Topological Interpretation of Levinson's Theorem

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  $\pi : E \to A$ surjective algebra morph., $J \cong \ker \pi$.

  $$J \hookrightarrow E \xrightarrow{\pi} A$$

- Boundary maps, e.g. $\mathrm{ind} : K_1(A) \to K_0(J)$, give rise to equations between topologically quantised physical quantities, one related to the system described by J the other to that by A.
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- I will show here that Levinson's theorem is of that type.

Levinson's theorem

Consider $H = H_0 + V$ on $L^2(\mathbb{R}^d)$

- H_0 is "nice" free motion (no bound states) (e.g. $H_0 = -\Delta, \partial, \cdots$)
- V (decaying) potential creating finitely many bound states
- $\sigma(H_0) = \sigma_{ac}(H_0) = \sigma_{ac}(H) = I_{H_0}$ (interval)

Scattering operator $S = S(H_0)$, $S(E)$ the scattering matrix (unitary)

Time delay at energy E is $iS^*(E)S'(E)$.

Levinson's theorem

Theorem

Integrated time delay $=$ *number of bound states* $+$ *corrections.*

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Integrated time delay = number of bound states + corrections.

$$\frac{i}{2\pi}\int_{\sigma(H_0)} (\mathrm{tr}(S^*(E)S'(E)) - reg.)\, dE = \mathrm{Tr}(P_b) + corr.$$

$$corr. = \begin{cases} \frac{1}{2} & \text{if } \exists \text{ halfbound state} \\ 0 & \text{else} \end{cases} \qquad (d = 3)$$

tr trace on $L^2(\mathbb{S}^{d-1})$, Tr trace on $L^2(\mathbb{R}^d)$, P_b bound state projection.
Halfbound state (0-energy resonance): $H\Psi = 0$ for $\Psi \in L^2_{loc}(\mathbb{R}^d)\backslash L^2(\mathbb{R}^d)$ but in some weighted L^2

Usual proofs involve complex analysis but it is topology!

Topological version of Levinson's theorem 1

Compare evolution of $e^{-itH}\Psi$, $\Psi \in \mathrm{im} P_b^\perp$ with $e^{-itH_0}\Psi_\pm$, $\Psi_\pm \in L^2(\mathbb{R}^d)$ such that $\lim_{t\to\pm\infty} \|e^{-itH}\Psi - e^{-itH_0}\Psi_\pm\| = 0$.

- $\Omega_\pm := \mathrm{s}-\lim_{t\to\pm\infty} \left(e^{-itH}\right)^* e^{-itH_0}$ wave operators.
- $\Omega = \Omega_-$ an isometry intertwining dynamics of H_0 with that of $H|_{ac}$

$$\Omega^*\Omega = 1, \qquad \Omega\Omega^* = 1 - P_b$$

$$S = \Omega_+^*\Omega_- = \mathrm{s} - \lim_{t\to+\infty} \left(e^{-itH_0}\right)^* \Omega e^{-itH_0}$$

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- May also consider $C(S^1, \mathcal{K}(\mathcal{H}')^+)$ in place of $C(S^1)$.
- Part of $\pi(\Omega)$ should be related to the scattering oper. S so that part of the winding number is integrated time delay.
- Eigenvalues may be embedded. No gap condition needed!
- Conceptual clearness.
- Topologically more involved models possible.

New formulae for wave operators

The condition on the wave operator is the difficult analytical part!

Theorem ([Kellendonk, Richard (d=1) 2009][Richard, Tiedra (d=3) 2013])
$H_0 = -\Delta$ *sur* $L^2(\mathbb{R}^d)$ *(*$d = 1, 3$*)*, $V(x)|1 + x^2|^{\rho_d} \in L^2(\mathbb{R}^d)$.

$$\Omega = 1 + R(A)(S(H_0) - 1) + \textit{compact}$$

$A = \frac{1}{2}(\vec{x} \cdot \vec{\nabla} + \vec{\nabla} \cdot \vec{x})$ *(gen. dilation)*, $R(A) = \oplus_{l \in \mathbb{N}} R_l(A)$ *(angular mom.)*

$$R_0(A) = \frac{1}{2}\left(1 + \tanh(\pi A) - i \cosh(\pi A)^{-1}\right) P_{s-wave}$$

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- There are results in $d = 2$ in the absense of half bound states.
- Bellissard & Schulz-Baldes have studied $H_0 =$ Laplacian on a lattice.

Some non-commutative topology

$\mathcal{H}$ inf. dim. sep. Hilbert space, $\mathcal{K}(\mathcal{H})$ compact operators.
W isometry of codim 1. $W^*W = 1$, $WW^* = 1 - \text{proj. of rank 1}$.

$$\begin{array}{ccccc}
\mathcal{K}(\mathcal{H}) & \hookrightarrow & \mathcal{B}(\mathcal{H}) & \xrightarrow{\pi} & \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \\
\| & & \cup & & \bigcup \\
\mathcal{K}(\mathcal{H}) & \hookrightarrow & C^*(W) & \xrightarrow{\pi} & \pi(C^*(W)) \cong C(\mathbb{S}^1)
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Theorem (Atkinson)

$F \in \mathcal{B}(\mathcal{H})$ is Fredholm whenever $\pi(F)$ is invertible.

Theorem (Index theorem; Gochberg, Krein)

If F is Fredholm then $\operatorname{ind}(F) = -\operatorname{wind}(\pi(F))$.

- index and winding number are homotopy invariant and characterise uniquely the homotopy classes.

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- $\pi(F) = \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \circ \Gamma_4$ (concatenation to restrictions on sides)

$$\begin{aligned} \Gamma_1(A) &= \mathrm{s} - \lim_{s \to -\infty} e^{isA} F(A, B) e^{-isA} \\ \Gamma_2(B) &= \mathrm{s} - \lim_{t \to +\infty} e^{itB} F(A, B) e^{-itB} \end{aligned}$$

similarily for Γ_3, Γ_4.

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$\text{wind}(\pi(F))=w_1+w_2+w_3+w_4$,

$$w_i=\epsilon_i\frac{1}{2\pi\imath}\int_{-\infty}^{+\infty}\Gamma_i^{-1}(x)\Gamma_i'(x)dx,\quad \epsilon_1=\epsilon_2=1,\ \epsilon_3=\epsilon_4=-1$$

differentiability and integrability assumed.

M as energy-scale space

Specify $B = \frac{1}{2}\ln(-\Delta)$, A generator of scaling (dilation).
$\pi(\Omega) = \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \circ \Gamma_4$ with

$$\begin{aligned}
\Gamma_2(H_0) &= s - \lim_{t\to+\infty} e^{it\frac{1}{2}\ln H_0}\Omega e^{-it\frac{1}{2}\ln H_0} = S(H_0) \\
\Gamma_4(H_0) &= 1 \\
\Gamma_1(A) &= s - \lim_{s\to+\infty} e^{-isA}\Omega e^{isA} \quad \text{rescale } p \to e^{-s}p \\
&\overset{here}{=} 1 + R(A)(S(0)-1) = P_{hb}^{\perp} + (1-2R_0(A))P_{hb} \\
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So $w_3 = w_4 = 0$,

$$
\begin{aligned}
w_2 &= \text{integrated time delay} \\
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(1) Topol. invariants in QM arise as elements of $K(A)$ where A is a natural C^*-algebra to which H is affiliated.

$K_1(A)$ abelian group generated by homotopy classes of unitaries in A (or $M_n(A)$, $n \in \mathbb{N}$).

$K_0(A)$ Grothendieck group of the abelian semigroup generated by homotopy classes of projections in A (or $M_n(A)$, $n \in \mathbb{N}$).

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(2) Need to get numbers! These arise from homomorphisms $\tau : K(A) \to \mathbb{C}$.

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(3) Need to give these numbers a physical interpretation.

(4) When are they integer?

Find Fredholm operator F s.th. $\tau(g) = \mathrm{ind} F$ ($g \in K(A)$, $\tau \in ht(A)$).

Boundary maps

(5) A topological relation between two physical systems (algebras) A and J is given by an extension E: $J \hookrightarrow E \xrightarrow{\pi} A$.

From $J \hookrightarrow E \to A$ we get

$$\delta : K_i(A) \to K_{i+1}(J), \quad \delta^* : ht_i(J) \to ht_{i+1}(A)$$

such that

$$\tau(\delta x) = \delta^* \tau(g)$$

Examples

1. Bulk edge correspondances.
   $A = C^*(\text{bulk}) = \mathcal{B} \rtimes_B \mathbb{R}^2$,
   $g = g_F$ the class in K_0 of the Fermi proj. (supposed in a gap).

$$\begin{array}{ccc} C^*(\text{edge}) & \hookrightarrow \text{Wiener-Hopf} \xrightarrow{\pi} & C^*(\text{bulk}) \\ \text{wind}_k^{\|} & & \text{chern} = \delta^* \text{wind}_k^{\|} \\ \sigma_{edge} & = & \sigma_H = \text{chern}(g_F) \end{array}$$

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2. Levinson's theorem. g is the class in K_1 of $\pi(\Omega)$.

$$\begin{array}{rcl} C^*(\text{bounded}) & \hookrightarrow \text{Toeplitz} \xrightarrow{\pi} & C^*(\text{scattered}) \\ \text{Tr} & & \text{wind}_E \\ \text{number bound states} & = & \text{integr. time delay + corr.} \end{array}$$

Higher degree Levinson's theorems

Add degrees of freedom.

$H = H(y)$ for $y \in Y$ a $2n$ dim. closed manifold (top. space).

$$C(Y, \mathcal{K}(\mathcal{H})) \hookrightarrow \mathcal{E} \xrightarrow{\pi} C(S^1 \times Y, \mathcal{K}(\mathcal{H}'))$$

$\mathcal{E} = \mathrm{Toeplitz} \otimes C(Y)$. $P_b = P_b(y)$ vector bundle over Y.

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If $\Omega \in \mathcal{E}$

$$\mathrm{chern}_{2n}([P_b]_0) = \delta^* \mathrm{chern}_{2n}([\pi(\Omega)]_1)$$

(degree $2n$ Levinson theorem), explicitely ($n = 1$)

$$\begin{aligned} \frac{1}{2\pi i} \int_Y \mathrm{Tr}(P_b dP_b\, dP_b) &= \frac{1}{24\pi^2} \int_{\sigma(H_0)\times Y} \mathrm{tr}((S^* - 1)\mathbf{d}S\, S^*\mathbf{d}S\, S^*\mathbf{d}S) \\ &+ \text{ similar terms with } \quad \Gamma_i, \quad i = 1, 3, 4 \end{aligned}$$

d exterior differential on Y, $\mathbf{d}$ exterior differential on $\mathbb{R}^+ \times Y$.

Higher degree Levinson's theorems

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$$\begin{array}{ccccc} C^*(\text{bounded}) \otimes C(Y) & \hookrightarrow \text{Toeplitz} \otimes C(Y) & \xrightarrow{\pi} & C^*(\text{scattered}) \otimes C(Y) \\ \text{chern}_{2n} & & & \text{wind}_{2n+1} \\ \text{chern nb. of bd state bundle} & = & & ? \end{array}$$

- "Adiabatic curvature and the S-matrix" Sadun & Avron 1996 contains elements of a higher Levinson's theorem.
- I provide an example where the above identity is not trivial.

Aharonov Bohm point interaction

$$H_\alpha = \left(\imath\nabla + \alpha\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\right)^2$$

on $C_c^\infty(\mathbb{R}^2\backslash\{0\})$.

$$H_\alpha = \oplus_{m\in\mathbb{Z}} H_{\alpha,m}, \quad H_{\alpha,m} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(m+\alpha)^2}{r^2}$$

- If $c = |m+\alpha| \geq 1$ then $H_{\alpha,m}$ is essentially self-adjoint.
- If $c = |m+\alpha| < 1$ then $H_{\alpha,m}$ deficiency index $(1,1)$.
- $H_{\alpha=0}$ one parameter family of δ-interactions.
- For $\alpha \in (0,1)$, H_α describes a four parameter family of δ-interactions with magnetic flux tube at 0 ($B = \alpha\delta$).

$$(1-U)\begin{pmatrix} a_0 \\ a_{-1} \end{pmatrix} = 2i(1+U)\begin{pmatrix} \alpha b_0 \\ (1-\alpha)b_{-1} \end{pmatrix}$$

$\psi_m(r) = a_m r^{-c} + b_m r^c + o(r^c)$, $U \in U(2)$.

So $H = H_\alpha^U$, $U \in U(2)$, $\alpha \in (0,1)$.

Aharonov Bohm point interaction

- Free Hamiltonian is $-\Delta$, $\sigma_{ac}(H_\alpha^U) = \sigma(-\Delta) = \mathbb{R}_+$.
- Number of eigenvalues of H_α^U equals number of eigenvalues of U with positive imaginary part.

Theorem (Kellendonk & Pankrashkin & Richard 2011)

Let $\lambda_i \in \mathbb{C}$, $|\lambda_i| = 1$, $\Im(\lambda_1) < 0 < \Im(\lambda_2)$.
$Y = Y_{\lambda_1,\lambda_2} = \{U \in U(2) | U \text{ has eigenvalues } \lambda_1, \lambda_2\}$.

1. $Y \ni U \mapsto \Omega = \Omega(H_\alpha^U, -\Delta)$ *is continuous*
2. $\Omega \in \mathcal{E} = \text{Toeplitz} \otimes C(Y, M_2(\mathbb{C}))$,
3. $P_b = P_b^U$ *defines a non-trivial line bundle over* Y *with chern number* 1.