Topological properties in solids probed by Experiment

- Quantum Hall effect
- 2D TIs
- Weak Topological Insulators
- Strong topological insulators
- …
What is Topology?

Wikipedia: Topology is the study of continuity and connectivity

... homeomorphism ....
fiber bundles ...
..... Pontryagin classes
Haussdorf dimension
... Massey product
What is Topology?

Idea: distinguish geometrical objects by integer numbers

Example: Winding number of closed path

1. Define Complex plane

2. Path coordinate $t \in [0,1)$

3. Find topological Invariant

4. Proof correctness

$$z(t) = |z(t)| \cdot e^{i\phi(t)}$$

$$Q(z) = \frac{1}{2\pi i} \int_0^1 \frac{dz(t)/dt}{z(t)} dt$$

$$Q(z) = \frac{1}{2\pi i} \int_0^1 \frac{d}{dt} \log (z(t)) dt$$

$$= \frac{1}{2\pi} [\log (z(t))]_0^1 = \frac{1}{2\pi i} \log \left( \frac{|z(1)|e^{i\phi(1)}}{|z(0)|e^{i\phi(0)}} \right)$$

$$= \frac{1}{2\pi i} \cdot (\log (e^{i\phi(1)}) - \log (e^{i\phi(0)})) = \frac{\phi(1) - \phi(0)}{2\pi}$$
Topology in Solids: quantum Hall effect

- accurate measurement of $h/e^2$ (precision: $10^{-10}$)

Thouless et al., PRL 49, 405 (82)

$$j_y = -e \int \int_{MBZ} \frac{dk_x dk_y}{(2\pi)^2} \sum_{\alpha} v_{y}^{\alpha}$$

$$= \frac{E_x e^2}{h} \int \int_{MBZ} \frac{dk_x dk_y}{(2\pi)i} \sum_{\alpha} \left( \langle \frac{\partial u^{\alpha}}{\partial k_y} | \frac{\partial u^{\alpha}}{\partial k_x} \rangle - \langle \frac{\partial u^{\alpha}}{\partial k_x} | \frac{\partial u^{\alpha}}{\partial k_y} \rangle \right)$$

Chern number is a distinct integer, if the system is gapped, i.e. a band is either completely occupied or completely empty

$\rho_{xy} = \frac{h}{i e^2}$

InGaAs/AlGaAs

$\tau = 30$ mK

$n = 2.8 \times 10^{16}$ m$^{-2}$

$\mu = 3.4$ m$^2$/Vs
Chern number = integer: the argument

1) Define magnetic Brillouin zone (MBZ) by integer number of flux quanta inside each unit cell

⇒ wave function has zeros inside the unit cell (Aharonov Bohm phase ≠ 0)

2) Combination with required periodicity of MBZ requires a phase mismatch around the zero for a particular real space $x$

3) Integral of the gradient of the phase mismatch along the interface has to be single valued: 0, 2π, 4π ...

4) By Stokes theorem, this is identical to the Chern number

$$\sum_{\alpha} \left( \frac{\partial u^\alpha}{\partial k_y} \frac{\partial u^\alpha}{\partial k_x} - \frac{\partial u^\alpha}{\partial k_x} \frac{\partial u^\alpha}{\partial k_y} \right)$$

Requirement: the band must be full, such that the MBZ is densely occupied

Münks, Master thesis. MSU

Kohmoto, Ann. Phys. 160, 343 (85)
Chern number = integer: filling the band

**Band degeneracy: eB/h**

![Graph showing Landau levels and energy vs. magnetic field]

- Landau level 0
- Landau level 1
- Landau level 2

**Energy [meV]**

- 0
- 5
- 10
- 15
- 20
- 25
- 30

**magnetic field [T]**

- 0
- 2
- 4
- 6
- 8
- 10

\[ \rho_{xy} = \frac{B}{en} \]

- h/e^2
- h/2e^2
- h/3e^2

\[ \sigma_{xy} \frac{ie^2}{\hbar} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{1}{2\pi i} \left( \begin{pmatrix} \frac{\partial \phi_0}{\partial \theta} & \frac{\partial \phi_0}{\partial \varphi} \end{pmatrix} - \begin{pmatrix} \frac{\partial \phi_0}{\partial \varphi} & \frac{\partial \phi_0}{\partial \theta} \end{pmatrix} \right) \]

**Ground state Wave function**

Conductance with \( E_F \) at localized states does not depend on boundary conditions ⇒

Prange, The QHE

**Landau level energy diagram**

- Landau level 0
- Landau level 1
- Landau level 2

Energy [meV]

- 0
- 5
- 10
- 15
- 20
- 25
- 30

**Magnetic field [T]**

- 0
- 2
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\( \sigma_{xy} \frac{e^2}{\hbar} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{1}{2\pi i} \left( \begin{pmatrix} \frac{\partial \phi_0}{\partial \theta} & \frac{\partial \phi_0}{\partial \varphi} \end{pmatrix} - \begin{pmatrix} \frac{\partial \phi_0}{\partial \varphi} & \frac{\partial \phi_0}{\partial \theta} \end{pmatrix} \right) \)

**Boundary phase factors**

No QHE

\( E_F \)

G=1

QHE

\( E_F \)

\( G=1 \)

Adding disorder

- Extended states
- Localized states
- Edge states

5/35
Quantum Hall winding number in real space

2D LDOS at B = 12 T, 0.3 K

Corbino-Geometry

Prediction: one more flux quantum = one node encircles the flux = winding number of zeros

One node (0) per flux quantum in extended state

Extended state probed by STM (C=1)

PRL 101, 256802 (08)

Arovas et al. PRL 60, 619 (85)
Where is the charge of the quantized Hall voltage (bulk insulating)?

Answer: at the topological phase boundary, where different Chern numbers clash.

Laughlins argument: one more flux moves charge from inner to outer rim without energy cost (WF identical).

⇒ ....⇒ one chiral edge state per Chern number.
Seeing the edge state

Edge state = „metallic“ area of high compressibility

Scanning SET image (2.2 T)

Scanning capacitance image

Suddards et al.
NJP 14, 083015 (12)

- Local potential changes
  SET conductance
- Metallic edge state screens backgate potential for SET

courtesy A. Yacoby (Harvard)
Part II
2D Topological Insulators
(B = 0 T)
Topology in 2D at B = 0 T

Make a band gap in 2D by mixing two bands with different parity

Inverted bands

+ k-mixing

M from spin orbit

Splitting from k-p

k-space

Formally:

Spin 1

\[ H = \begin{pmatrix} h(k) & 0 \\ 0 & h^*(-k) \end{pmatrix} \]

Spin 2

\[ h(k) = \epsilon(k) I_{2 \times 2} + d_a(k) \sigma^a \]

Pauli matrix for s,p

\[ d_a(k) = (Ak_x, -Ak_y, M(k)) \]

\[ M(k) = M - B(k_x^2 + k_y^2) \]

Nodal line in k-space

topological number for one "spin"

\[ \sigma_{xy} = -\frac{1}{8\pi^2} \int \int dk_x dk_y \hat{d} \cdot \hat{\partial}_x \hat{d} \times \hat{\partial}_y \hat{d} \cdot \frac{e^2}{h} \]

\[ \Delta \sigma_{xy} = +/- 1 \text{ for } 0 < M < 4B \ (+: \text{Spin 1, - Spin2}) \]
Experiment: non-trivial topology at $B = 0$ T

Tuning sign of $M$ by $z$-confinement

$\chi_{1,2}^2 = k_x^2 + F \pm \sqrt{F^2 - (M^2 - E^2)/B_+B_-}$

$F = \frac{A^2 - 2(MB + ED)}{2B_+B_-}$

Scanning SQUID (3 K)

4-point resistance

$G = 0.01 \, e^2/h$

$T = 30 \, \text{mK}$

$M < 0$

$M > 0$

Nowack et al., Nature Mat. 12, 787 (13)

$G = 2 \, e^2/h$

$500 \, \text{nA/µm}$

$30 \, \mu\text{m}$

1st transport: Ong et al. PRB 28, 2289(83)

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1st transport: Ong et al. PRB 28, 2289(83)
Scanning tunneling spectroscopy ?
(LDOS with high resolution)

Heterostructure 2DES

- tunneling current: $10^{-50}$ A
- STS-Resolution: 100 nm

ca. 100 nm

2D TI

- tunneling current: $10^{-10}$ A
- STS-Resolution < 0.1 nm
Stacked 2D topological insulators = weak 3D topological insulators

First experimental weak TI: $\text{Bi}_{14}\text{Rh}_3\text{I}_9$ cleaved at the dark side

Kane et al., PRL 95, 226801 (05)

Rasche et al., Nature Mat. 12, 422 (13)
Probing spin transport in 2D TI

Non local voltage meas. (2D HgTe)

Strong signal if both areas TI

small signal, if one area =TI
one area = bulk

Brüne et al., Nature Mat. 8, 485 (12)
Quantum anomalous Hall effect

$H_{\uparrow}(k) = H^*_{\downarrow}(-k)$

$H_{\uparrow}(k) \neq H^*_{\downarrow}(-k)$

$\mathcal{H} = \begin{pmatrix} h(k) & 0 \\ 0 & h^*(-k) \end{pmatrix}$
Quantum anomalous Hall effect (Exp.)

A ferromagnetic 2D TI

Chang et al., Science 340, 167 (13)
Part III
3D Topological Insulators
(B = 0 T)
Kramers pair movement in 2D ribbon

Spin moved from left to right with band gap in bulk = spin pol. edge state required at $E_F$

$\begin{align*}
\text{(Pfaffian vs. Determinant at TRIM)}
\end{align*}$

Physical realization for $k_y$ movement

Edge state = bulk band property

States important for movement (Pfaffian vs. Determinant at TRIM)
3D Topological Insulators

Strong TI

Weak TI

0: (001)  0: (011)  0: (111)  1: (111)

TRIM

Dark surface

required surface states at $E_F$, all spin polarized and time reversal invariant

only relative Bloch wave function phases at TRIMs matter

Bulk inversion symmetry of crystal $\Rightarrow$
Sign at TRIM $= \text{product of parities of all states below the gap}$

$E(k)$ dispersion

TRIM $= \text{Time reversal invariant momenta (} k = -k \text{)}$
Materials: 3D Topological Insulators

Bulk inversion symmetry of crystal ⇒ Sign at TRIM = product of parities of all states below the gap

⇒ Band inversion (= exchanged parity) at 1 TRIM (typically Γ)

Energy levels at Γ (Bi$_2$Se$_3$)

Conduction band

Good means to invert bands

Spin-orbit interaction electron-electron interaction

...
**Exp. proof: 3D Topological Insulators**

*Zhang et al., Nature Phys. 5, 438 (09)*

**Our contribution**

Spin polarization (disentangled): 85%

*PRB 86, 235106 (12) Sb$_2$Te$_3*

**ARPES Bi$_2$Te$_3**

**E (k) with spins**

*Science 325, 178 (09) Bi$_2$Se$_3*
### Materials: topological insulators

<table>
<thead>
<tr>
<th>Type</th>
<th>Material</th>
<th>Band gap</th>
<th>Bulk transport</th>
<th>Remark</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D, v = 1</td>
<td>CdTe/HgTe/CdTe</td>
<td>&lt; 10 meV</td>
<td>insulating</td>
<td>high mobility</td>
<td>[26]</td>
</tr>
<tr>
<td>2D, v = 1</td>
<td>AlSb/InAs/GaSb/AlSb</td>
<td>~4 meV</td>
<td>weakly insulating</td>
<td>gap is too small</td>
<td>[64]</td>
</tr>
<tr>
<td>3D (1;111)</td>
<td>Bi$_{1-x}$Sb$_x$</td>
<td>&lt; 30 meV</td>
<td>weakly insulating</td>
<td>complex S.S.</td>
<td>[31, 35]</td>
</tr>
<tr>
<td>3D (1;111)</td>
<td>Sb</td>
<td>semimetal</td>
<td>metallic</td>
<td>complex S.S.</td>
<td>[34]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$_2$Te$_3$</td>
<td>0.3 eV</td>
<td>metallic</td>
<td>simple S.S.</td>
<td>[79]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$_2$Se$_3$</td>
<td>0.17 eV</td>
<td>metallic</td>
<td>distorted S.S.</td>
<td>[80, 81]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Sb$_2$Te$_3$</td>
<td>0.3 eV</td>
<td>metallic</td>
<td>heavily p-type</td>
<td>[82]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$_2$Te$_2$Se</td>
<td>~0.2 eV</td>
<td>reasonably insulating</td>
<td>$\rho_{xx}$ up to 6 $\Omega$ cm</td>
<td>[96, 99, 101]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>(Bi$_2$S)$_2$Te$_3$</td>
<td>&lt; 0.2 eV</td>
<td>moderately insulating</td>
<td>mostly thin films</td>
<td>[168]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$_{2-x}$Sb$<em>x$Te$</em>{3-y}$Se$_y$</td>
<td>&lt; 0.3 eV</td>
<td>reasonably insulating</td>
<td>Dirac-cone engineering</td>
<td>[103, 104, 187]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$<em>2$Te$</em>{1.6}$S$_{1.4}$</td>
<td>0.2 eV</td>
<td>metallic</td>
<td>n-type</td>
<td>[185]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$<em>{1.1}$Sb$</em>{0.9}$Te$_2$S</td>
<td>0.2 eV</td>
<td>moderately insulating</td>
<td>$\rho_{xx}$ up to 0.1 $\Omega$ cm</td>
<td>[185]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Sb$_2$Te$_2$Se</td>
<td>?</td>
<td>metallic</td>
<td>heavily p-type</td>
<td>[96]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>Bi$_2$(Te,Se)$_2$(Se,S)</td>
<td>0.3 eV</td>
<td>semi-metallic</td>
<td>natural Kawaiolite</td>
<td>[186]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>TlBiSe$_2$</td>
<td>~0.35 eV</td>
<td>metallic</td>
<td>simple S.S., largest gap</td>
<td>[87-89]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>TlBiTe$_2$</td>
<td>~0.2 eV</td>
<td>metallic</td>
<td>distorted S.S.</td>
<td>[89]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>TlBi(S,Se)$_2$</td>
<td>&lt; 0.35 eV</td>
<td>metallic</td>
<td>topological P.T.</td>
<td>[93, 94]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>PbBi$_2$Te$_4$</td>
<td>~0.2 eV</td>
<td>metallic</td>
<td>S.S. nearly parabolic</td>
<td>[106, 109]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>PbSb$_2$Te$_4$</td>
<td>?</td>
<td>metallic</td>
<td>p-type</td>
<td>[106]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>GeBi$_2$Te$_4$</td>
<td>0.18 eV</td>
<td>metallic</td>
<td>n-type</td>
<td>[96-98]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>PbBi$_2$Te$_7$</td>
<td>0.2</td>
<td>metallic</td>
<td>heavily n-type</td>
<td>[110]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>GeBi$_2$Te$_7$</td>
<td>?</td>
<td>?</td>
<td>no data published yet</td>
<td>[111]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>(PbSe)$_5$(Bi$_2$Se$_3$)$_6$</td>
<td>0.5 eV</td>
<td>metallic</td>
<td>natural heterostructure</td>
<td>[114]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>(Bi$_2$)(Bi$_2$Se$_2$6S$_4$)</td>
<td>semimetal</td>
<td>metallic</td>
<td>(Bi)$_n$(Bi$_2$Se$<em>3$)$</em>{3n}$ series</td>
<td>[112]</td>
</tr>
<tr>
<td>3D (1;000)</td>
<td>(Bi$_2$)(Bi$_2$Te$_3$)$_2$</td>
<td>?</td>
<td>?</td>
<td>no data published yet</td>
<td>[111]</td>
</tr>
<tr>
<td>3D TCI</td>
<td>SnTe</td>
<td>0.3 eV (4.2 K)</td>
<td>metallic</td>
<td>Mirror TCI, $n_M = -2$</td>
<td>[54]</td>
</tr>
<tr>
<td>3D TCI</td>
<td>Pb$_{1-x}$Sn$_x$Te</td>
<td>&lt; 0.3 eV</td>
<td>metallic</td>
<td>Mirror TCI, $n_M = -2$</td>
<td>[140]</td>
</tr>
<tr>
<td>3D TCI</td>
<td>Pb$<em>{0.77}$Sn$</em>{0.23}$Se</td>
<td>invert with T</td>
<td>metallic</td>
<td>Mirror TCI, $n_M = -2$</td>
<td>[138]</td>
</tr>
<tr>
<td>3D (1;111)?</td>
<td>SmB$_6$</td>
<td>20 meV</td>
<td>insulating</td>
<td>possible Kondo TI</td>
<td>[118-121]</td>
</tr>
<tr>
<td>3D (0;001)?</td>
<td>Bi$_{14}$Rh$_4$I$_9$</td>
<td>0.27 eV</td>
<td>metallic</td>
<td>possible weak 3D TI</td>
<td>[123]</td>
</tr>
<tr>
<td>3D (1;000)?</td>
<td>KBiPt ($R$ = Lu, Dy, Gd)</td>
<td>zero gap</td>
<td>metallic</td>
<td>evidence negative</td>
<td>[130]</td>
</tr>
<tr>
<td>Weyl S.M.?</td>
<td>Nd$<em>2$(Ir$</em>{1-x}$Rh$_{x}$)$_2$O$_7$</td>
<td>zero gap</td>
<td>metallic</td>
<td>too preliminary</td>
<td>[135]</td>
</tr>
</tbody>
</table>
Detecting prohibited backscattering

Backscattering prohibited by destructive interference

Berry phase: $-\pi/2$

Berry phase: $+\pi/2$

Li et al., Rev. Mod. Phys 83, 1057 (11)

Joint DOS from ARPES

Experiment

Joint DOS without backscatter

Bi$_{1-x}$Sb$_x$

$dl/dV$ at 0 mV

Fourier transform

STM map of standing electron waves

Roushan et al.
Nature 460, 1106 (09)
3D TI: tuning $E_F = E_D$

Mixing $\text{Bi}_2\text{Se}_3$ and $\text{Sb}_2\text{Te}_3$

$\Rightarrow$ towards devices

Y. Ando, J. Phys. Soc. Jap. 82, 102001 (13)
3D TI: magnetotransport

2D type transport

Topological surface state transport

Bi$_2$Se$_3$

$\theta = 57.5^\circ$

Bi$_2$Te$_2$Se$_1$

phase factor of oscillations due to Berry phase

Y. Ando, J. Phys. Soc. Jap. 82, 102001 (13)
3D TI spin transport

Spin accumulation at interface induces torque $\tau$ on ferromagnet

Fit to equation:

$$\frac{d\hat{m}}{dt} = -\gamma \hat{m} \times (\hat{B}_{\text{ext}} - \mu_0 M_{\text{eff}} m_z \hat{z}) + \alpha \hat{m} \times \frac{d\hat{m}}{dt} + \gamma \tau || \frac{\hat{m} \times (\hat{x} \times \hat{m})}{|\hat{x} \times \hat{m}|} + \gamma \tau \perp \frac{\hat{x} \times \hat{m}}{|\hat{x} \times \hat{m}|}$$

Torque by spin accumulation

Spin torque per current density

$\hat{\theta} = \frac{\hat{T} 2e}{j_y \hbar}$

Melnik et al., Nature 511, 449 (14)
Towards switchable Topological Insulators

crystalline Ge$_2$Sb$_2$Te$_5$

$E_F$

$E_D$

Valence band

Spin ARPES perp $k_{||}$, in plane

surface state

-K

+K

26.0

25.5

25.0

24.5

kinetic energy [eV]

0.2 0.0 0.2

$k_{||}$ [Å$^{-1}$]

APL 103, 243109 (13)

0.5 ns Loke et al., Science 336, 1566

1 fJ/bit Xiong et al., Science 332, 568

conducting

insulating

Phase change materials (Ge$_x$Sb$_y$Te$_z$)

CD/DVD

PCRAM

since 2008

since 1996

2011: 512 MB
Topological Crystalline Insulators

Idea: use point group symmetries in Brillouin zone

4 non-equivalent TRIMs at L1-L4 with inverted band gap = trivial Z2

L3, L4 on mirror plane:
classify mirror parities: $n_+ - n_-$
$\Gamma$: +1  L3, L4: -1

$\Rightarrow$ surface states for any surface with mirror symmetry between $\Gamma$ and $\bar{X}$

DFT: band inversion at L removed in PbTe

SnTe

ARPES Pb$_{0.6}$Sn$_{0.4}$Te

Xu et al., Nature Com. 3, 1192 (12)

Hsieh et al., Nature Com. 3, 982 (12)
Even more general?
Towards a periodic table of topology

<table>
<thead>
<tr>
<th>AZ class</th>
<th>Space of projectors in momentum space</th>
<th>BL class</th>
<th>$N_{f}^{\text{min}}$</th>
<th>Fermionic replica NL$\sigma$M target space</th>
<th>Topological or WZW term</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>${Q(k) \in G_{m,m+n}(\mathbb{C})}$</td>
<td>0</td>
<td>1</td>
<td>$U(2N) \times U(N)$</td>
<td>Pruisken</td>
</tr>
<tr>
<td>AI</td>
<td>${Q(k) \in G_{m,m+n}(\mathbb{C})</td>
<td>Q(k)^* = Q(-k)}$</td>
<td>4_</td>
<td>2</td>
<td>$Sp(2N) \times Sp(N) \times Sp(N)$</td>
</tr>
<tr>
<td>AII</td>
<td>${Q(k) \in G_{2m,2(m+n)}(\mathbb{C})</td>
<td>(i\sigma_y)Q(k)^*(-i\sigma_y) = Q(-k)}$</td>
<td>3_</td>
<td>1</td>
<td>$O(2N) \times O(N) \times O(N)$</td>
</tr>
<tr>
<td>AIII</td>
<td>${q(k) \in U(m)}$</td>
<td>1 or 2</td>
<td>1 or 2</td>
<td>$U(N) \times U(N) \times U(N)$</td>
<td>WZW</td>
</tr>
<tr>
<td>BDI</td>
<td>${q(k) \in U(m)</td>
<td>q(k)^* = q(-k)}$</td>
<td>9_</td>
<td>2</td>
<td>$U(2N) \times Sp(N)$</td>
</tr>
<tr>
<td>CII</td>
<td>${q(k) \in U(2m)</td>
<td>(i\sigma_y)q(k)^*(-i\sigma_y) = q(-k)}$</td>
<td>9_</td>
<td>2</td>
<td>$U(2N) \times O(2N)$</td>
</tr>
<tr>
<td>D</td>
<td>${Q(k) \in G_{m,2m}(\mathbb{C})</td>
<td>\tau_x Q(k)^* \tau_x = -Q(-k)}$</td>
<td>3_</td>
<td>1</td>
<td>$O(2N) \times U(N)$</td>
</tr>
<tr>
<td>C</td>
<td>${Q(k) \in G_{m,2m}(\mathbb{C})</td>
<td>\tau_y Q(k)^* \tau_y = -Q(-k)}$</td>
<td>4_</td>
<td>2</td>
<td>$Sp(N) \times U(N)$</td>
</tr>
<tr>
<td>DIII</td>
<td>${q(k) \in U(2m)</td>
<td>q(k)^T = -q(-k)}$</td>
<td>5 or 7</td>
<td>1 or 2</td>
<td>$O(2N) \times O(2N) \times O(2N)$</td>
</tr>
<tr>
<td>CI</td>
<td>${q(k) \in U(m)</td>
<td>q(k)^T = q(-k)}$</td>
<td>6 or 8</td>
<td>2 or 4</td>
<td>$Sp(N) \times Sp(N) \times Sp(N)$</td>
</tr>
</tbody>
</table>
Summary

Topological indices:
Integer bulk property requiring robust non-trivial transversal conductivity which implies boundary states at $E_F$

Experimentally realized:
- Quantum Hall effect (80’s): GaAs, Si, Graphene, …
  - 2D Topological insulator: HgTe, InAs/GaSb
  - Quantum anomalous Hall effect: BiCrSbTe
    - Weak 3D topological insulator: BiRhl
- **Strong 3D topological insulator** (many examples, mostly SO, but also Kondo ?)
  - Topological crystalline insulators: SnTe

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