Platonic QHE

Index

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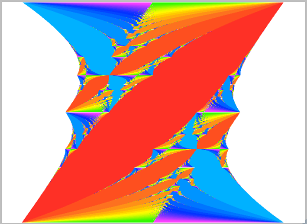
ESI, 2014

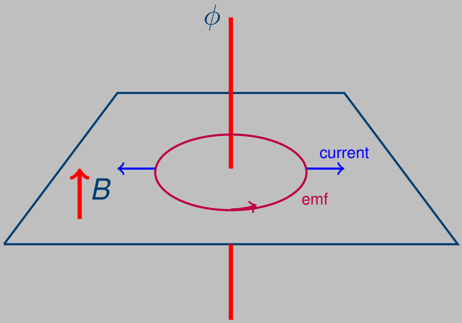


Outline

Laughlin pump

- emf: $-\dot{\phi}$
- Radial current: $\underbrace{\sigma}_{Hall}\dot{\phi}$
- Charge transported by fluxon:

$$\sigma \int \dot{\phi}\, dt = 2\pi\,\sigma \in \mathbb{Z}$$



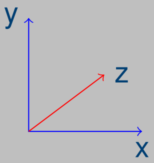
Losing charge in Hilbert space

Landau in the plane

$$z = x + iy, \quad \bar{z} = x - iy$$

$$2\partial = \partial_x - i\partial_y, \quad 2\bar{\partial} = \partial_x + i\partial_y$$

$$D = \bar{\partial} + \frac{Bz}{4}$$



$$H = \underbrace{\frac{1}{2}\left(p - \frac{1}{2}B \times x\right)^2}_{Landau} = 2D^*D + \frac{B}{2}, \quad \underbrace{[D, D^*] = \frac{B}{2}}_{ladder}$$

Holomorphic tricks

- Lowest Landau level

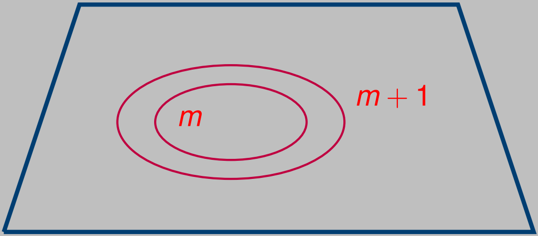
$$0 = D\,|\psi\rangle = \left(\bar\partial + \frac{Bz}{4}\right)|\psi\rangle$$

-

$$|\psi_0\rangle = e^{-B|z|^2/4}, \quad D\,|\psi_0\rangle = 0$$

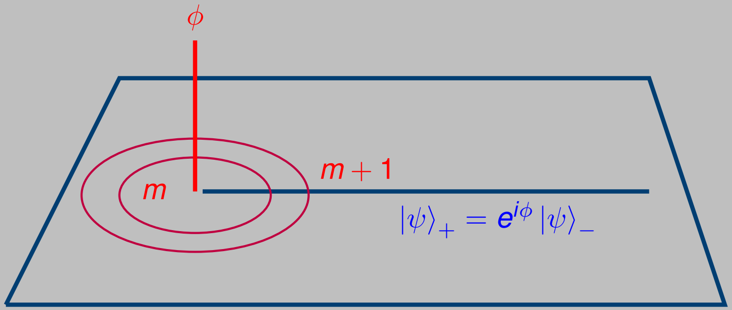
-

$$|\psi_m\rangle = z^m\,|\psi_0\rangle\,, \quad D\,|\psi_m\rangle = 0, \quad m \in \mathbb{Z}_+$$



Spectral Flow

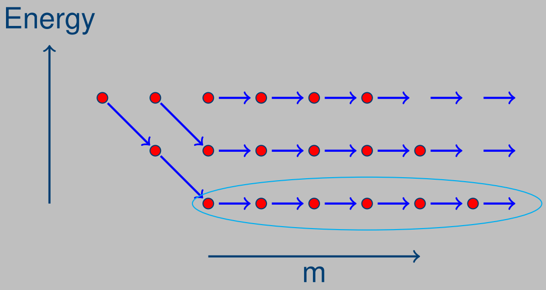
- Flux as bc



- Spectral flow: $m \to m + \frac{\phi}{2\pi}$

Spectral flow

Hilbert hotel

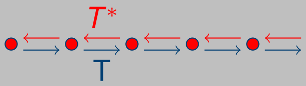


Flux insertion pushes unit charge (per Landau level) to infinity

Fredholm Index

$$
\begin{aligned}
Index T &= \dim \ker T - \dim \ker T^* \\
&= \dim \ker T^* T - \dim \ker TT^*
\end{aligned}
$$

- $T^* T = \mathbb{1} \Longrightarrow \dim \ker T^* T = 0$
- $TT^* = \mathbb{1} - |0\rangle \langle 0|$
  $\Longrightarrow \dim \ker TT^* = 1$
- $Index\ T = 0 - 1 = -1$



Fredholm Index

Careless commutators

$$Spec(TT^*) \setminus 0 = Spec(T^*T) \setminus 0$$

$$Tr(\mathbb{1} - T^*T) = \sum_j \left(1 - |t_j|^2\right)$$

$$\dim\ker\ T^*T - \dim\ker\ TT^* = Tr\,(\mathbb{1} - T^*T) - Tr\,(\mathbb{1} - TT^*)$$

If $\mathbb{1} - T^*T$ has a trace

$$Index\ T = Tr\,[T^*, T]$$

Relative index

Non-commutative Pythagoras

$$\underbrace{C = P - Q}_{cos}, \quad \underbrace{S = P - Q_\perp}_{sin}$$

Non-commutative Pythagoras

$$\underbrace{CS + SC = 0}_{non-commutative}, \quad \underbrace{C^2 + S^2 = \mathbb{1}}_{Pythagoras}$$

$$CS = (P - Q)(P - Q_\perp) = P - PQ_\perp - QP = PQ - QP$$

$$C^2 = (P - Q)^2 = P - QP - PQ + Q = Q_\perp P + P_\perp Q$$

$$C \leftrightarrow S \Leftrightarrow Q \leftrightarrow Q_\perp$$

$\infty - \infty \in \mathbb{Z}$

Comparing infinite projections

Relative index: Assume $Tr|P-Q|^{2m} < \infty$ .Then $\forall n \geq m$

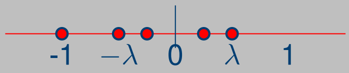
$$\underbrace{Tr(P-Q)}_{C}{}^{2n+1} = \dim\ker(P-Q-1) - \dim\ker(P-Q+1)$$

Proof:

$$C\,|\psi\rangle = \lambda\,|\psi\rangle \Rightarrow SC\,|\psi\rangle = \lambda S\,|\psi\rangle$$

$$-C(S\,|\psi\rangle\,) = -\lambda S\,|\psi\rangle$$

What if $S\,|\psi\rangle = 0$



$$0 = \langle S\psi|S\psi\rangle = \langle\psi|\,\mathbb{1} - C^2\,|\psi\rangle = 1-\lambda^2$$

$$Tr(P-Q)^{2n+1} = \sum \lambda_j^{2n+1} = deg(1) - deg(-1)$$

Fredholm=Relative index

$$(P-Q)^2P=(P-Q)Q_\perp P=PQ_\perp P$$

Theorem:

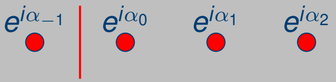
$$\underbrace{\mathit{Index}\ PUP}_{U:\mathit{Range}\,P\to\mathit{Range}\,P} = \mathit{Tr}(P-Q)^3, \quad Q=U^*PU$$

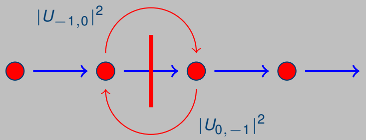
$$T=PUP, \quad T^*T=PQP$$

$$\begin{aligned}\dim\ker T^*T-\dim\ker TT^* &= \mathit{Tr}(\underbrace{P}_{\mathbb{1}}-PQP)-\mathit{Tr}(\underbrace{P}_{\mathbb{1}}-UQPQU^*)\\ &= \mathit{Tr}\,PQ_\perp P-\mathit{Tr}\,QP_\perp Q\\ &= \mathit{Tr}\,(P-Q)^3\end{aligned}$$

Quantized flows

Kitaev





$$\underbrace{\begin{pmatrix} e^{i\alpha_{-1}} & 0 & 0 & 0 \\ 0 & e^{i\alpha_0} & 0 & 0 \\ 0 & 0 & e^{i\alpha_1} & 0 \\ 0 & 0 & 0 & e^{i\alpha_2} \end{pmatrix}}_{Ind=0} \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{Ind=1}$$

Quantized flow

$$\sum_{j<0,k\geq 0} |U_{jk}|^2 - \sum_{j\geq 0,k<0} |U_{jk}|^2 \in \mathbb{Z}$$

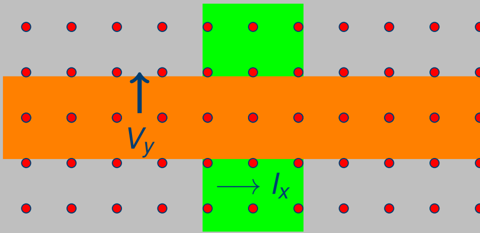
Driving and Response

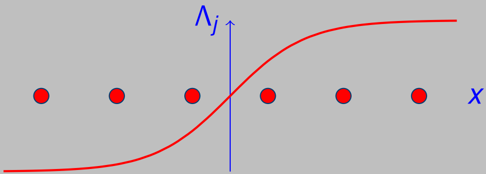
Hall effect

$$H(\phi_1, \phi_2) = U(\phi_1, \phi_2) H U^*(\phi_1, \phi_2)$$

$$U(\phi_1, \phi_2) = e^{i(\phi_1 \Lambda_1 + \phi_2 \Lambda_2)}$$

Finite voltage, not field



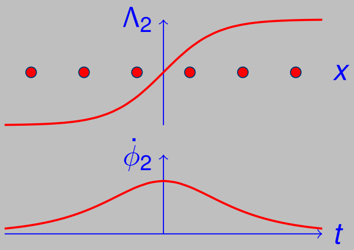


ϕ_2 control: Voltage

- $U(\phi_2) = e^{i\phi_2 \Lambda_2(y)}$
- Driving voltage:

$$\int_{\infty}^{\infty} E \cdot dy = \dot{\phi}_2 \underbrace{\left(\Lambda_2(\infty) - \Lambda_2(-\infty)\right)}_{=1}$$

$$= \dot{\phi}_2$$



Voltage

- Linear response=Adiabatic driving

Robust

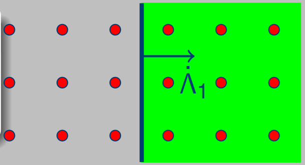
Precise shape of $\Lambda(x)$ and $\phi(t)$ irrelevant

Currents

Conserved currents

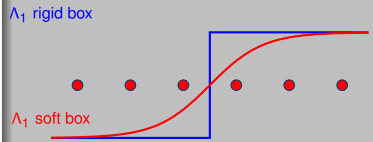
Unitary family

$$H(\phi_1) = U(\phi_1) H U^*(\phi_1), \quad U(\phi_1) = e^{i\phi_1 \Lambda_1}$$



Current: Conservation law of charge

$$\partial_1 H = i[\Lambda_1, H(\phi_1)] = \underbrace{-\dot{\Lambda}_1}_{\textit{rate of charge in box}}$$



Adiabatic curvature

Adiabatic curvature for unitary families

$$\Omega_{12} = i\, Tr\, P(\Lambda_1 P_\perp \Lambda_2 - \Lambda_2 P_\perp \Lambda_1)P$$

Proof:

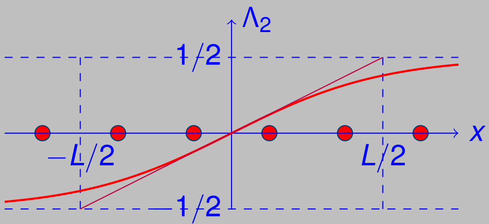
$$\begin{aligned} P[\partial_1 P, \partial_2 P]P &= -P\big[[\Lambda_1, P], [\Lambda_2, P]\big]P \\ &= -P\big[[\Lambda_1, P_\perp], [\Lambda_2, P_\perp]\big]P \\ &= P(\Lambda_1 P_\perp \Lambda_2 - \Lambda_2 P_\perp \Lambda_1)P \end{aligned}$$

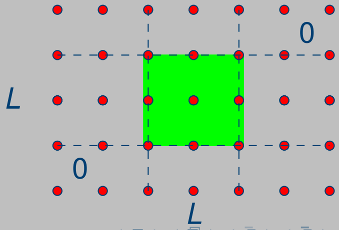
Adiabatic curvature=2π Kubo

P short range $\wedge$ translation invariant, $L \to$ *infty*

$$\Omega_{12} = \overbrace{\lim_{L\to\infty} \frac{i}{L^2}\, Tr_{L\times L}\, P(X_1 P_\perp X_2 - X_2 P_\perp X_1)P}^{2\pi Kubo}$$

$$= \frac{i}{L^2} \int_{-L/2}^{L/2} \prod_{j=1}^{3} d^2x_j\, \langle x_j | \, P \, | x_{j+1} \rangle)(x_2 \times x_3), \quad x_4 = x_1$$





2π Kubo=Index

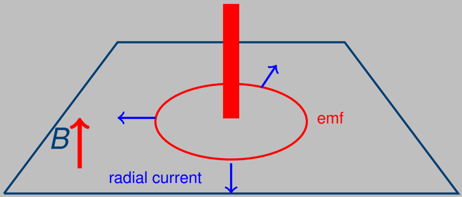
Index(PUP)

P=Fermi-Projection

$$\mathit{Index}(PUP) = 2\pi \; \mathit{Kubo} \qquad \underbrace{U = \frac{z}{|z|}}_{AB\ \mathit{fluxon}}$$

Landau : $|\psi_m\rangle \sim z^m e^{-|z|^2}$

$$PUP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \end{pmatrix}$$



Relative index in 2-D

- $$Q = UPU^*, \quad P - Q = [P, U]U^*, \quad U = \frac{z}{|z|}$$

- $$\langle z_1 | \, (P - Q) \, | z_2 \rangle = \langle z_1 | \, P \, | z_2 \rangle \left(1 - \frac{U(z_1)}{U(z_2)} \right)$$

- $$Tr \, (P - Q)^3 = \int \prod_{j=1}^{3} dz_j \, \langle z_j | \, P \, | z_{j+1} \rangle \left(1 - \frac{U(z_j)}{U(z_{j+1})} \right), \quad z_4 = z_1$$

P Translation invariant (Ergodic)

Relative index= Kubo

$$Tr\,(P-Q)^3 = \underbrace{\int d^2y\, d^2z\, \langle 0|\, P\, |y\rangle \langle y|\, P\, |z\rangle \langle z|\, P\, |0\rangle\ y \times z}_{2\pi Kubo}$$

Elements of proof:

- Both sides cubic in P
- Translation invariance:

$$\prod_{j=1}^{3} \langle z_j|\, P\, |z_{j+1}\rangle = \prod_{j=1}^{3} \langle z_j + a|\, P\, |z_{j+1} + a\rangle$$

- 2 (of 6) integration over an explicit function:

$$\int d^2a \prod_{j=1}^{3} \left(1 - \frac{U(z_j + a)}{U(z_{j+1} + a)}\right)$$

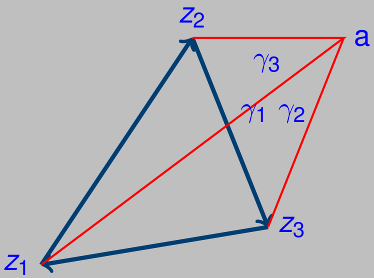
Relative index= Kubo

The world most complicated formula for area of triangles

$$\int d^2a \prod_{j=1}^{3}\left(1-\frac{U(z_j-a)}{U(z_{j+1}-a)}\right) = \underbrace{i\int d^2a\left(\sum \sin\gamma_j\right)}_{computation} = \underbrace{2\pi i\, Area(z_1,z_2,z_3)}_{magic}$$

- Connes:
  - Convergence,
  - Dimension analysis
  - Translation invariance
- Colin de Verdier:

$$\sum \gamma_j = \begin{cases} 2\pi & a \text{ in triangle} \\ 0 & a \text{ outside} \end{cases}$$



Platonic

Omissions

- Localization
- FQHE
- Chern Simons
- K-theory
- Bulk Edge duality
- Hofstadter butterfly
- Diophantine equations
- Hall viscosity
- Spin transport
- Open Systems

References

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- Bellissard, Schultz-Baldes, Prodan
- Aizenman, Graf, Wartzel
- Hastings and Michalakis
- Read, Taylor, Haldane

Acknowledgments

R. Seiler, B. Simon, G.M. Graf, M. Fraas, L. Sadun, O. Kenneth, J. Bellissard