Classification of Symmetry Protected Topological Phases in Interacting Systems

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Outline

- Intrinsic topological(IT) order and symmetry protected topological(SPT) order.

- 1D SPT phases in interacting bosonic systems.

- 2D and 3D SPT phases in interacting bosonic systems.

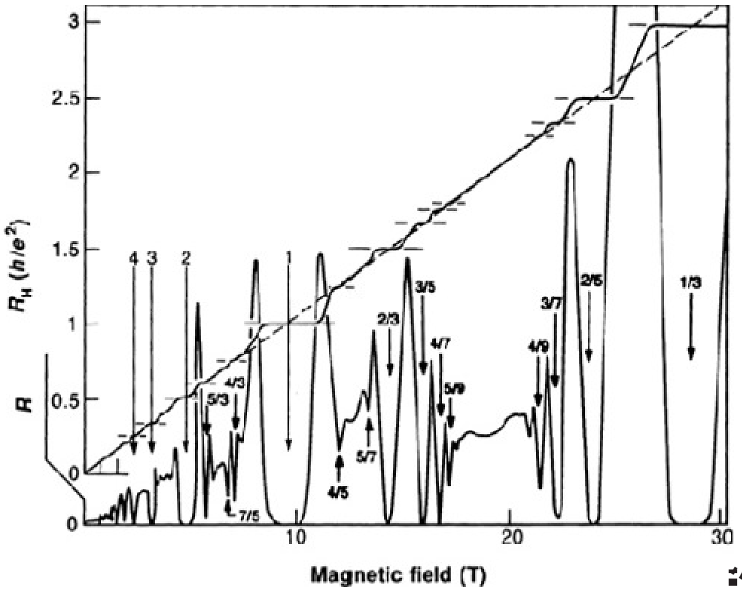
- SPT phases in interacting fermionic systems.

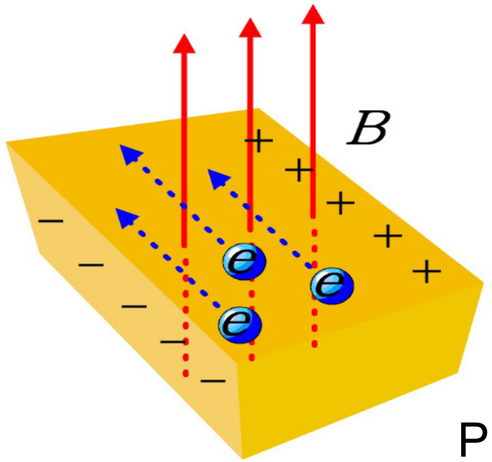
- Summary and outlook.

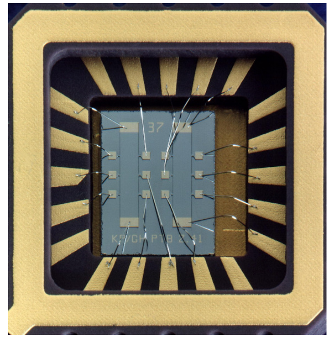
Topological phenomena in strongly correlated systems

Fractional Quantum Hall Effect

D C Tsui, *et al* 1982



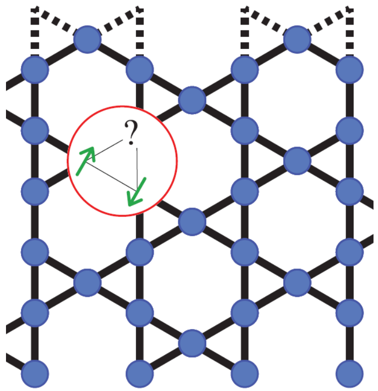


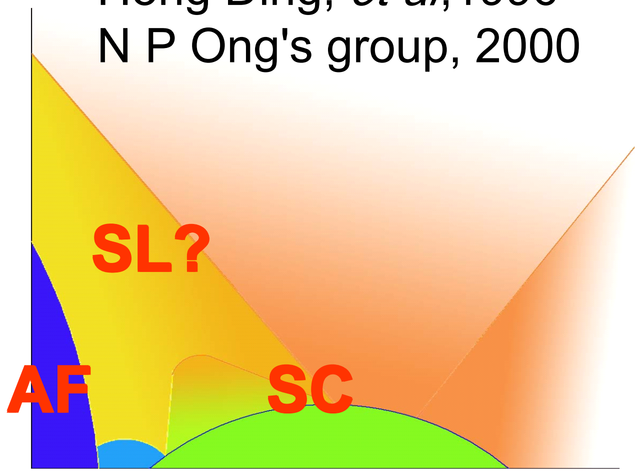


P W Anderson, 1987
Hong Ding, *et al*,1996
N P Ong's group, 2000

Spin liquid

- Frustrated magnets
- High-Tc cuprates





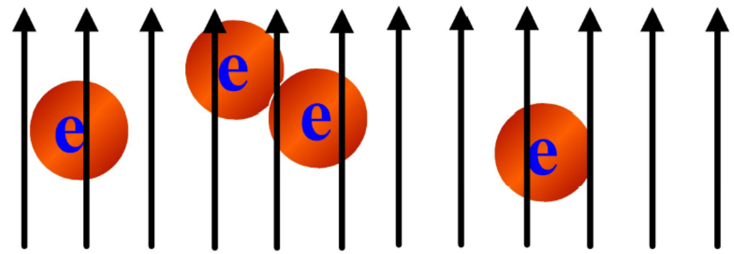
S Yan, D Huse and S White Science, 2011

New phases of matter: intrinsic topological order X.-G. Wen, 1989

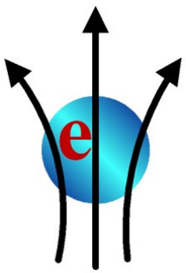
- Can have the same symmetry as disordered systems.
- Gapped ground state without long range correlations.
- Ground state degeneracy depends on the topology of the manifold.
- Ground state degeneracy is robust against any local perturbations.
- Excitations carry fractional statistics.
- Protected chiral edge states (chiral topological order, e.g. FQHE).

Topological terms for intrinsic topological order

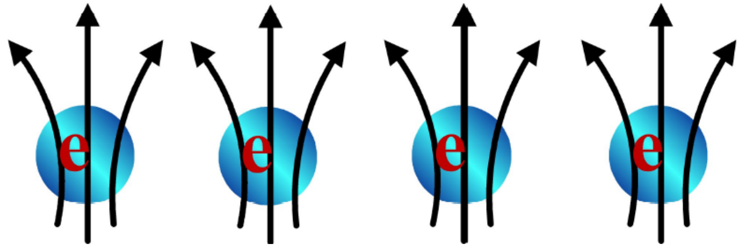
FQHE $\Psi_3 = \prod_{i<j}(z_i - z_j)^3 \mathrm{e}^{-\frac{1}{4}\sum_i |z_i|^2}$



$$\mathcal{L}_{\text{eff}} = \frac{2m+1}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

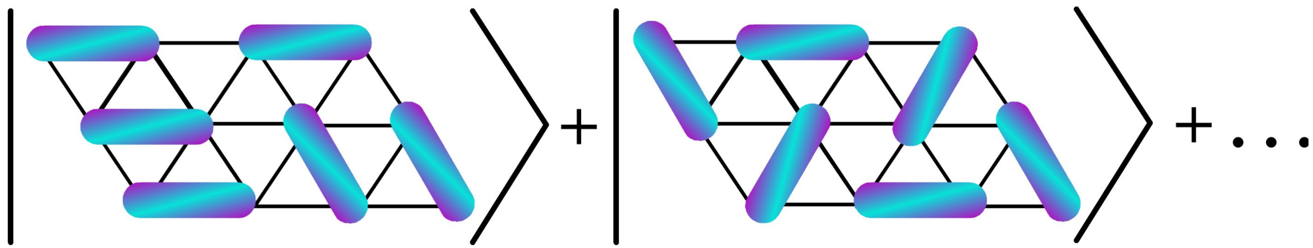


R B Laughlin 1983
E Witten, 1989
S C Zhang, *et al* 1989
X G Wen, *et al* 1989



Z_2 spin liquid R. Moessner and S. L. Sondhi 2001

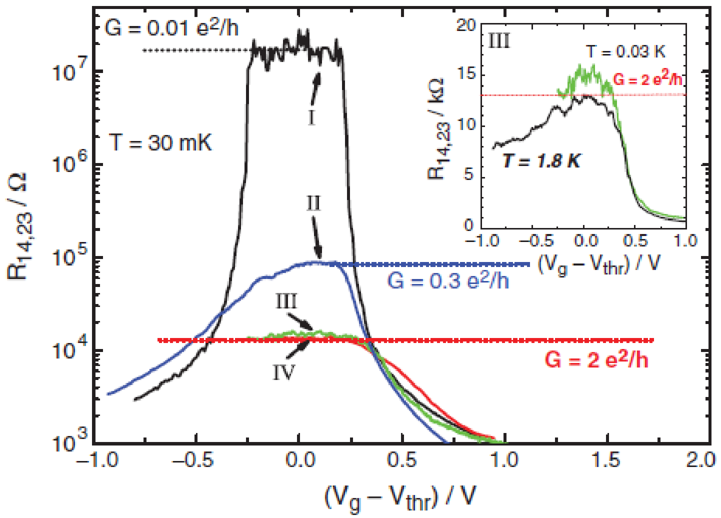
$|\Psi_{RVB}\rangle =$

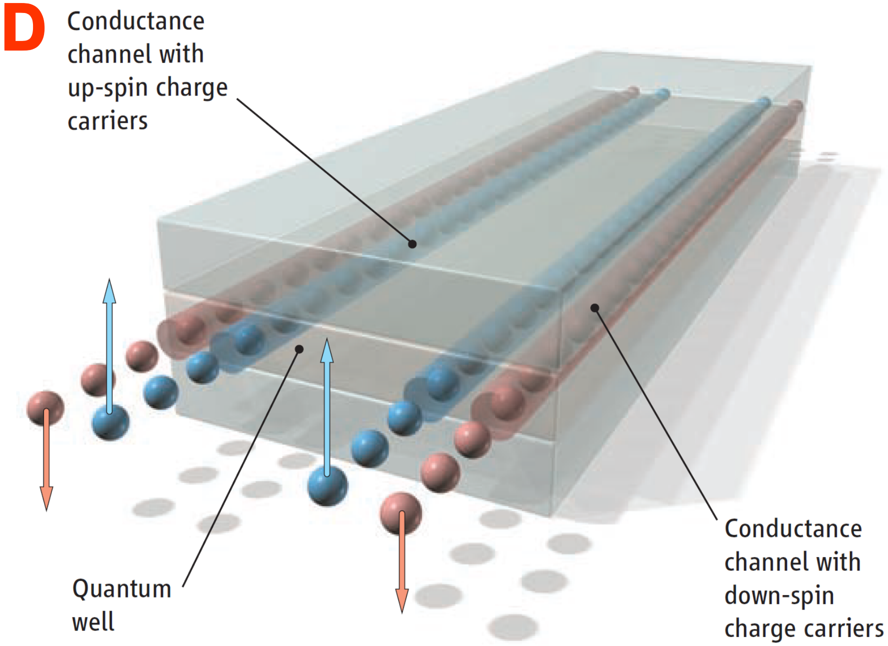


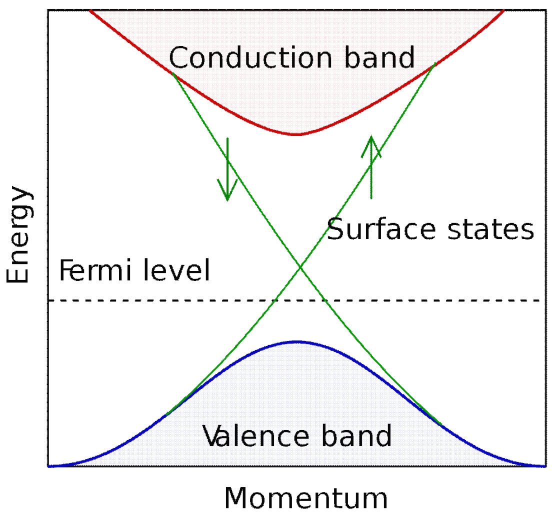
$$L = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda}, \qquad I, J = 1, 2 \qquad K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Symmetry protected topological(SPT) phenomena

Topological insulator in 2D/3D

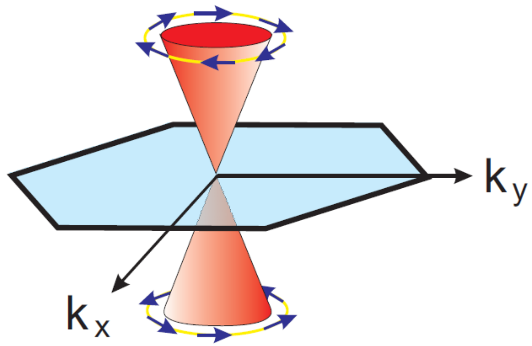






C L Kane, *et al*, 2005
B A Bernevig, et al 2006
W Molenkamp's group 2007
M Zahid Hasan, *et al*, 2008

from Wikipedia



New phases of matter: symmetry protected topological order

Z C Gu and X G Wen, 2009

- Can have the same symmetry as trivial disordered systems.

- Gapped ground state without long range correlations.

- Excitations do not carry fractional statistics.

- Indistinguishable from trivial disordered systems if symmetry is broken in bulk.

- Stable against any local perturbations preserving symmetry.

- Protected gapless edge states if symmetry is not (spontaneously or explicitly)broken on the edge.

SPT phase in strongly interacting 1D model

Spin one Haldane chain realizes 1D topological order(even with strong interaction)

$$H = \sum_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1} + U(S_i^z)^2)$$

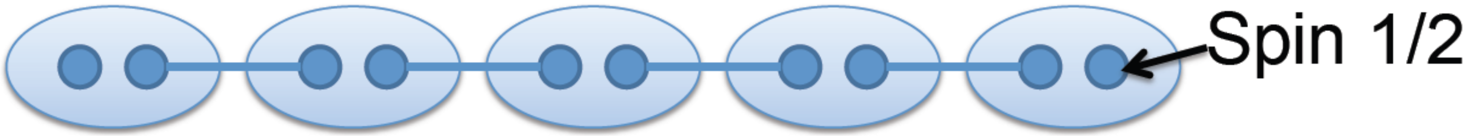
- stable up to U~1

But Haldane phase requires symmetry protection!

- Haldane phase can be protected by many kinds of symmetries: time reversal, spin rotation, etc...

Z C Gu and X G Wen, 2009, F Pollmann, *et al*, 2010

Fixed point wavefunction: spin-(1/2,1/2) dimer model

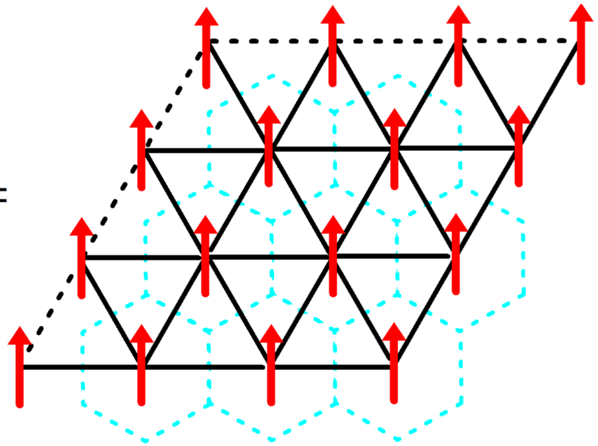


Z C Gu and X G Wen, 2009

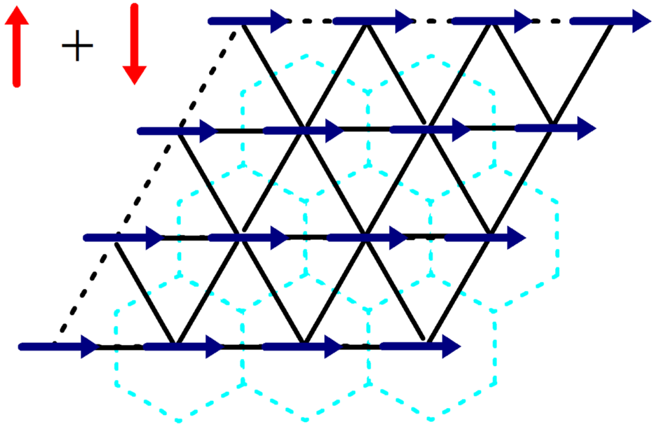
The key observation: edge states form projective representation of the symmetry group!

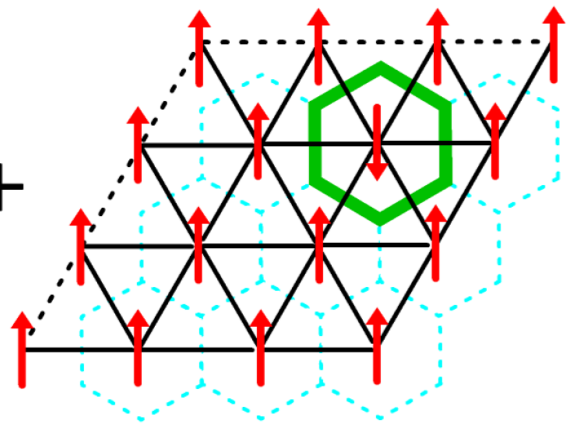
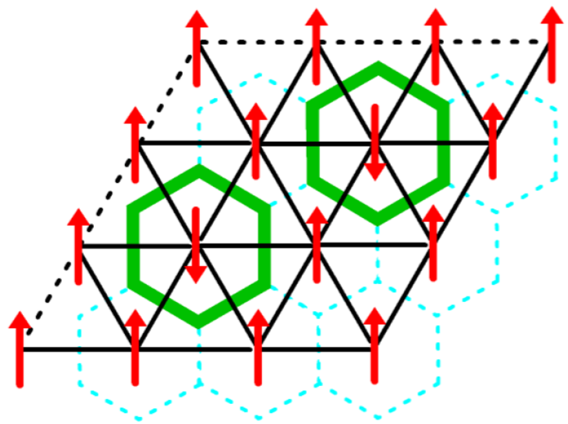
A revisit of transverse Ising model:

$$H = -\sum_{\langle pq \rangle} \sigma_p^z \sigma_q^z - t \sum_p \sigma_p^x$$

$|\Psi_{Ferro}\rangle =$ 

$\rightarrow \; = \; \uparrow + \downarrow$

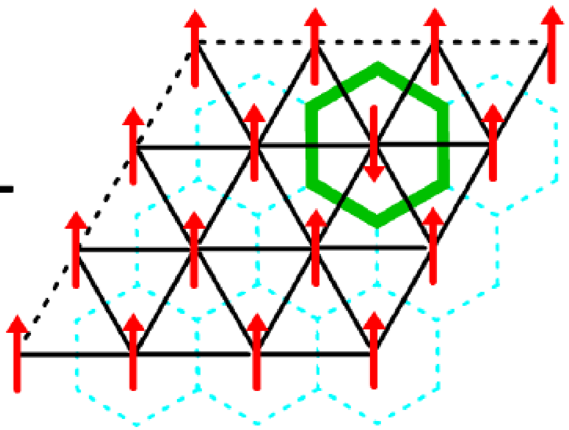
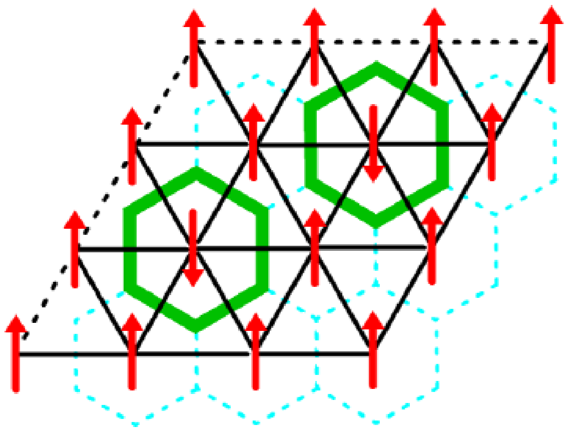
$|\Psi_0\rangle =$ 

$|\Psi_0\rangle = +$  $+$  $+ \ldots$

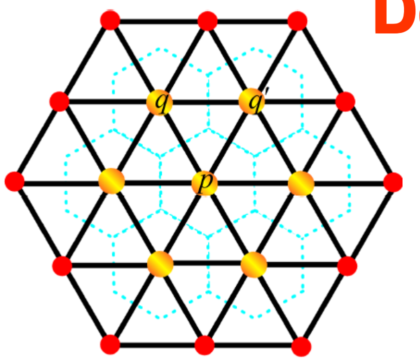
An example of Ising SPT phase in 2D

How many different paramagnetic phases? **Two!**

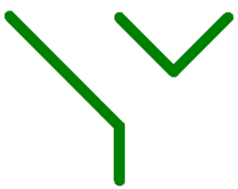
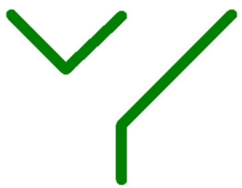
(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

$|\Psi_1\rangle = -$  $+$  $+ \ldots$

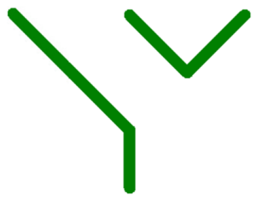
$$\widetilde{\sigma}_p^x = \sigma_p^x \prod_{\langle qq'p \rangle} i^{\frac{1+\sigma_q^z \sigma_{q'}^z}{2}}; \qquad [\widetilde{\sigma}_p^x, \widetilde{\sigma}_{p'}^x] = 0$$



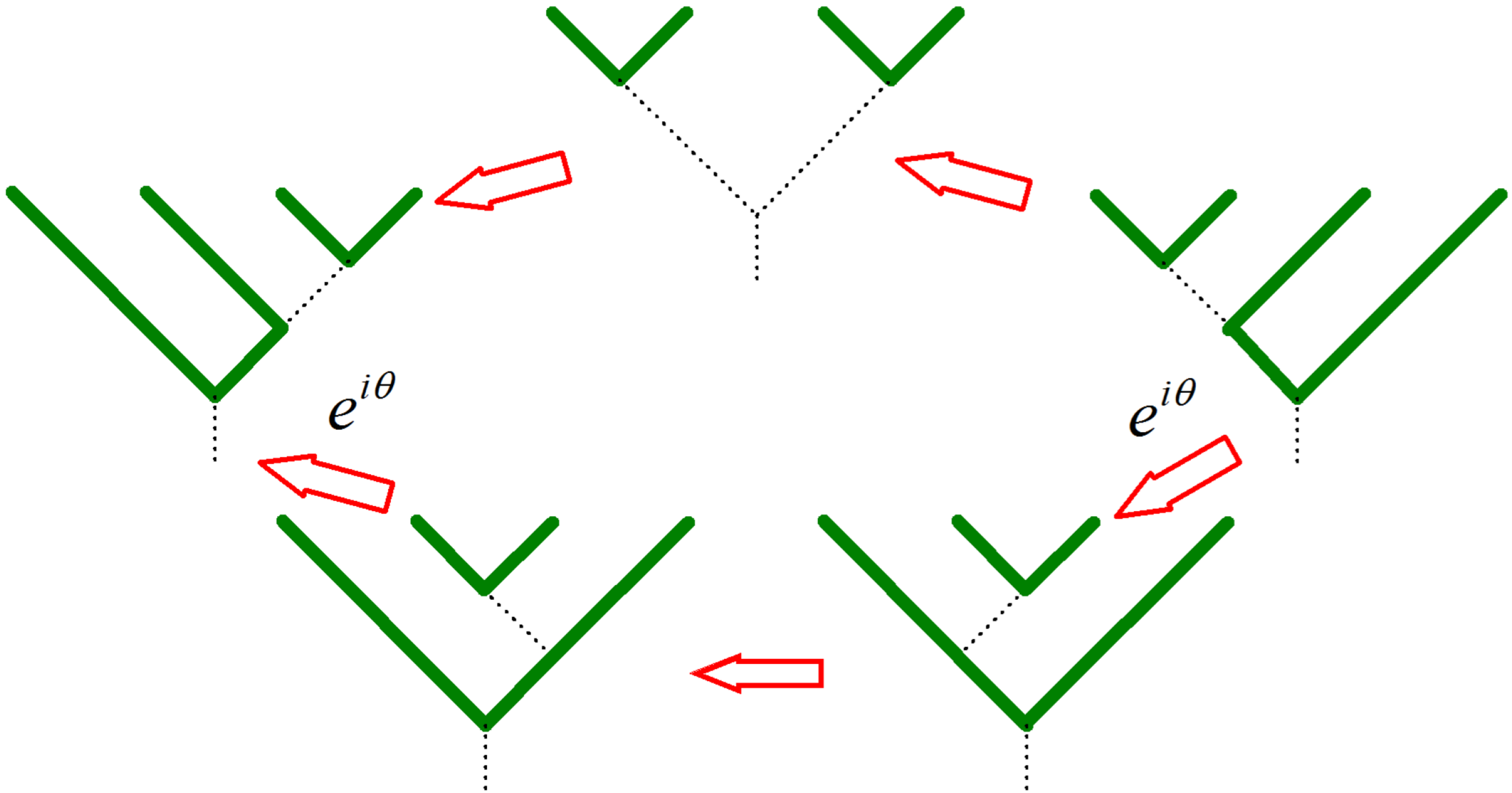
Domain deformation rule

 $= -$ 

But why not?

 $= e^{i\theta}$ 

Topologically consistent condition for fixed point wavefunction



$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

Duality between Ising model and Z_2 gauge model

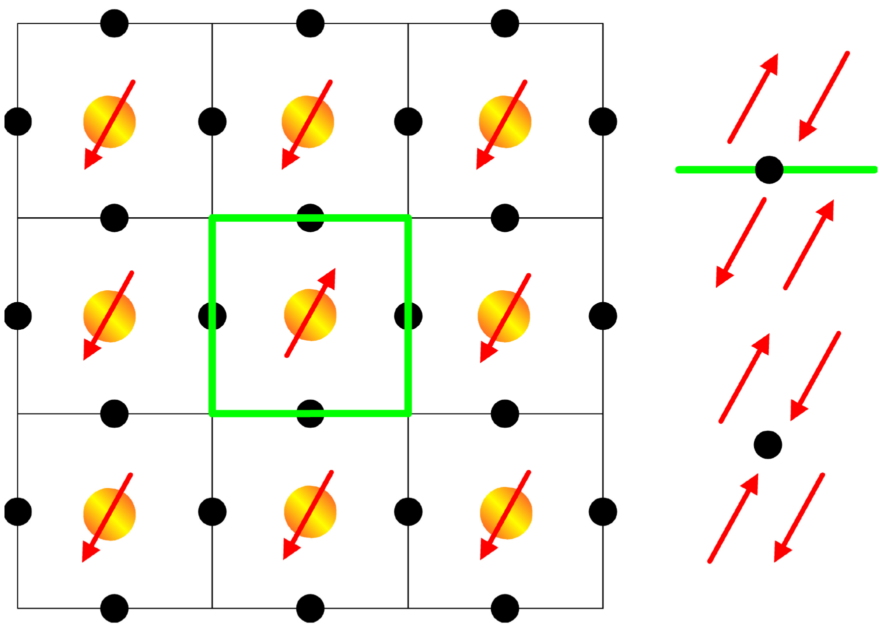
$$H_{Z_2} = U\sum_v \left(1-\prod_{l\in v}\tau_l^z\right) - t\sum_p\prod_{l\in p}\tau_l^x \Longleftrightarrow H_{\mathrm{Ising}} = -\sum_p \sigma_p^x$$

$$\tau_l^z = \sigma_p^z\sigma_q^z$$

$$\sigma_p^x = \prod_{l\in p}\tau_l^x$$

$$H'_{Z_2} = U\sum_v \left(1-\prod_{l\in v}\tau_l^z\right) - t\sum_p\prod_{l\in p}\tau_l^x - h\sum_l \tau_l^z$$

$$H = -t\sum_p \sigma_p^x - h\sum_{\langle pq\rangle}\sigma_p^z\sigma_q^z$$



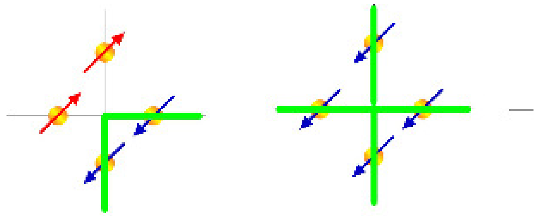
Duality map requires Z_2 symmetry to be preserved!

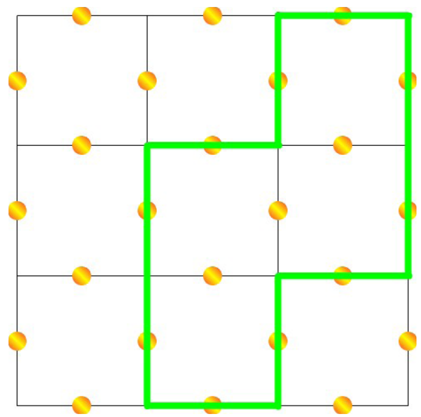
- String condensation corresponds to domain wall condensation

Z_2 gauge model(toric code model)

$$H_{Z_2} = U\sum_v \left(1 - \prod_{l\in v}\tau_l^z\right) - t\sum_p \prod_{l\in p}\tau_l^x$$

Kitaev 2003, M. Levin and X.G. Wen 2005

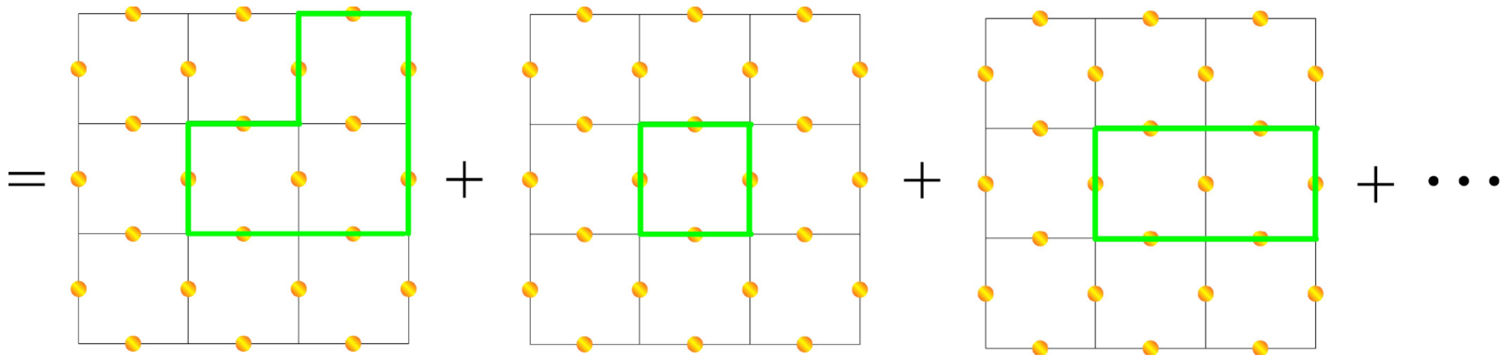




$$|\uparrow\rangle \rightarrow \text{no string}; \quad |\downarrow\rangle \rightarrow \text{one string}$$

Ground state

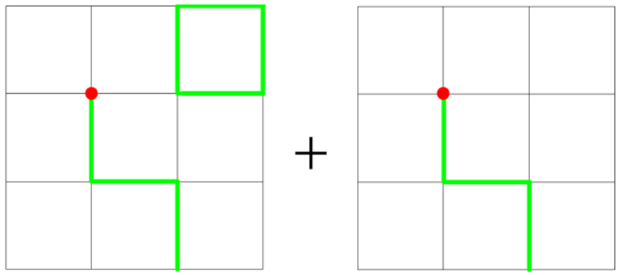
$$|\Psi_{Z_2}\rangle = \sum |X_{\text{close}}\rangle$$



Topological properties

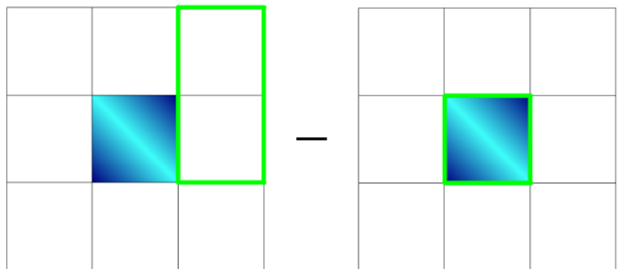
The same topological order as Z_2 spin liquid

e =



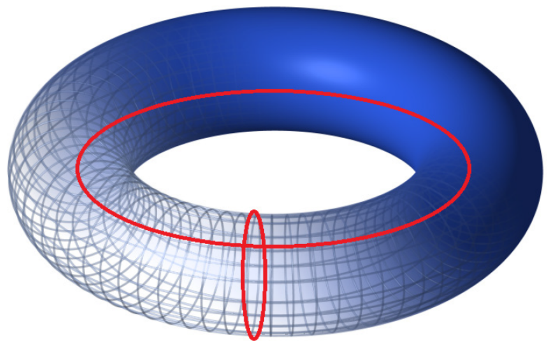
$+\cdots$

m =



$+\cdots$

Four-fold ground state degeneracy on a torus

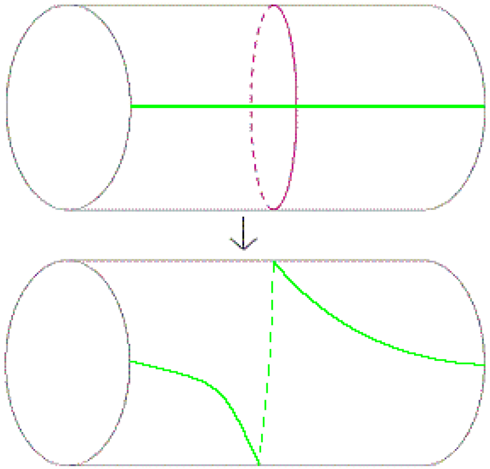


Quasi-particle in toric code model:

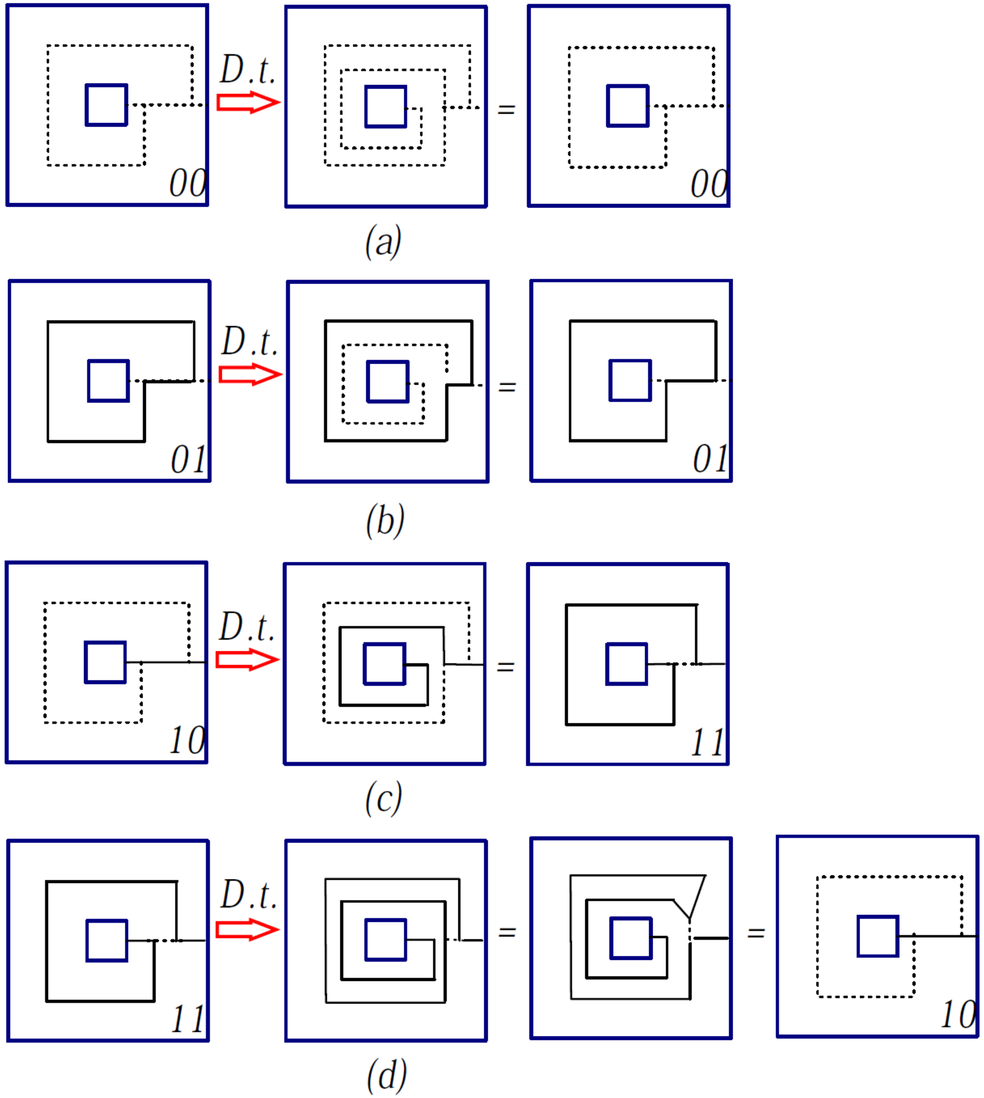
1, e, m, f=em

Dehn twist and T matrix

Dehn twist:



from Wikipedia



T matrix:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(\mathrm{e}^{\mathrm{i}\,\theta_i}) = (1, 1, 1, -1)$$

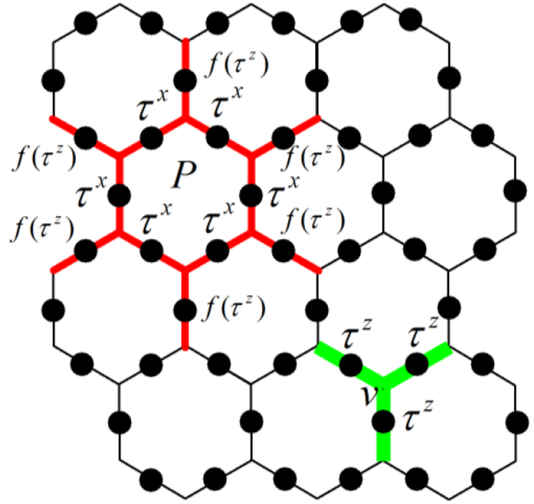
The twisted toric code: double semion model

M. Levin and X.G. Wen 2005

$$H_{dsemion} = U \sum_v \left(1 - \prod_{l \in v} \tau_l^z\right) - t \sum_p \left(\prod_{l \in p} \tau_l^x \prod_{l \in \text{leg of } p} i^{\frac{1+\tau_l^z}{2}} \right)$$

$$|\Psi_{\text{dsemion}}\rangle = \sum_{X \text{closed}} (-)^{n(X)} |X\rangle$$

$$f(x) = \frac{1+x}{2}$$



T matrix:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$(e^{i\theta_i}) = (1, 1, i, -i)$$

- End of string is a semion or anti-semion.

Quasi-particle types in double semion model:

1, s, $\underline{s}$, b=s$\underline{s}$

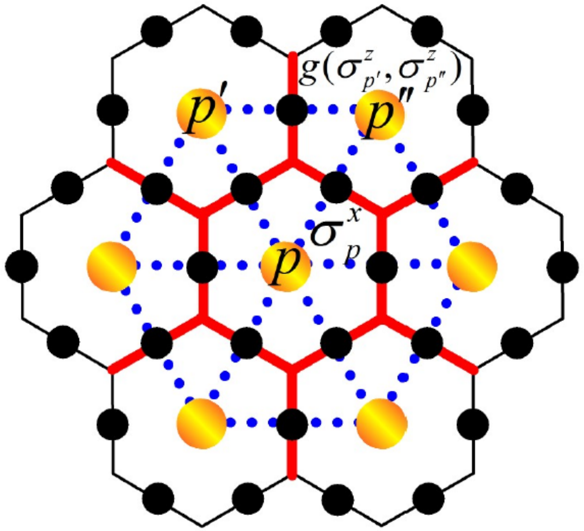
Dual theory of double semion model

$$H_{dsemion} = U\sum_v \left(1-\prod_{l\in v}\tau_l^z\right) - t\sum_p \left(\prod_{l\in p}\tau_l^x \prod_{l\in \text{leg of } p} i^{\frac{1+\tau_l^z}{2}}\right)$$

$$\tau_l^z = \sigma_{p'}^z\sigma_{q'}^z$$

$$\sigma_p^x = \prod_{l\in p}\tau_l^x$$

$$H_{\text{twistIsing}} = -\sum_p \widetilde{\sigma}_p^x = -\sum_p \left(\sigma_p^x \prod_{\text{sites}\in \text{p}} i^{\frac{1+\sigma_{p'}^z\sigma_{p''}^z}{2}}\right)$$



The dual theory of double semion model is an SPT ordered phase!

$$H = -\alpha\sum_p \sigma_p^x - (1-\alpha)\sum_p \widetilde{\sigma}_p^x$$

$$H' = -U\sum_v\prod_{i\in v}\sigma_i^z - \alpha\sum_p\prod_{i\in p}\sigma_i^x - (1-\alpha)\sum_p\left(\prod_{i\in p}\sigma_i^x\prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}}\right)$$

Different SPT orders

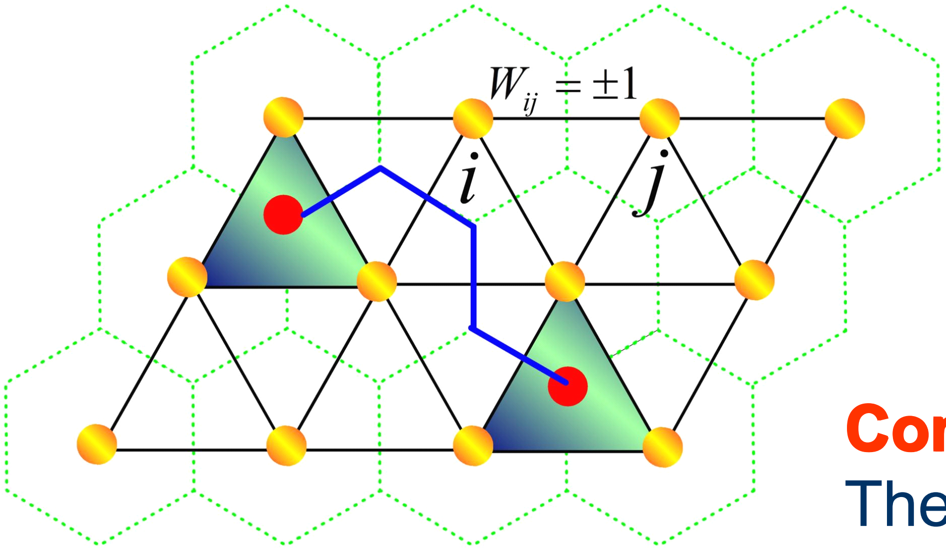
Different (intrinsic) topological orders

M Levin and Z.-C. Gu
(Phys. Rev. B 86, 115109 (2012))

Bulk response and the nature of gapless edge

Assume that Ising spins carry Z_2 gauge charge and can couple to background Z_2 gauge field

Z_2 gauge flux carries semion statistics!



$$\widetilde{W}_\beta |0\rangle = |0\rangle$$

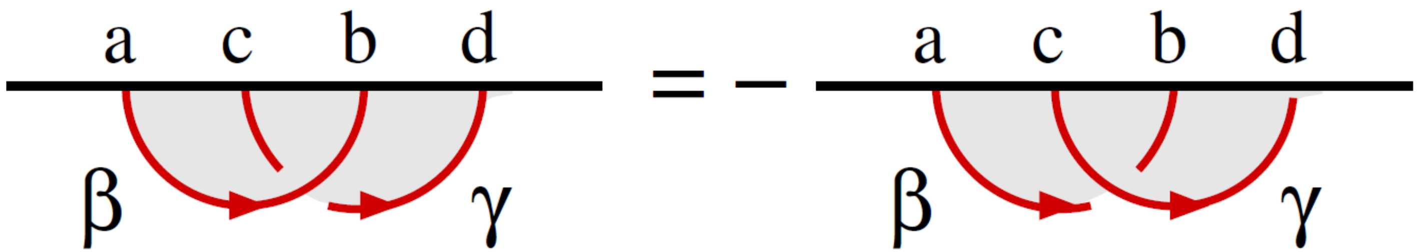
$$\widetilde{W}_\gamma |0\rangle = |0\rangle$$

$$\tilde{W}_\beta \tilde{W}_\gamma = -\tilde{W}_\gamma \tilde{W}_\beta$$

Contradiction

There is No 1D representation!

Non-trivial statistics of flux leads to degenerate edge states!



Do we have a systematic way to classify SPT phase?

- In-equivalent projective representations are classified by second group cohomology class, which classifies all 1D SPT phases.

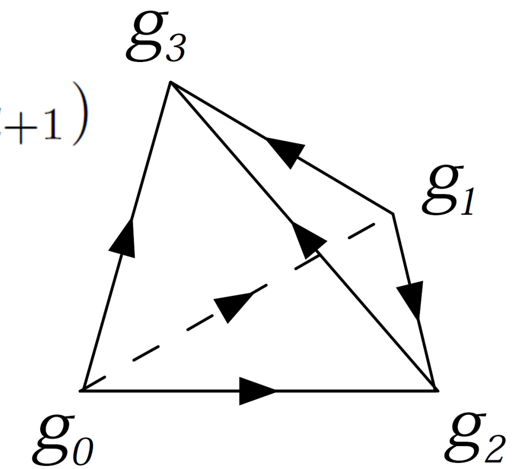
- In-equivalent flux statistics of G are classified by third group cohomology class, which classifies all 2D SPT phases.
 (R. Dijkgraaf and E. Witten, 1990)

Conjuncture: Does d+1-th group cohomology class classify dD SPT phases? Why group cohomology?

Group cohomology classifies topological Berry phase terms of discrete nonlinear sigma model with gauge anomaly on their boundary!

(bosonic) SPT phases in any dimensions with any symmetry

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_i\}} \prod_{d+1-\text{simplex}} \nu_{d+1}^{s_{01\cdots d}}(g_0, g_1, \cdots, g_{d+1})$$



- Branched(vertex ordered) d+1-simplex

$$\nu_{d+1} : G \times G \times \cdots \times G \mapsto U(1)$$

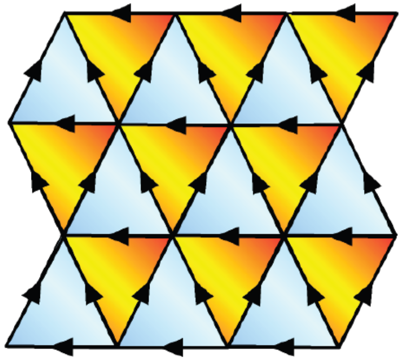
$$\nu_{d+1}(gg_0, gg_1, \cdots, gg_{d+1}) = \nu_{d+1}(g_0, g_1, \cdots, g_{d+1})$$

co-cycle condition: $$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{d+2}) = 1$$

- **SPT orders in bosonic systems are classified by d+1 group cohomology $\mathcal{H}^{1+d}[G, U(1)]$ in d spacial dimension.**

- Each element gives rise to an exactly solvable hermitian Hamiltonian with a unique ground state on closed manifold.

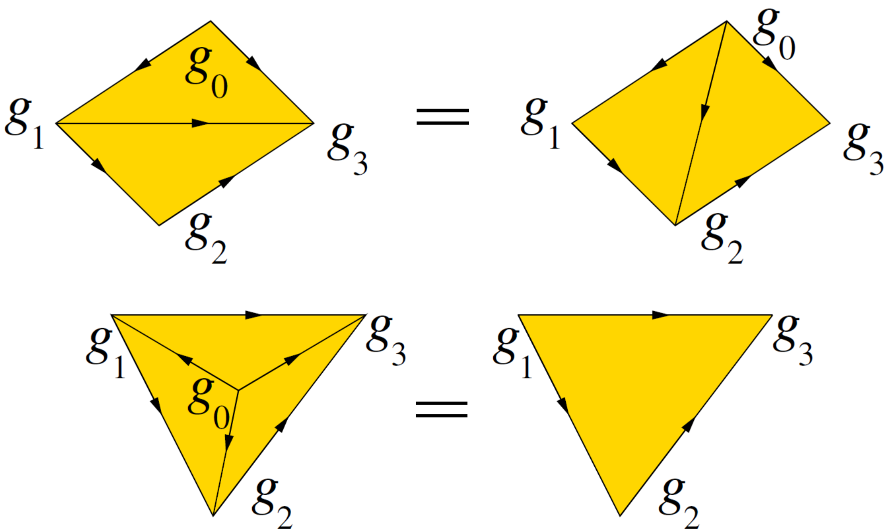
An example of 1+1D case



$$Z = |G|^{-N_v} \sum_{\{g_i\}} \mathrm{e}^{-S(\{g_i\})}$$

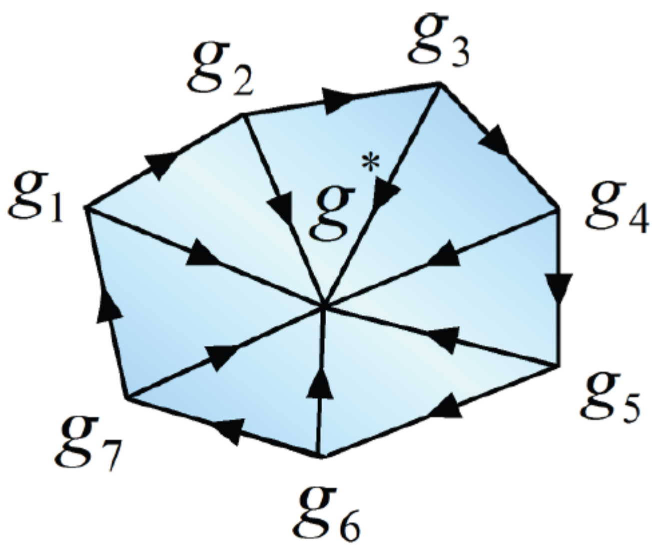
$$\mathrm{e}^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$

Topological invariant



$$\frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1$$

Fixed point wavefunction



$$\Psi(\{g_i\}_M) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$$

Classifications of bosonic SPT phases

Just like we use group representation theory to classify symmetry breaking phases, we use group cohomology theory to classify bosonic SPT phases.

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

Z_2^T means time reversal $T^2 = 1$

X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen (Science 338, 1604 (2012))

Basic concepts of classifying SPT phases in interacting fermion systems

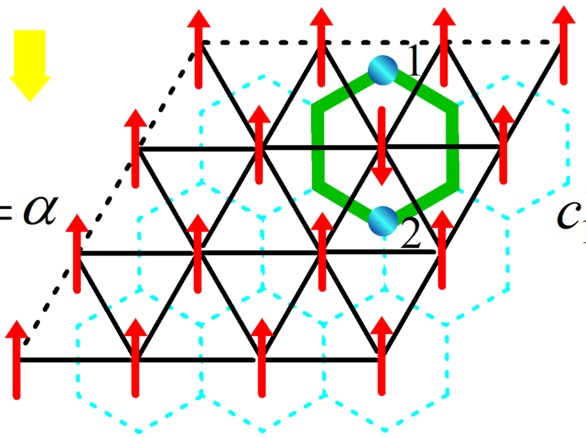
- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry. (Xie Chen, Z C Gu, X G Wen, Phys. Rev. B 84, 235128 (2011))

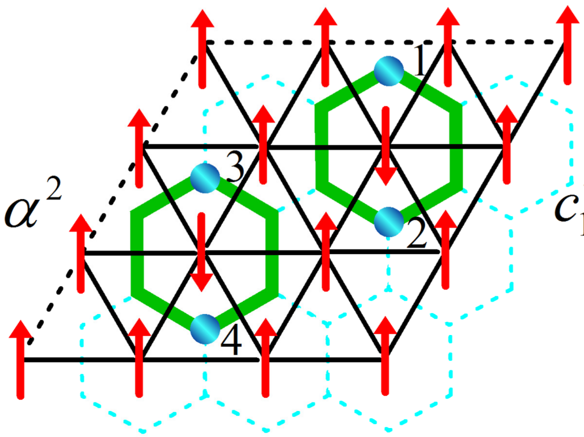
- The statistics of the gauge flux is still a good way to understand the classification scheme in 2D.
(Meng Cheng and Zheng-Cheng Gu, Phys. Rev. Lett. 112, 141602(2014))

- Discrete topological nonlinear sigma model can be generalized into interacting fermion systems.

- Lead to the discovery of new mathematics --- a (special) group super-cohomology theory, which can be regarded as a square root of group cohomology class.
(Z.-C. Gu, X.-G. Wen, arXiv:1201.2648)

An example of intrinsic fermionic Ising SPT phase in 2D

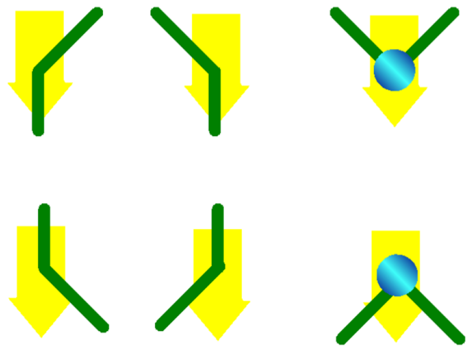
$|\Psi_f\rangle = \alpha$ 

$c_1^+c_2^+ + \alpha^2$ 

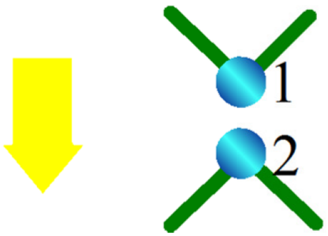
$c_1^+c_2^+c_3^+c_4^+ + \dots$

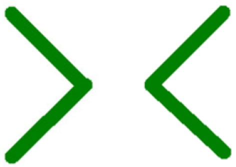
$\alpha = \pm i$

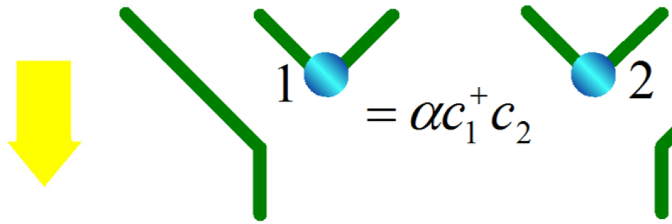
Domain decoration rule:



Domain deformation rule:

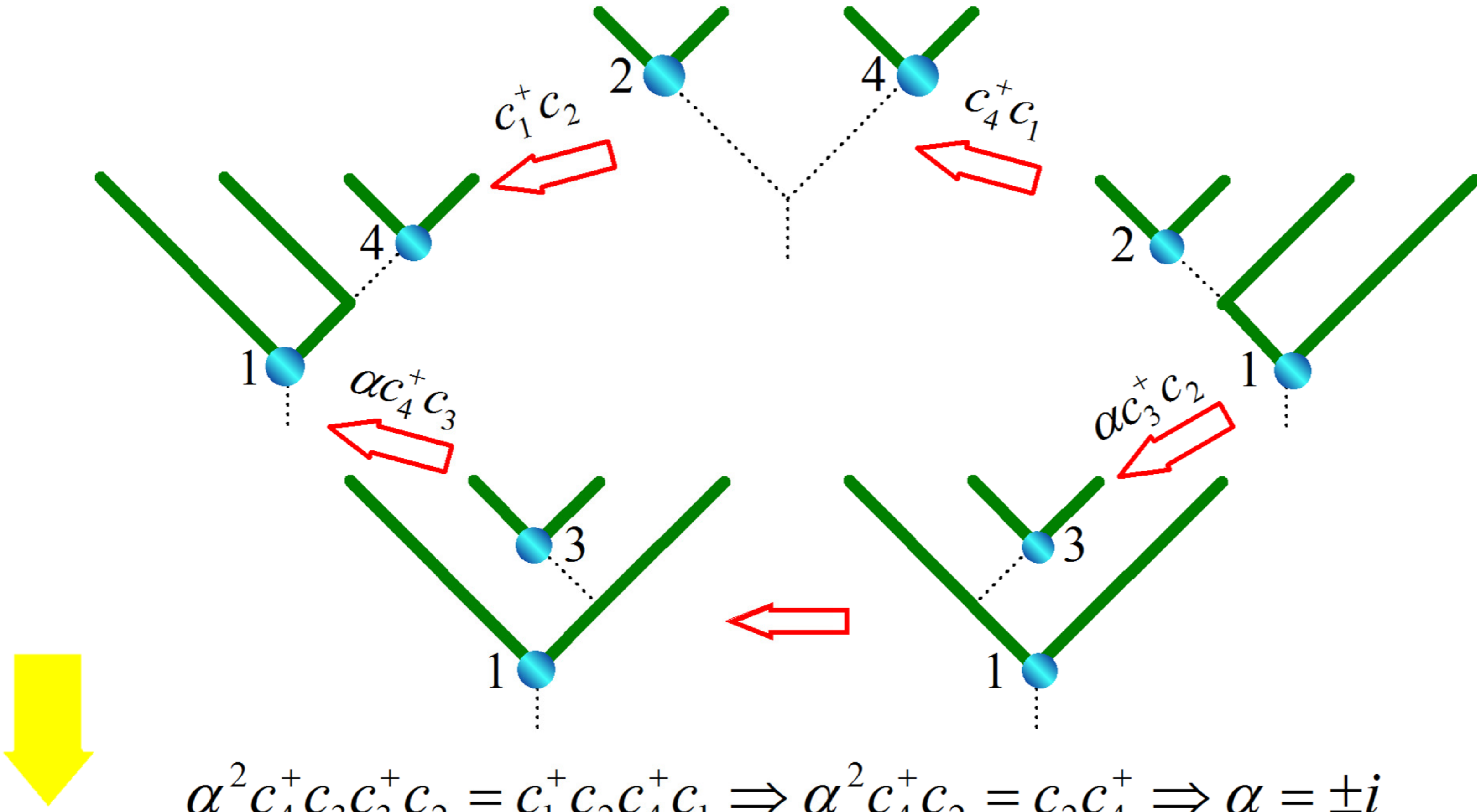


$= \alpha c_2^+ c_1^+$




$= \alpha c_1^+ c_2$


Topologically consistent condition for fixed point wavefunction



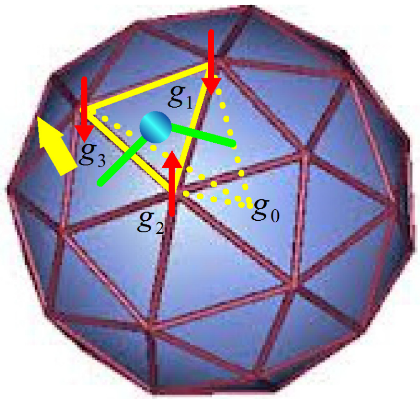
$$\alpha^2 c_4^+ c_3 c_3^+ c_2 = c_1^+ c_2 c_4^+ c_1 \Rightarrow \alpha^2 c_4^+ c_2 = c_2 c_4^+ \Rightarrow \alpha = \pm i$$

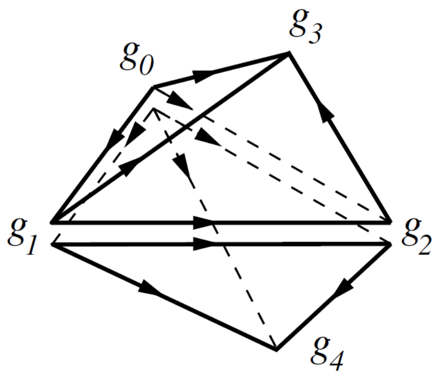
Z_2 gauge flux carries anyon statistics (exchange phase $\pm i\pi/4$)!

(Z.-C. Gu, Zhenghan Wang and X.-G. Wen arXiv:1309.7032,(2013))

The concept of Grassmann valued topological Berry phase

The domain decoration picture for wavefunction implies Grassmann graded amplitude for partition function





$$\mathcal{V}_3^+(g_0,g_1,g_2,g_3)=\nu_3^+(g_0,g_1,g_2,g_3)\times$$
$$\theta_{(1,2,3)}^{n_2(g_1,g_2,g_3)}\theta_{(0,1,3)}^{n_2(g_0,g_1,g_3)}\bar{\theta}_{(0,2,3)}^{n_2(g_0,g_2,g_3)}\bar{\theta}_{(0,1,2)}^{n_2(g_0,g_1,g_2)}$$

$$\mathcal{V}_3^-(g_0,g_1,g_2,g_4)=\nu_3^-(g_0,g_1,g_2,g_4)\times$$
$$\theta_{(0,1,2)}^{n_2(g_0,g_1,g_2)}\theta_{(0,2,4)}^{n_2(g_0,g_2,g_4)}\bar{\theta}_{(0,1,4)}^{n_2(g_0,g_1,g_4)}\bar{\theta}_{(1,2,4)}^{n_2(g_1,g_2,g_4)}$$

Arbitrary dimension

$$\mathcal{V}_d^+(g_0,g_1,...,g_d)\in M_f,\quad g_i\in G_b\quad G_b^{1+d}\to M_f$$

$$\mathcal{V}_d^+(g_0,g_1,...,g_d)=\nu_d^+(g_0,g_1,g_2,g_3,g_4,...,g_d)\times$$
$$\theta_{(1234...d)}^{n_{d-1}(g_1,g_2,g_3,g_4,...,g_d)}\theta_{(0134...d)}^{n_{d-1}(g_0,g_1,g_3,g_4,...,g_d)}...\times$$
$$\bar{\theta}_{(0234...d)}^{n_{d-1}(g_0,g_2,g_3,g_4,...,g_d)}\bar{\theta}_{(0124...d)}^{n_{d-1}(g_0,g_1,g_2,g_4...,g_d)}...$$

Total symmetry

$$G=G_b\otimes Z_2^f$$

Z_2 graded structure

$$n_{d-1}(g_i,g_j,...,g_k)=0,1$$

$$\sum_{i=0}^{d}n_{d-1}(g_0,...,\hat{g}_i,...,g_d)=\ \text{even}$$

Fermionic topological nonlinear sigma model

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}$$

$$\equiv \int \prod_{(ij...k)} \mathrm{d}\theta_{(ij...k)}^{n_{d-1}(g_i,g_j,...,g_k)} \mathrm{d}\bar{\theta}_{(ij...k)}^{n_{d-1}(g_i,g_j,...,g_k)} \times$$

$$\prod_{\{xy...z\}} (-)^{m_{d-2}(g_x,g_y,...,g_z)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}(g_a,g_b,...,g_c)$$

$$\boxed{\begin{aligned} & n_{d-1}(g_1,g_2,...,g_d) \\ &= \sum_{i=1}^{d} m_{d-2}(g_1,...,\hat{g}_i,...,g_d) \bmod 2 \end{aligned}}$$

Super co-cycle condition(consistent domain deformation rules)

Topological invariant conditions enforce $\nu_{d+1}^{\pm}$ can be expressed by m_{d-1} and ν_{d+1} that satisfies:

$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0,\cdots,g_{i-1},g_{i+1},\cdots,g_{d+2}) = (-)^{f_{d+2}}$$

Example in 2+1D:

$$\boxed{\begin{aligned} f_1(g_0,g_1) &= 0; \\ f_2(g_0,g_1,g_2) &= 0; \\ f_3(g_0,g_1,...,g_3) &= 0; \end{aligned}}$$

$$\boxed{\begin{aligned} \nu_3^+(g_0,g_1,g_2,g_3) &= (-)^{m_1(g_0,g_2)}\nu_3(g_0,g_1,g_2,g_3), \\ \nu_3^-(g_0,g_1,g_2,g_3) &= (-)^{m_1(g_1,g_3)}/\nu_3(g_0,g_1,g_2,g_3) \end{aligned}}$$

$$\boxed{f_4(g_0,g_1,...,g_4) = n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)}$$

The inequivalent solution of n_d can be classified by $\mathcal{H}^d[G_b,\mathbb{Z}_2]$.

A (special) group super-cohomology theory

f_{d+2} is the Steenrod square Sq^2 of n_d, which maps:

$$n_d \in \mathcal{H}^d(G_b, \mathbb{Z}_2) \to f_{d+2} \in \mathcal{H}^{d+2}(G_b, \mathbb{Z}_2)$$

- The Steenrod square, one of the most novel structures in algebraic topology enters fermionic SPT phases!

Compute group super-cohomology class by using short exact sequence

d_{sp}	short exact sequence
0	$0 \to \mathcal{H}^1[G_b, U_T(1)] \to \mathscr{H}^1[G_f, U_T(1)] \to \mathbb{Z}_2 \to 0$
1	$0 \to \mathcal{H}^2[G_b, U_T(1)] \to \mathscr{H}^2[G_f, U_T(1)] \to \mathcal{H}^1(G_b, \mathbb{Z}_2) \to 0$
2	$0 \to \mathcal{H}^3[G_b, U_T(1)] \to \mathscr{H}^3[G_f, U_T(1)] \to B\mathcal{H}^2(G_b, \mathbb{Z}_2) \to 0$
3	$0 \to \mathcal{H}^4_{\text{rigid}}[G_b, U_T(1)] \to \mathscr{H}^4[G_f, U_T(1)] \to B\mathcal{H}^3(G_b, \mathbb{Z}_2) \to 0$

A valid graded structure must be obstruction free:

$$B\mathcal{H}^d[G_b, \mathbb{Z}_2] \equiv \{n_d | n_d \in \mathcal{H}^d[G_b, \mathbb{Z}_2] \text{ and } (-)^{f_{d+2}} \in \mathcal{B}^{d+2}[G_b, U(1)]\}$$

$$\mathcal{H}^4_{\text{rigid}}[G_b, U_T(1)] = \mathcal{H}^4[G_b, U_T(1)]/\Gamma,$$

Γ is a subgroup of $\mathcal{H}^4[G_b, U_T(1)]$ generated by $(-)^{f_4}$.

Classify fermionic SPT phases by using (special) group super cohomology theory

Interacting fermionic SPT phases					
$G_f \backslash d_{sp}$	0	1	2	3	Example
"none"$=Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	superconductor
$Z_2 \times Z_2^f$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	
$Z_2^T \times Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	supercond. with coplanar spin order
$Z_{2k+1} \times Z_2^f$	$\mathbb{Z}_{4k+2}$	$\mathbb{Z}_1$	$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_1$	
$Z_{2k} \times Z_2^f$	$\mathbb{Z}_{2k} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{4k}$	$\mathbb{Z}_1$	

Z_2^T means time reversal $T^2 = 1$ (Z.-C. Gu, X.-G. Wen, arXiv:1201.2648)

- The 2+1D classifications are consistent with (spin) Chern-Simons theory approach. (Meng Cheng and Zheng-Cheng Gu, Phys. Rev. Lett. 112, 141602(2014))

- The 3+1D topological superconductor with $T^2=1$ time reversal symmetry can not be obtained by K-theory classification for free fermion systems.

- The 3+1D topological superconductor with $T^2=1$ time reversal symmetry can not be realized as bosonic SPT phase either.

Towards a complete classification

Bosonic SPT phases

- Does the group super cohomology class give rise to a complete classification for bosonic SPT phases or not?

It is complete in 1+1D and 2+1D, but not in 3+1D and higher dimensions.

- Physically, this is because gauge-gravitational mixture anomaly exists in 3+1D and higher dimensions, and such a new anomaly can not be characterized by cohomology theory.

Example in 4+1D with a U(1) symmetry

- Gauge-gravitational mixture anomaly with topological response:

$$\mathbf{Z}_0(\text{sym.twist}, \mathcal{M}^5) \;=\; \exp[\mathrm{i}\frac{k}{3}\int_{\mathcal{M}^5} F \wedge \mathrm{CS}_3(\Gamma)]$$

(Juven Wang, Z C Gu and X G Wen, arXiv:1405.7689)

Cobordism theory can describe gauge-gravitational mixture anomaly! (Anton Kapustin arXiv:1404.6659)

Fermionic SPT phases

- Does the (special) group super cohomology class give rise to a complete classification for fermionic SPT phases or not?

It is complete in 1+1D, but incomplete in 2+1D. A general group super cohomology theory is very desired.

- Our recent work shows that there are 8 different fermionic 2D SPT phases protected by Ising symmetry.

(Z.-C. Gu and M. Levin, Phys. Rev. B 89, 201113(R) (2014))

- In a recent work, we find a (generic) group super-cohomology theory in 2+1D, which might give rise to a complete classification of fermionic SPT phases in 2+1D. (M Cheng and Z C Gu, to appear)

$$0 \to \mathscr{H}^3[G_f, U_T(1)] \to \mathscr{H}^3_{\text{general}}[G_f, U_T(1)] \to H^1(G_b, \mathbb{Z}_2) \to 0$$

- In 3+1D, we sill need to understand the generic group super-cohomology theory and even a super cobordism theory.

$$0 \to \mathscr{H}^{d+1}[G_f, U_T(1)] \to \mathscr{H}^{d+1}_{\text{general}}[G_f, U_T(1)] \to ? \to 0$$

- To describe gravitational-gauge mixture anomaly in fermion systems, we even need a super cobordism theory.

Other applications and future work

Application in high energy physics

- By studying $T^2=-1$ topological superconductor, we find that a pair of topological Majorana zero modes carry fractionalized C,P,T symmetries, with $T^4=-1$,$P^4=-1$,$C^4=-1$.

- By further assuming a Majorana neutrino is made up of four topological Majorana zero modes at cutoff scale, we naturally explained the origin of three generations of neutrinos and obtained the neutrino mass mixing matrix from a first principle. Mixing angles are intrinsically close to experimental data. Exact neutrino masses are predicted according to current neutrino oscillation data. (Z C Gu, arXiv:1308.2488, arXiv: 1403.1869)

Future directions

- SPT phases protected by supersymmetry.

- Twisted supersymmetry and a quantum theory of gravity.

- SPT phases as topologically stable and universal resources of measurement based quantum computation.