Homotopy Theory of Topological Insulators

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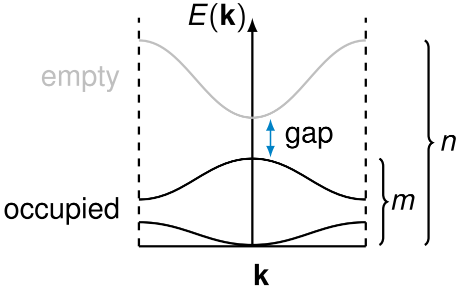

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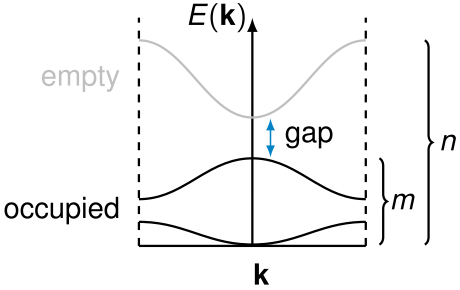

- Insulator ground state given by assignment

$$\mathbf{k} \mapsto V(\mathbf{k}) = \text{subspace spanned by } m \text{ occupied states}$$

Mathematical descriptions

Classifying map

$$
\begin{aligned}
T^d &\to \mathrm{Gr}_m(\mathbb{C}^n) \\
\mathbf{k} &\mapsto V(\mathbf{k})
\end{aligned}
$$

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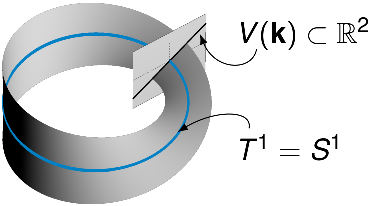
- Example: $d = 1$, $n = 2$ and $m = 1$



Topological Phases

"Two ground states are in the same topological phase if one can be $\left\{ \begin{array}{c} \text{adiabatically} \\ \text{continuously} \end{array} \right\} \left\{ \begin{array}{c} \text{deformed into} \\ \text{connected to} \\ \text{interpolated into} \end{array} \right\}$ the other."

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- Isomorphic as vector bundles: $\mathrm{Vect}_m^{\mathbb{C}}(T^d) = [T^d, \mathrm{Gr}_m(\mathbb{C}^\infty)]$
- Stably equivalent: $\tilde{K}(T^d) = [T^d, \mathrm{Gr}_\infty(\mathbb{C}^\infty)]$

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Homotopy classes $\supset$ Iso. classes of vector bundles $\supset$ K-groups

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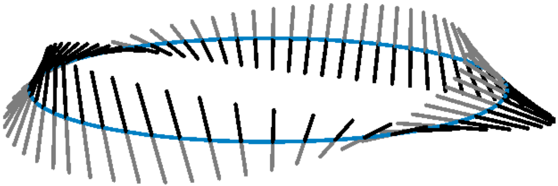
1. More topological phases
2. More restrictive definition of “strong” topological insulators

1. More topological phases

Symmetry $\mathcal{T} \circ \mathcal{I}$ in $d = 1$ with $m = 1$ of $n = 2$ bands occupied:

$$S^1 \to \mathrm{Gr}_1(\mathbb{R}^2)$$
$$(x, y) \mapsto \text{line through } (x, y)$$



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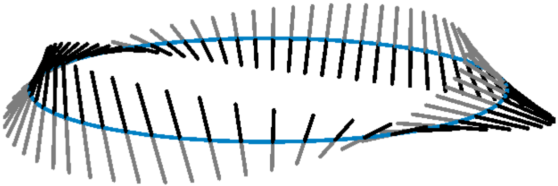

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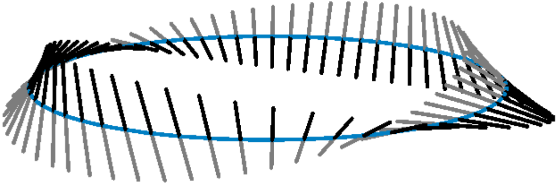

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S^3 $\qquad$ $\mathrm{Gr}_1(\mathbb{C}^2)$

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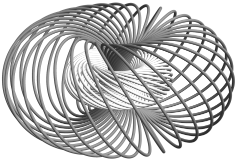

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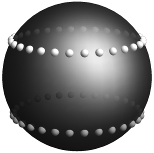

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Symmetry class	Dimension		
	1	2	3
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AIII	$\mathbb{Z}$	0	$\mathbb{Z}$
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
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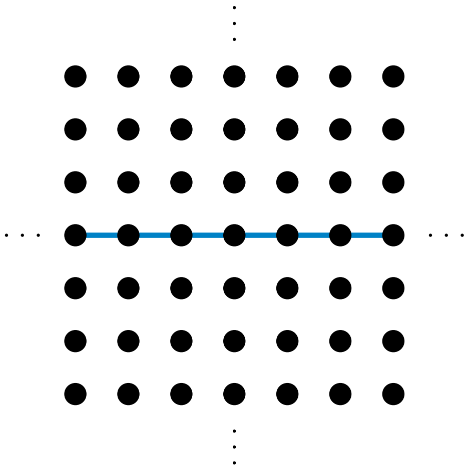

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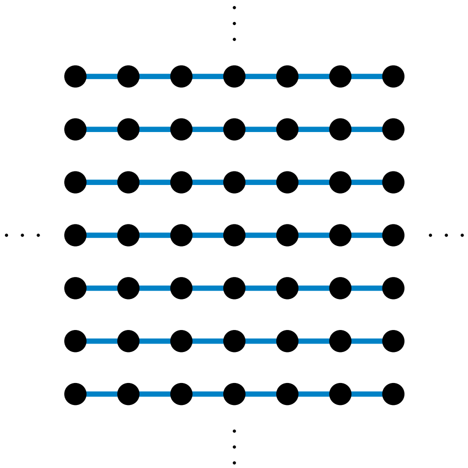

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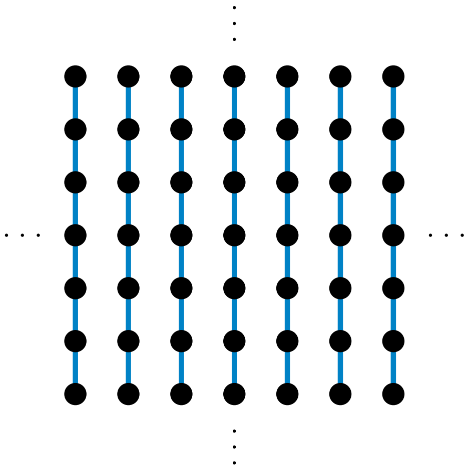

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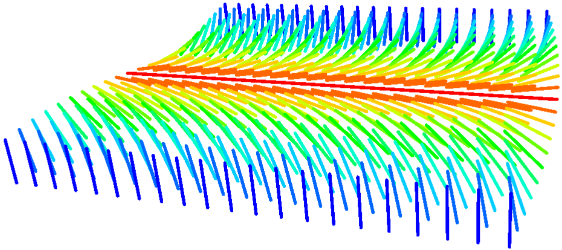

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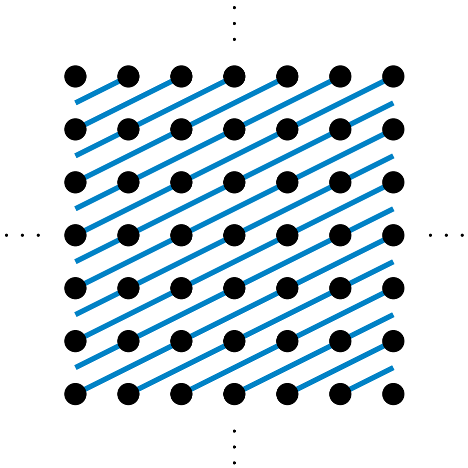

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$$= \underbrace{[S^d, \mathrm{Gr}_m(\mathbb{C}^n)]}_{\text{strong}} \times \underbrace{\prod_{p=1}^{d-1} [S^p, \mathrm{Gr}_m(\mathbb{C}^n)]^{\binom{d}{p}}}_{\text{weak}}$$

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Example $d = 3$, $m, n \gg 1$:

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Interpretation: Chern insulators stacked orthogonal to vector $(n_1, n_2, n_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

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Example $d = 3$, $m = 1$, $n = 2$:

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[T^3, \mathrm{Gr}_1(\mathbb{C}^2)] &= \{(n_0; n_1, n_2, n_3) \mid n_1, n_2, n_3 \in \mathbb{Z}, \\
&\qquad n_0 \in \mathbb{Z} \text{ for } n_1 = n_2 = n_3 = 0, \\
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New definition of “strong”: image of injective map

$$[S^3, \mathrm{Gr}_1(\mathbb{C}^2)] \hookrightarrow [T^3, \mathrm{Gr}_1(\mathbb{C}^2)]$$

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New definition of “strong”: image of injective map

$$[S^3, \mathrm{Gr}_1(\mathbb{C}^2)] \hookrightarrow [T^3, \mathrm{Gr}_1(\mathbb{C}^2)]$$

Generalises to all d and $\mathbb{Z}_2$-equivariant classes[RK,Guggenheim 2014]

Strong and weak

Example with symmetry $\mathcal{T} \circ \mathcal{I}$ for $d = 2$, $m = 1$, $n = 3$:

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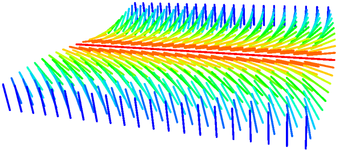

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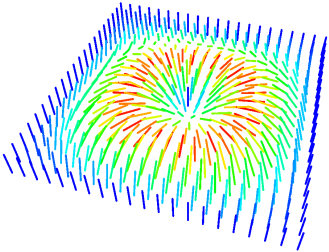

(reset)

Strong and weak

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$$[S^1, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{Z}_2 \text{ (Moebius band)}$$
$$[S^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{N} \text{ (Skyrmions)}$$

$$\begin{aligned}[T^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \{(n_0; n_1, n_2) \mid\ & n_1, n_2 \in \mathbb{Z}_2, \\ & n_0 \in \mathbb{N} \text{ for } n_1 = n_2 = 0, \\ & n_0 \in \mathbb{Z}_2 \text{ otherwise}\}\end{aligned}$$

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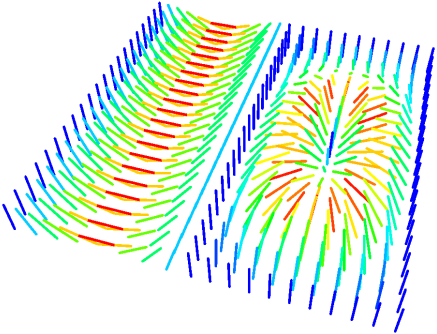
Breakdown of strong invariant $\mathbb{N} \to \mathbb{Z}_2$



(reset)

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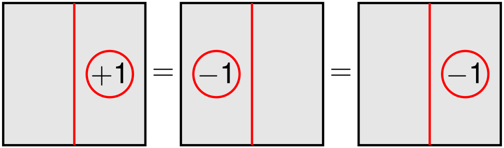

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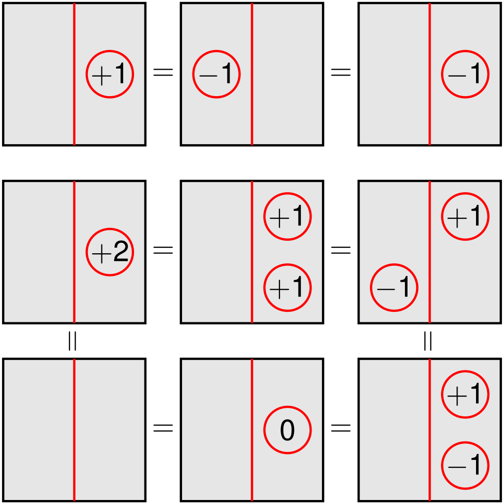
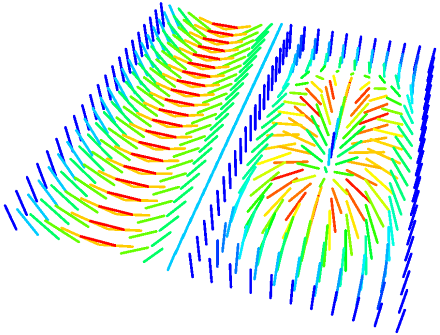


Stacked realisation



(reset)

Conclusion

- Homotopy classes $\supset$ Vect. bundle iso. classes $\supset$ K-groups
- Stability typically reached quickly, but some exceptions exist (Hopf insulator/superconductor)
- Only non-trivial maps from S^d are strong in general

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Thank you!