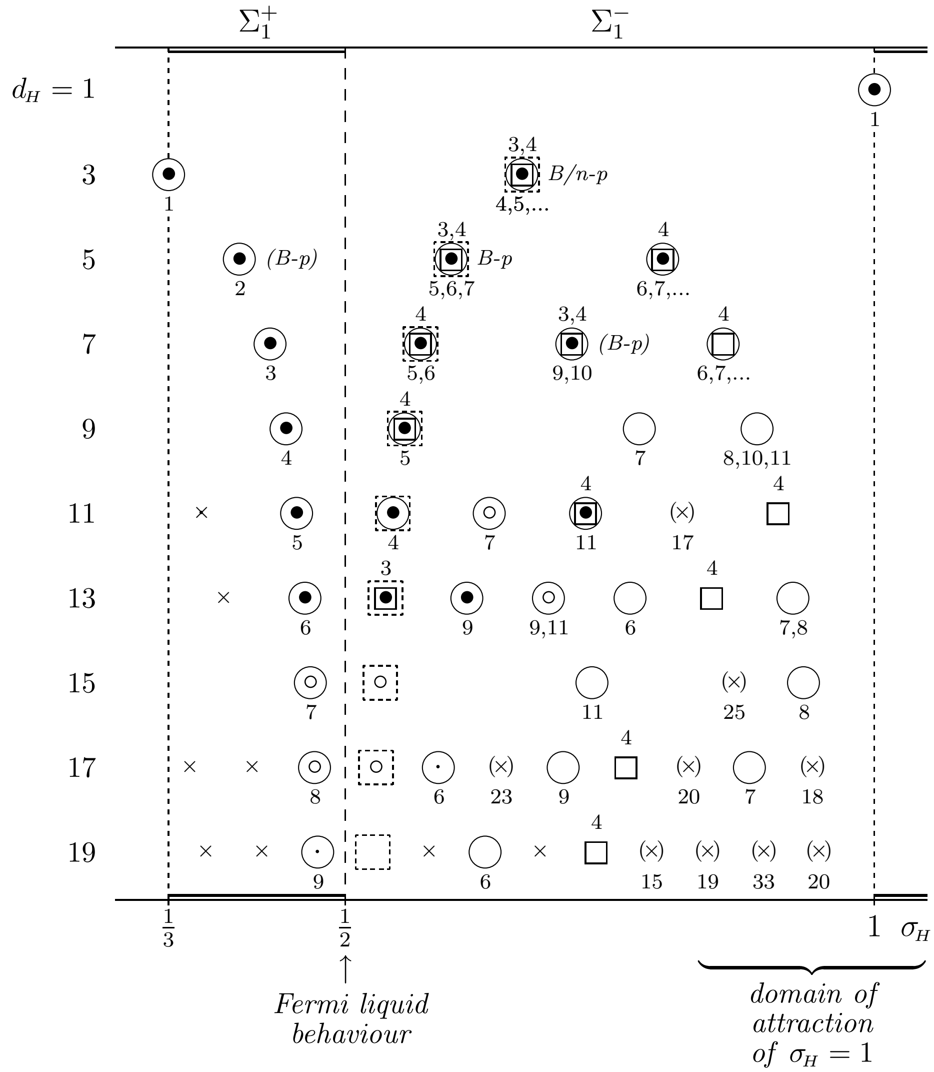

Figure 8.1. *Compilation of L-minimal ($\ell_{max} = 3$) chiral quantum Hall lattices (CQHLs) with odd-denominator Hall fractions σ_H in the interval $1/3 \leq \sigma_H \leq 1$.*

The experimental status of the Hall fractions displayed here is indicated, for single-layer systems, by "$\bullet$, $\circ$", and "$\cdot$", as in Fig. 4.1. Superposed on the interval $1/3 \leq \sigma_H \leq 1$ of that figure is a list of different L-minimal CQHLs: "$\bigcirc$" indicates maximally symmetric, L-minimal CQHLs of dimension $N \leq 11$ (where the corresponding dimensions are given below the symbols); "$\square$" indicates generic, indecomposable, L-minimal CQHLs of low dimension, $N \leq 4$ (the respective dimensions are given above the symbols). For fractions decorated with "$(\times)$", there are no low-dimensional ($N \leq 4$), L-minimal CQHLs. However, there are maximally symmetric ones in "high" dimensions (with the lowest such dimension indicated below the symbols). At fractions with "$\times$", there are neither low-dimensional, L-minimal CQHLs, nor maximally symmetric ones in "higher" dimensions. In addition, "$\square$" stands for non-chiral QH lattices that are "charge-conjugated" to the maximally symmetric, L-minimal CQHLs in Σ_1^+.

• *The Subwindows* Σ_p^+. Adopting the hypothesis of L-minimality, the uniqueness theorem tells us that, in the subwindows Σ_p^+, *no* (chiral) QH fluids can be found with Hall conductivities $\sigma_H \neq N/(2pN+1)$, $N, p = 1, 2, \ldots$. Taking a look at Fig. 4.1, remarkable agreement between this theoretical prediction and experimental data is found: QH fluids have been observed at $N/(2N+1)$, $N = 1, \ldots, 9$, in Σ_1^+; at $N/(4N+1)$, $N = 1, 2$, and 3, in Σ_2^+; and, as already mentioned, at just one value of $N/(6N+1)$, namely $N = 1$, in Σ_3^+. As we have mentioned in Subsect. 8.3, one may recognize these Hall fractions as the ones of the "basic Jain states" (Jain, 1989, 1990, 1992). One can show (Fröhlich, Studer, and Thiran, 1995) that, at the above fractions, the proposals of the hierarchy schemes (Haldane, 1983; Halperin, 1984; Jain and Goldman, 1992) and of our L-minimal-CQHL scheme coincide. The additional insight our approach offers is that all these proposals have a *unique* status as L-minimal, chiral QH fluids!

A closer inspection of Fig. 4.1 shows that, in the subwindows Σ_p^+, there seems to be *only one* fraction, $\sigma_H = 4/11$, at which a weak signal of a QH fluid has been reported, and which does *not* belong to the set of fractions described by the uniqueness theorem. The corresponding experimental data (reported only once) are somewhat controversial; see Hu (1991). Theoretically, a QH fluid at $\sigma_H = 4/11$ is predicted by the Haldane-Halperin and the Jain-Goldman hierarchy scheme at *low(!)* "level" 2 and 3, respectively. These two proposals can be shown (Fröhlich, Studer, and Thiran, 1995) to belong to one and the *same* universality class of QH fluids described by a *non-L-minimal*, two-dimensional (primitive) CQHL which, in some sense, provides the "simplest" example of a non-L-minimal CQHL. This fraction marks thus an interesting plateau value where further experiments might challenge the hierarchy schemes and/or our working hypothesis of L-minimality.

It has been pointed out in the literature (Hu, 1991) that the *absence* in the data of Fig. 4.1 of a QH fluid at $\sigma_H = 5/13$ is quite remarkable. Indeed, this fraction is conspicuous by its absence from the list of observed Hall fractions (in single-layer systems) with denominator $d_H = 13$, which are $\sigma_H = 3/13$, $4/13$, $6/13$, $7/13$, $8/13$, and $9/13$. Theoretically, the Haldane-Halperin and the Jain-Goldman hierarchy schemes predict a QH fluid with $\sigma_H = 5/13$ at *low(!)* "level" 3 and 2, respectively. These two proposals correspond to a *non-chiral* QH lattice. We note that, in addition, there is an (inequivalent) *chiral*, but *non-L-minimal* QH lattice in three dimensions with $\sigma_H = 5/13$; see Fröhlich, Studer, and Thiran (1995). This fraction is thus another interesting plateau value where the hierarchy schemes and/or the L-minimality assumption can be tested further.

By a similar reasoning process, in the first subwindow Σ_1^+, all fractions in the open intervals $N/(2N+1) < \sigma_H < (N+1)/(2N+3)$, $N = 1, 2, \ldots$, are interesting plateau values for testing the L-minimality assumption. To be explicit, we do *not* expect stable QH fluids to form at $\sigma_H = 4/11(!)$, $5/13(!)$, $6/17$, $7/19$, $8/21, \ldots$ in $(1/3, 2/5)$, and at $\sigma_H = 7/17$, $8/19, \ldots$ in $(2/5, 3/7)$. Note that, with the help of the shift maps $\mathcal{S}_1$ and $\mathcal{S}_2$ (see (8.56)), these predictions can be translated into predictions in the subwindows Σ_2^+ and Σ_3^+.

• *The Subwindows* Σ_p^-, $p = 1, 2, \ldots$. In the "complementary" subwindows, Σ_p^-, defined by $1/(2)p \leq \sigma_H < 1/(2p-1)$, $p = 1, 2, \ldots$, we do *not* have a complete classification of L-minimal CQHLs. Nevertheless, we can make interesting observations for these subwindows by exploiting the bijection theorem of the previous subsection and the (partial) classification results given in Fröhlich, Studer, and Thiran (1995); see Fig. 8.1.

First, we note that the experimental data in $\Sigma_1^- := [1/2, 1)$ (see Fig. 4.1) can hardly be interpreted as a complete "mirror image" of the data in the interval $(0, 1/2]$, as one would expect if charge conjugation were at work *in general*. Second, comparing, the data in the two complementary subwindows Σ_1^- and Σ_1^+, we find, besides the prominent series of fractions $\sigma_H = n/(2n-1)$, $n = 2, \ldots, 9$, "mirroring" the unique fractions in Σ_1^+ (i.e., $\sigma_H = 1 - \sigma_H'$), data points at $\sigma_H = 4/5$, $5/7$, $7/11$, $8/11$, $8/13$, $9/13$, and possibly at $10/17$. This is a first *experimental* indication that the sets of QH fluids appearing in the complementary subwindows Σ_p^+ and Σ_p^-, $p = 1, 2, \ldots$, are "*structurally distinct*", as our theory predicts.

We may ask to which extent the experimental data in Fig. 4.1 also support the one-to-one correspondences predicted by the bijection theorem between QH fluids in the different subwindows Σ_p^-. We can act "formally" with the shift maps $\mathcal{S}_{p-1}$, $p = 2$ and 3, of the bijection theorem on the fractions σ_H given in Σ_1^- of Fig. 4.1, for example, $\mathcal{S}_{p-1} : n/(2n-1) \mapsto n/(2pn-1)$; see (8.56). The resulting fractions σ_H that we obtain in the two subwindows Σ_2^- and Σ_3^- are fully *consistent* with the experimental data given in Fig. 4.1. Experimentally observed are the fractions $\sigma_H = n/(4n-1)$, $n = 2, 3, 4$, and $\sigma_H = 4/13$ (very weakly) in $\Sigma_2^- = [1/4, 1/3)$, and only one fraction in $\Sigma_3^- = [1/6, 1/5)$, namely $\sigma_H = n/(6n-1)$, with $n = 2$.

If the QH fluids corresponding to points in Σ_1^- were described by L-minimal CQHLs then, by the logic of the bijection theorem, we would predict the formation of (chiral) QH fluids at $\sigma_H = 4/13(!)$, $5/17$, $5/19, \ldots$ in Σ_2^-, and at $\sigma_H = 3/17$, $4/21, \ldots$ in Σ_3^-. These are thus interesting plateau-values for experimentation.

What do we know *explicitly* about L-minimal CQHLs in the subwindows Σ_p^-? As mentioned above, the analysis presented in Fröhlich, Studer, and Thiran (1995) contains, in particular, a complete classification of all *low-dimensional* ($N \leq N_* = 4$) and of all maximally symmetric, L-minimal CQHLs with $\sigma_H \leq 1$.

• *Summary of Classification Results for the Fundamental Subdomain* Σ_1^-. The upshot of our analysis (see Fig.8.1) is that, in Σ_1^-, natural proposals for QH fluids at the fractions of the series $\sigma_H = n/(2n-1)$, $n = 2, 3, \ldots$, are provided by the *charge-conjugation picture*, meaning that the corresponding QH fluids are *composite*. They consist of an electron-rich subfluid with a partial Hall fraction $\sigma_{(1)} = 1$, and of a hole-rich subfluid corresponding to an L-minimal CQHL of the $su(N)$-series in Σ_1^+ with partial Hall fraction $\sigma_{(2)} = -N/(2N+1)$, where $N = n - 1$. This is, however, *not* the full story! It is a *structural property* of the subwindows Σ_p^- that, at a given Hall fraction σ_H, one typically finds *more than one* L-minimal CQHL realizing that fraction. This is much in contrast to the *unique* realization of the fractions $\sigma_H = N/(2N+1)$ in the complementary subwindow Σ_1^+.

For example, at the *finite* series of fractions $\sigma_H = n/(2n-1)$, $n = 2, \ldots, 7$, one finds maximally symmetric, L-minimal CQHLs in dimensions $N = 10 - n$ which are based on the root lattices of the exceptional Lie algebras E_{9-n}, similarly to the way in which the $su(N)$-QH lattices in Σ_1^+ are based on the root lattices of $su(N)$; see Subsect. 8.5. While the last two members of this finite series, $\sigma_H = 6/11$ and $7/13$, are realized by unique, low-dimensional ($N = 4$ and 3, respectively), L-minimal CQHLs, the higher dimensional members of this "E-series" of CQHLs contain several embedded L-minimal, chiral "*QH (sub)lattices*" of lower dimensions. All these QH (sub)lattices (although not necessarily maximally symmetric) represent possible proposals for QH fluids at the corresponding Hall fractions σ_H.

Let us define precisely what we mean by a "QH lattice embedding":

Definition 8.6.1. *A QH lattice* $(\Gamma', \mathbf{Q}' \in \Gamma'^*)$ *is embedded into another QH lattice* $(\Gamma, \mathbf{Q} \in \Gamma^*)$ *if (i) both QH lattices correspond to the same Hall fraction, i.e.,* $\sigma'_H = <\mathbf{Q}', \mathbf{Q}'> = <\mathbf{Q}, \mathbf{Q}> = \sigma_H$, *(ii)* Γ' *is a sublattice of* Γ, *and (iii) the two charge vectors* $\mathbf{Q}'$ *and* $\mathbf{Q}$ *are compatible in the sense that all multi-electron/hole states described by* $(\Gamma', \mathbf{Q}')$ *remain physical states when viewed (via the lattice embedding* $\Gamma' \subset \Gamma$*) as states described by* $(\Gamma, \mathbf{Q})$. *In particular, all the electric charges stay the same, i.e.,* $<\mathbf{Q}', \mathbf{q}'> = <\mathbf{Q}, \mathbf{q}'>$, *for all* $\mathbf{q}' \in \Gamma' \subset \Gamma$.

At the level of symbols (see (8.33)), we write such embeddings as

$$ {}_{N'}\!\left(\frac{n_H}{d_H}\right)^{g'}_{\lambda'} [\ell'_{min}, \ell'_{max}] \;\hookrightarrow\; {}_{N}\!\left(\frac{n_H}{d_H}\right)^{g}_{\lambda} [\ell_{min}, \ell_{max}] \,. \tag{8.70} $$

Note that, as an immediate consequence of definition (8.38), $\ell'_{min} \geq \ell_{min}$.

Physically, a QH fluid described by the QH lattice $(\Gamma', \mathbf{Q}')$ embedded into another lattice $(\Gamma, \mathbf{Q})$ is characterized by a *restricted* set of possible multi-electron/hole excitations, as compared to the corresponding set of the fluid associated with the lattice $(\Gamma, \mathbf{Q})$. Furthermore, since the neutral sublattice (see (8.52)) of $(\Gamma', \mathbf{Q}')$ is a sublattice of the neutral sublattice of $(\Gamma, \mathbf{Q})$, the embedded fluid exhibits a (global) symmetry group G' (see (8.63)) which is a *subgroup* of G, the symmetry group exhibited by the fluid associated with $(\Gamma, \mathbf{Q})$. Thus, in this precise sense, *the embedded fluid exhibits a more restricted symmetry than the one it embeds into.* Put differently, *passing from a QH fluid to an embedded subfluid corresponds to a "reduction or breaking of symmetries".* [As a mathematical aside, we remark that the study of embeddings of maximally symmetric CQHLs into one another is equivalent to the study of regular conformal embeddings of level-1 Kac-Moody algebras and the respective branching rules. For recent results on the latter subject, see, e.g., Schellekens and Warner (1986), Bais and Bouwknegt (1987), and Kac and Sanielevici (1988).] Experimentally, symmetry breaking might be realized in *phase transitions* that are driven, at a given Hall fraction, by varying external control parameters. Hence, it is most interesting to see at which fractions in Σ_1^- such "*structural*" *phase transitions* can be expected within our framework.

As an example (see Appendix D in Fröhlich, Studer, and Thiran (1995)), one finds 13 and 5 QH sublattices embedded into the QH lattice of the E-series at $\sigma_H = 2/3$ (E_7) and $3/5$ (E_6), respectively. The *composite* L-minimal CQHL at $\sigma_H = 2/3$ which consists of two Laughlin subfluids with partial Hall fraction $\sigma_{(i)} = 1/3,\ i = 1, 2$, is the lowest-dimensional ($N = 2$) QH sublattice of the E_7-QH lattice at $\sigma_H = 2/3$. All the other QH sublattices at $\sigma_H = 2/3$ are *indecomposable*. (Recall that we have mentioned in Thm. 8.4.6 that all L-minimal CQHLs with $\sigma_H < 2/3$ are indecomposable.) In $\Sigma_1^- = [1/2, 1)$, complex embedding patterns of L-minimal CQHLs are found at the fractions $\sigma_H = 2/3,\ 3/5,\ 4/5,\ 4/7,\ 5/7,\ 5/9$, and at the even-denominator fraction $\sigma_H = 1/2$! These fractions are interesting in the light of the data in Fig. 4.1, where phase transitions are indicated at $\sigma_H = 2/3,\ 3/5$, and possibly at 5/7, driven by an added in-plain component of the external magnetic field (Clark *et al.*, 1990; Engel *et al.*, 1992; Sajoto *et al.*, 1990), and at $\sigma_H = 2/3$, driven by changing the density of charge carriers in the system (Eisenstein, Stormer *et al.*, 1990b); see also the data reported in Suen *et al.* (1994) on phase transitions in wide-single-quantum-well systems!

[Clearly, by the uniqueness of the L-minimal CQHLs in the subwindows Σ_p^+ and our heuristic stability principle, we do *not* expect structural phase transitions there. So, what about a possible indication of a magnetic field driven phase transition at $\sigma_H = 2/5$? As a matter of fact, in Fröhlich and Studer (1993b) (see also Fröhlich, Studer, and Thiran (1995)) we have argued that $\sigma_H = 2/5$ is the most likely plateau value where we may expect a phase transition from a *spin-polarized* to a *spin-singlet* QH fluid. While, structurally, the two phases are described by *one and the same* L-minimal CQHL, the phase transition corresponds to a change from an *internal* $SU(2)$-symmetry to a *spatial* $SU(2)_{spin}$-symmetry.]

We complete our short review of results derived in Fröhlich, Studer, and Thiran (1995) – see Fig. 8.1 – by mentioning that to *all* data points in Σ_1^- (including the fraction $\sigma_H = 1/2$) one can associate *at least one* L-minimal CQHL that is either generic (without special symmetry properties) and low-dimensional ($N \leq 4$), maximally symmetric with dimension $N \leq 9$ (based on the root lattice of a simple or semi-simple Lie algebra), or charge-conjugated to an $su(N)$-lattice in Σ_1^+. Within these three subclasses of CQHLs, predictions of *new* QH fluids are made at $\sigma_H = 6/7,\ 10/13,\ 10/17(!),\ 13/17,\ 10/19,\ 12/19,\ 14/19, \ldots$, and at the *even-denominator* fractions $\sigma_H = 3/4$ and $5/8$. The CQHLs that yield even-denominator Hall fractions have a structure that can naturally be interpreted as describing *double-layer/component* QH *systems*. Furthermore, staying within these three subclasses, we do *not* expect *stable(!)* QH fluids to form at $\sigma_H = 9/11,\ 11/17,\ 14/17,\ 13/19$, and $15/19$ in Σ_1, where we have omitted fractions with $d_H \geq 21$ and within the "domain of attraction" of the most stable Laughlin fluid at $\sigma_H = 1$. *None* of these fractions has been observed experimentally! These predictions are rather different from those of the standard hierarchy schemes (Haldane, 1983; Halperin, 1984; Jain and Goldman, 1992); for further discussions, see Fröhlich, Studer, and Thiran (1995).

• *Phase Transitions at* $\sigma_H = 1$. An interesting question to ask is whether one should expect to observe *phase transitions at* $\sigma_H = 1$. The unique L-minimal ($\ell_{max} = 1$) CQHL at $\sigma_H = 1$ is the one-dimensional "Laughlin lattice" with $m = 1$; see example **(b)** in Subsect. 8.3. Thus, any other CQHL realizing this fraction necessarily has to be *non-L-minimal* ($\ell_{max} \geq 3$), a fact that suggests a *markedly reduced stability* for the corresponding fluids, as compared to the (L-minimal) Laughlin fluid! Moreover, by Thm. 8.4.4, we know that any other indecomposable CQHL at this fraction exhibits a charge parameter λ strictly larger than 1. By an argument similar to the one in (8.48), this leads to the prediction of *fractional* charges in these fluids! For the purpose of illustration, we give the lowest-dimensional examples of such CQHL's classified in Appendix C of Fröhlich, Studer, and Thiran (1995). Using the symbol introduced in (8.33), one finds the following embeddings of non-L-minimal CQHLs at $\sigma_H = 1$:

$$ {}_2(1)_2^4\ [3,3] \hookrightarrow \left\{ \begin{array}{c} {}_3(1)_2^6\ [3,3] \\ \\ {}_3(1)_2^8\ [3,3] \end{array} \right\} \hookrightarrow {}_5(1)_2^8\ [3,3] \hookrightarrow \ \ldots\ . \tag{8.71} $$

This "chain of embeddings", with the corresponding possibilities of structural phase transitions, is particularly interesting in the light of the recent experimental data given in Murphy *et al.* (1994). There, evidence for a phase transition between different QH fluids at $\sigma_H = 1$ has been reported. The phase transition appears to be driven by an in-plane magnetic field, $\mathbf{B}_c^{\parallel}$, and is observed in *double-layer* QH systems. This matches our theoretical predictions: In (8.71), e.g., the first two CQHLs, (the lattice with symbol ${}_2(1)_2^4$ and the one with symbol ${}_3(1)_2^6$), both are natural candidates for describing double-layer/component QH fluids. The first one can be shown to exhibit a discrete $\mathbb{Z}_2$ layer symmetry, while the second one can be seen to exhibit a continuous $A_1 = su(2)$ layer symmetry; see Fröhlich, Studer, and Thiran (1995). Furthermore, since for all lattices in (8.71) the charge parameter λ equals 2, we would expect, as mentioned above, that quasi-particles with fractional charge 1/2 exist in the corresponding QH fluids. An experimental investigation of this prediction would seem to be revealing and is encouraged!

• *Concluding Remarks.* By the bijection theorem, *all* the statements about L-minimal *chiral* QH lattices in $\Sigma_1^- := [1/2, 1)$ have their precise analogues in the shifted subwindows $\Sigma_p^- := [1/(2p), 1/(2p-1))$, for $p = 2, 3, \ldots$. For example, interpreting the phase transitions observed at $\sigma_H = 2/3,\ 3/5$, and possibly $5/7$ in Σ_1^- as *structural phase transitions*, we predict analogous transitions at the Hall fractions $\sigma_H = 2/7,\ 3/11,\ (5/17)$ in Σ_2^-, and at $\sigma_H = 2/11,\ 3/17,\ (5/27)$ in Σ_3^-.

When acting with the shift maps $\mathcal{S}_p$ on the composite, *mixed-chirality* QH lattices at $\sigma_H = n/(2n-1),\ n = 2, 3, \ldots$, corresponding to the charge-conjugation picture, we obtain composite *non-euclidean* QH lattices which are however *not* primitive (where primitivity has been defined at the end of Subsect. 8.1). A discussion of non-primitive QH lattices and the corresponding QH fluids has been given in Appendix E in Fröhlich, Studer, and Thiran (1995). In particular, we have described composite, non-euclidean (but "factorizable") QH lattices corresponding to the hierarchy fluids in the windows $\Sigma_p^-,\ p = 1, 2, 3$.

For the complementary subwindows $\Sigma_p^+ := [1/(2p+1), 1/(2p))$, the one-to-one correspondences between the different sets of L-minimal CQHLs are contained in the classification result of the uniqueness theorem discussed above.

These results may suffice to convince the reader that there is a significant *structural asymmetry* between the sets of QH fluids with Hall conductivities in the two complementary subwindows Σ_p^+ and Σ_p^-, for a given $p = 1, 2, \dots$, while there is a *structural similarity* among QH fluids with conductivities in the "+-windows" Σ_p^+ and among the ones with conductivities in the "−-windows" Σ_p^-, for different values of p.

After more than ten years since its discovery (Tsui, Stormer, and Gossard, 1982), the fractional QH effect in the interval $0 < \sigma_H \leq 1$ is thus still an interesting field of experimental and theoretical research. In our recent work, summarized in this section, we have argued that the QH-lattice approach provides an efficient instrument for describing universal properties of QH fluids. In our analysis of *physically relevant* QH lattices we have assumed chirality and L-minimality as basic properties. New and refined experimental data in the neighbourhood of the various plateau-values discussed above would either support or question these hypotheses, and hence could lead to further progress in the understanding of this fascinating effect. The idea, however, that the physics of incompressible QH fluids at large distance scales and low frequencies can be encoded into pairs of an integral lattice Γ and a primitive vector $\mathbf{Q}$ in its dual lattice should be regarded as a firmly established theoretical result! (Number theory makes its appearance in condensed matter physics.)

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